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Determination of Young's Modulus in the Circumferential Direction of Corn Stalks

K. Murawski[A] *and J. Kukiełka*[B]

[A] School of Physical Sciences, Flinders University of South Australia, Bedford Park, S.A. 5042, Australia.
[B] Department of Physics AR, Akademicka 13, 20-033 Lublin, Poland.

Abstract

A corn stalk is assumed to be a tube with strong anisotropic properties. For this model a set of four fundamental equations is written which describes the behaviour of one-dimensional pressure waves of small amplitude and long wavelength in an infinite thin-walled compliant tube with walls of elastic rings. The reductive Taniuti–Wei (1968) method is developed in order to reduce this set to Burgers' equation. Finally, formulae which describe the velocity of pressure shocks are derived. The analysis of these formulae along with associated measurements allows calculation of Young's modulus in the circumferential direction for corn stalks.

1. Introduction

The knowledge of certain relationships between soil, machines and plants is required when developing agricultural machinery. It is difficult to establish these relationships because the effects of the activity of machines both on soil and plants are not fully known. Deriving physical models of individual plants is necessary since this allows the behaviour of biological materials to be described in a mathematical way. This permits various loading conditions to be studied. So far physical models of this nature have not worked successfully. The difficulty results mainly from insufficient knowledge of the process of the strain of biological materials, and their anisotropic and heterogeneous structure. Besides, rheological models of these materials are time varying and cells may be destroyed by repeated loadings.

Experimental research carried out so far (Gawda and Haman 1983; Gowin and Haman 1984) has described Young's modulus in the longitudinal direction and has assumed full heterogeneity of the investigated samples. But the actual medium of the corn stalk possesses anisotropic properties in the circumferential direction, which current methods have been unable to estimate.

The main purpose of the present paper is to propose a new method which addresses this question and specifically enables us to determine the elastic properties in the circumferential direction of the corn stalk on the basis of shock pressure propagation.

Knowledge of Young's modulus is useful in order to predict the behaviour of corn stalks subject to wind forces. Often the stalk constitutes a valuable industrial material and therefore knowledge of its elastic properties is of considerable use.

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2. Model of the Corn Stalk: Fundamental Equations

The medium of the stalk has a complicated and heterogeneous structure and in order to analyse the mechanics of this structure some simplifying assumptions must be made. However, future investigations of more realistic and thus more complicated models appear feasible. Nevertheless, simple models can give significant insight into the structure of the solutions of more realistic and complex models.

In this paper it is assumed that the stalk is an infinite tube having a constant volume density and constant but different physical properties in the circumferential, longitudinal and radial directions. The air filling the tube is assumed to be an ideal gas which undergoes barotropic conversion during pressure shock propagation, independent of its viscosity. One-dimensional irrotational motion of the air in the tube can be assumed because investigated samples of the stalks possess large values of aerodynamic slenderness and the internal diameter $2a$ is small compared with the wavelength λ of the pressure disturbance. If we denote the characteristic amplitude of the wave by L we may write

$$L \ll 2a \ll \lambda . \tag{1}$$

The proposed method of determination of Young's modulus in the circumferential direction of the corn stalk is based on Lamb (1980) and Bhatnagar (1979). These authors were partially interested in wave propagation in an infinite tube, although Lamb did not take the viscosity and compressibility of the medium into account. Bhatnagar used the energy conservation equation and the state equation in the form of $p = p(\rho, T)$, where T is the temperature of the fluid, ρ the density of the air filling the tube and p the pressure of the air inside. All quantities were averaged over a cross section.

Our starting point is the following set of equations:

(i) The conservation of mass equation given by

$$(\rho A)_t + (\rho V A)_x = 0; \tag{2}$$

(ii) Euler's equation

$$\rho V_t + \rho V V_x + p_x - \mu V_{xx} = 0; \tag{3}$$

(iii) Newton's equation (Lamb 1980)

$$\rho_m h a^2 A_{tt} + E h (A - \pi a^2) - 2\pi a^3 (p - q) = 0; \tag{4}$$

(iv) equation of state of the air

$$\rho(p) = \Psi p . \tag{5}$$

The symbols used here have the following meanings: A is the internal cross sectional area of the stalk, V the air velocity, Ψ a constant, E Young's modulus in the circumferential direction, ρ_m the volume density of the stalk wall, h the thickness of the stalk wall, a the average radius of the stalk at the undisturbed uniform state, q the outside pressure of the air, x the length coordinate of the stalk, and t the time. The subscripts x and t indicate partial differentiation.

