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Linearly Coupled Anharmonic Oscillators and Integrability

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Abstract

The Painlevé test for a linearly coupled anharmonic oscillator is performed. We show that this system does not pass the Painlevé test. This suggests that this system is not integrable. Moreover, we apply Ziglin's (1983) theorem which provides a criterion for non-existence of first integrals besides the Hamiltonian. Calculating numerically the maximal one-dimensional Lyapunov exponent, we find regions with positive exponents. Thus, the system can show chaotic behaviour. Finally we compare our results with the quartic coupled anharmonic oscillator.

1. Introduction

Recently, much attention has been paid to the relation between coupled quartic anharmonic oscillators and integrability (Bountis *et al.* 1982; Yoshida 1984; Carnegie and Percival 1984; Lakshmanan and Sahadevan 1985; Steeb and Kunick 1985; Steeb *et al.* 1985*b*, 1986*a*, 1986*b*; Steeb and Louw 1986). The different aspects (singular point analysis, stability analysis of periodic solutions, numerical treatment) for studying integrability have been discussed. Depending on the coupling constants and the energy value E one finds that the system can be integrable or non-integrable. Among the non-integrable systems we find those with chaotic behaviour.

In the present paper we investigate a linearly coupled anharmonic oscillator. The Hamiltonian H under investigation is given by

$$H(x,p) = \tfrac{1}{2}(p_1^2+p_2^2+Ax_1^2+Ax_2^2)+\tfrac{1}{4}a(x_1^4+x_2^4)+cx_1x_2\,, \tag{1}$$

where $a>0$ and $A\geqslant 0$. The equations of motion are

$$\ddot{x}_1 = -Ax_1-ax_1^3-cx_2\,, \qquad \ddot{x}_2 = -Ax_2-ax_2^3-cx_1\,. \tag{2a, b}$$

This system describes two linearly coupled anharmonic oscillators. The potential admits the discrete group C_{2v}.

First we perform a singular point analysis (Painlevé test) for the system (2). The system does not pass this test, which suggests that the system is not integrable. Then, we apply Ziglin's theorem in order to prove whether or not the system is integrable. Finally, we calculate numerically the maximal one-dimensional Lyapunov exponent. Since the motion is bounded and this value is positive for certain energy values E and coupling constants c, we find that the system can behave chaotically.

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2. Particular Solutions

For special values of A, a and c particular solutions of system (2) can be found. For $c = 0$ the equations of motion are uncoupled. The resulting equation $\ddot{x} = -Ax - ax^3$ can be solved in terms of elliptic functions (Davis 1962). For $c \neq 0$ we can find a periodic solution of system (2) by setting $x_1 = x_2 = x$. Let us assume that $A + c \geqslant 0$. Then the solution to the equation of motion $\ddot{x} = -(A+c)x - ax^3$ is given by

$$x(t) = C_1 \operatorname{cn}((A+c+aC_1^2)^{\frac{1}{2}}(t-t_0), k), \tag{3}$$

where $k = C_1\{2(A+c)/a + 2C_1^2\}^{-1/2}$ (k modulus of the elliptic function cn). The constants of integration are C_1 and t_0. For the special case $A + c = 0$ equation (3) simplifies to

$$x(t) = C_1 \operatorname{cn}(a^{\frac{1}{2}} C_1(t-t_0), 2^{-\frac{1}{2}}). \tag{4}$$

Another periodic solution can be found by setting $x_1 = -x_2 = x$. Consequently, we get $\ddot{x} = (-A+c)x - ax^3$. Again the solution can be expressed by elliptic functions.

The elliptic functions are periodic functions. When the differential equations described above are considered in the complex plane, the solution is given by the elliptic function cn with a complex argument. Its singularities are simple poles of order one and are distributed doubly periodically in the whole complex z-plane.

3. Painlevé Test

It is desirable to have a simple approach for deciding whether a dynamical system is integrable or not. For ordinary differential equations the so-called Painlevé test (also called singular point analysis) can serve 'in a certain sense' to decide between integrable and non-integrable cases [see Steeb and Louw (1986) for a detailed discussion and references therein]. The differential equation is considered in the complex time plane. Then the structure of the singularities (poles, algebraic branch points, etc.) of the solution of the differential equation is studied in the complex time plane.

The idea of the singular point analysis is due to Kowalevski (1889, 1891), who showed that the only algebraically completely integrable systems among rigid-body motions are Euler's rigid body, Lagrange's top and the Kowalevski top.

For performing the Painlevé test [cf. Steeb and Louw (1986), Steeb *et al.* (1985*a*) and Kunick and Steeb (1986) for details and also for references], we consider the quantities x_1, x_2 and t in the complex domain. For the sake of simplicity we do not change our notation. In the first step one tries to find a solution (or solutions) of system (2) expressed as a Laurent series. First, we determine the dominant behaviour of a Laurent series. We find three branches:

Branch I: Inserting

$$x_k(t) \propto a_{k0}(t-t_1)^{n_k} \tag{5}$$

($k = 1, 2$) into system (2) we find that $n_1 = n_2 = -1$ and $a_{10}^2 = a_{20}^2 = -2/a$. The coupling term cx_1x_2 does not play any role in the dominant behaviour. For the anharmonic oscillator $\ddot{x} = -Ax - ax^3$ (which has the Painlevé property and therefore passes the Painlevé test), we know that the resonances are given by $r_1 = -1$ and $r_2 = 4$. Consequently, system (2) has the resonances $r_1 = -1$ (twofold) and $r_2 = 4$ (twofold). The resonances are defined as follows: Inserting the ansatz $x_k(t) = a_{k0}(t-t_1)^{-1} + b_k(t-t_1)^{-1+r}$ into system (2) without the non-dominant terms yields, up to leading order in b, $Q(r)b = 0$, where $b = (b_1, b_2)^{\mathrm{T}}$ and $Q(r)$ is a 2×2 matrix whose elements depend on r. The roots of det Q are called resonances (sometimes also called Kowalevski exponents). The resonance at $r = -1$ corresponds to the arbitrariness of t_1. Inserting the ansatz (Laurent series)

$$x_k(t) = (t-t_1)^{-1} \sum_{j=0}^{\infty} a_{kj}(t-t_1)^j \tag{6}$$

($k = 1, 2$) into system (2), we find that $a_{11} = a_{21} = 0$ and

$$a_{12} = (ca_{20} + Aa_{10})/6, \qquad a_{22} = (ca_{10} + Aa_{20})/6, \tag{7a}$$

$$a_{13} = a_{23} = 0. \tag{7b}$$

At the resonances $r_2 = 4$ we find that a_{14} and a_{24} can be chosen arbitrarily. From this branch we cannot decide whether system (2) passes the Painlevé test. The general solution must contain four arbitrary constants since system (2) is given by two second-order ordinary differential equations. In the present case we have only three arbitrary constants, namely t_1, a_{14} and a_{24}.

Branch II: Let x_2 be less singular than x_1. Assuming that x_2 is small enough that the right-hand sides of (2a) and (2b) are dominated by x_1 only, then we obtain for the dominant behaviour

$$x_1(t) \propto a_{10}^2(t-t_1)^{-1}, \tag{8a}$$

$$x_2(t) \propto a_{20} + a_{21}(t-t_1) + (2c/a)(t-t_1)\ln(t-t_1). \tag{8b}$$

Branch III: Let x_1 be less singular than x_2. Putting $x_1 \rightleftharpoons x_2$ we find the result of branch II.

Consequently, system (2) does not pass the Painlevé test. This means we cannot find a Laurent expansion with four arbitrary constants [see Steeb and Louw (1986) for more details]. Only for $c = 0$ does the system pass the Painlevé test, and this case is trivial since we have two uncoupled anharmonic oscillators. In this special case the system (2) is integrable. For $c \neq 0$ we conjecture that the result that the system (2) does not pass the Painlevé test indicates that system (2) is not integrable for $c \neq 0$. Notice that we cannot apply the theorem of Yoshida (1983) which states: 'In order that a given similarity invariant system $\dot{x}_i = F_i(x)$ with rational right-hand sides is algebraically integrable, every possible resonance must be a rational number'.

The search for first integrals [besides the Hamiltonian (1)] is unsuccessful. We have considered the ansatz

$$h(x, p) = p_1^2 g_1 + p_2^2 g_2 + p_1 p_2 g_3 + p_1 g_4 + p_2 g_5 + g_6, \tag{9}$$

where the g_i are functions of x_1 and x_2 only. Furthermore, the search for symmetry generators of the Cartan form (Steeb 1983)

$$\alpha = \sum_{i=1}^{2} p_i \, \mathrm{d}x_i - H(x, p)\, \mathrm{d}t \tag{10}$$

is unsuccessful. This means that, besides $S = \partial/\partial t$ which is related to the Hamiltonian (1), no further symmetry generator arises.

4. Ziglin's Theorem

Recently, Ziglin (1983) gave necessary conditions for the existence of a given number $r > 0$ of additional meromorphic first integrals for the system in terms of the monodromy group of the system in variations along some phase curve of the system. This theorem can be applied here. In the following we consider the special case $A + c = 0$. The variational system of (2) is given by

$$\ddot{y}_1 = (-A - 3ax_1^2)y_1 - cy_2, \qquad \ddot{y}_2 = -cy_1 + (-A - 3ax_2^2)y_2. \tag{11a, b}$$

We put $x_1(t) = x_2(t) = x(t)$, and then $\ddot{x} + ax^3 = 0$. The solution, given by equation (4), has two independent periods in the complex time plane. The coefficient matrix can be diagonalised [see Yoshida (1986) for the technique]. Then, we obtain the normal variational equation and the tangential variational equation. Let $\ddot{\xi} + A(t)\xi = 0$ (Hill's equation) be the normal variational equation and let $X(t)$ be its fundamental system. The monodromy matrix $M(T_j)$ is defined by $X(t + T_j) = X(t)\, M(T_j)$ for each period T_j. Then Ziglin's theorem can be formulated as (Yoshida 1986): 'Suppose that a Hamiltonian system with two degrees of freedom permits a meromorphic first integral h in addition to the Hamiltonian H. Further, suppose that for a periodic solution the normal variational equation is written in the form of Hill's equation $\ddot{\xi} + A(t)\xi = 0$ with multiple periods T_j. Then all monodromy matrices $M(T_j)$, with non-resonant multipliers, must commute.'

Thus, this theorem can be applied here. We find that besides H there is no other functionally independent meromorphic first integral.

5. Numerical Studies

First of all let us introduce the one-dimensional Lyapunov exponent λ which serves to characterise chaos (Contopoulos *et al.* 1978). Given an autonomous system of ordinary differential equations of first order $\dot{x}_i = F_i(x)$ and the corresponding variational system

$$\dot{y}_i = \sum_{k=1}^{n} \frac{\partial F_i(x)}{\partial x_k} y_k, \tag{12}$$

any set of initial values $x_{10}, \ldots, x_{n0}$ and $y_{10}, \ldots, y_{n0}$ gives a solution $x_i(t, x_{10}, \ldots, x_{n0})$ and $y_i(t, x_{10}, \ldots, y_{n0})$. If the system $\dot{x}_i = F_i(x)$ is defined on a compact manifold and preserves a measure for almost all $x_{10}, \ldots, x_{n0}$ and for all $y_{10}, \ldots, y_{n0}$, the limit

$$\lim_{t \to \infty} \frac{1}{t} \ln \| y(t) \| =: \lambda(x_{10}, \ldots, y_{n0}) \tag{13}$$

exists where $\| y(t) \|$ denotes the norm of $y_1(t), \ldots, y_n(t)$. The quantities λ are called one-dimensional Lyapunov exponents. Given $x_{10}, \ldots, x_{n0}$, λ takes at most n different values, and the maximum of them is denoted by λ_{max}. If one chooses $y_{10}, \ldots, y_{n0}$ at random with a probability of 1, one obtains λ_{max}.

In the present case we put $\dot{x}_1 = x_3$ and $\dot{x}_2 = x_4$. Then (2) can be written as an autonomous system of first-order differential equations. The corresponding variational equations are given by

$$\dot{y}_1 = y_3 , \qquad \dot{y}_2 = y_4 ,$$

$$\dot{y}_3 = (-A-3ax_1^2)y_1 - cy_2 , \qquad \dot{y}_4 = -cy_1 + (-A-3ax_2^2)y_2 . \tag{14}$$

For our numerical treatment we put $A = 0$ and $a = 1$. The Toda–Brumer criterion which measures local instability indicates that for $x_1^2 x_2^2 < c^2/9$ the system is in the chaotic region, bearing in mind, however, that this criterion can lead to misinterpretations [see Eckhardt *et al.* (1986) for a detailed discussion and references therein]. Notice that the equations of motion are not scale invariant. This means for different energy shells we find different behaviour. The numerical calculations are performed using the Lie series (Gröbner and Knapp 1967), with the program tested for the special case of $c = 0$. Here we find that $\lambda_{max} = 0$, since we have a periodic motion. We let $x_{10} = 0 \cdot 7$, $x_{20} = 0 \cdot 8$ and $x_{30} = x_{40} = 0$ and, for c sufficiently small, we find that $\lambda_{max} = 0$. Thus, no chaotic motion is indicated. If we let $c = 2$ then we obtain $E = 1 \cdot 282425$ and the maximal one-dimensional Lyapunov exponent is given by $\lambda_{max} = 0 \cdot 10$. Also, for other trajectories with energy $E = 1 \cdot 282425$ chaotic motion is indicated, so that the motion is bounded and $\lambda_{max} > 0$. With increasing E the system becomes more and more chaotic.

6. Conclusions

Let us now compare our results with quartic coupled anharmonic oscillators, with

$$H(x,p) = \tfrac{1}{2}(p_1^2 + p_2^2) + \tfrac{1}{4}(x_1^4 + x_2^4) + \tfrac{1}{2}cx_1^2 x_2^2 . \tag{15}$$

The integrability for this Hamiltonian has been studied previously (Bountis *et al.* 1982; Yoshida 1984). The Hamiltonian equations of motion pass the Painlevé test if and only if $c = 0$, 1 and 3. In these cases the system is algebraically integrable. For $c = 0$ the system is decoupled, while for $c = 1$ and 3 there is a second first integral (besides the Hamiltonian). The first integrals are polynomials. In all other cases additional first integrals cannot exist. Here we can apply the theorem due to Yoshida (1983) and, consequently, the quartic coupled oscillator is different from the linearly coupled oscillator in which, for all $c > 0$, the system is not integrable. Yoshida (1986) applied Ziglin's theory to the Hamiltonian (15) and confirmed the results from the Painlevé analysis. Finally, we mention that the equations of motion of the Hamiltonian (15) are scale invariant. Owing to the scale invariance the calculation has to be done only for one energy shell. An open question is the study of the quantised version of the Hamiltonian (1), i.e. the investigation of 'quantum chaos' (Steeb and Louw 1986).

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