3. Derivation of Burgers' Equation

Our primary aim is to derive an approximate single equation from the fundamental set of equations presented in the previous section. For this purpose we apply a reductive Taniuti–Wei (1968) method. Assuming that A, V and p are slowly varying functions in a reference frame moving with speed V_0, we introduce the coordinate stretching

$$\xi = \epsilon(x - V_0 t), \qquad \tau = \epsilon^2 t, \tag{6}$$

where ϵ is a scale parameter proportional to the wave number, defined by

$$\epsilon = L/2a. \tag{7}$$

On the other hand, since we are concerned with weak nonlinear waves, we expand the dependent variables as power series in ϵ around the undisturbed uniform state:

$$A = A_0 + \epsilon A_1 + \dots, \qquad V = \epsilon V_1 + \epsilon^2 V_2 + \dots, \qquad p = q + \epsilon p_1 + \dots. \tag{8}$$

In terms of ξ and τ the set of equations (2)–(5) reduces to

$$\epsilon(pA)_\tau + \{pA(V - V_0)\}_\xi = 0, \tag{9}$$

$$p\{\epsilon V_\tau + (V - V_0) V_\xi\} + \frac{1}{\Psi} p_\xi = \epsilon \frac{\mu}{\Psi} V_{\xi\xi}, \tag{10}$$

$$\epsilon^4 A_{\tau\tau} - 2\epsilon^3 V_0 A_{\tau\xi} + \epsilon^2 V_0^2 A_{\xi\xi} + R_1 A + R_2 p = R_3, \tag{11}$$

where

$$R_1 = \frac{E}{\rho_m a^2}, \qquad R_2 = -\frac{2\pi a}{\rho_m h}, \qquad R_3 = \frac{\pi(Eh - 2aq)}{\rho_m h}, \tag{12}$$

and indices τ and ξ indicate partial differentiation. Substituting (8) into the above set of equations and equating all the coefficients of the various powers of ϵ to zero, we have

$$q(A_0 V_1 - V_0 A_1) - A_0 V_0 p_1 = 0, \qquad p_1 = q V_0 \Psi V_1, \qquad R_1 A_1 + R_2 p_1 = 0. \tag{13}$$

Hence, we obtain

$$A_1 = -\frac{R_2}{R_1} p_1 = -q V_0 \Psi \frac{R_2}{R_1} V_1, \tag{14}$$

and the formula which determines the velocity of the moving frame

$$V_0 = \left(\frac{Eh}{\Psi(Eh + 2aq)}\right)^{\frac{1}{2}}. \tag{15}$$

For the second power in ϵ, equations (9)–(11) may be rewritten as

$$\left(q-\frac{R_1}{R_2}A_0\right)A_{1\tau}+2\frac{R_1}{R_2}\left\{V_0+\frac{1}{\Psi V_0}\left(\frac{A_0 R_1}{qR_2}-1\right)\right\}A_1 A_{1\xi}$$
$$+q(A_0 V_{2\xi}-V_0 A_{2\xi})-A_0 V_0 p_{2\xi}=0, \tag{16}$$

$$-\frac{R_1}{\Psi V_0 R_2}A_{1\tau}-qV_0 V_{2\xi}+\frac{R_1^2}{q\Psi V_0 R_2^2}\left(\frac{1}{\Psi V_0}-V_0\right)A_1 A_{1\xi}+\frac{1}{\Psi}p_{2\xi}$$
$$=-\frac{\mu R_1}{qV_0 R_2 \Psi^2}A_{1\xi\xi}, \tag{17}$$

$$R_1 A_2+R_2 p_2=0. \tag{18}$$

Substituting (18) into (16), we get

$$\left(q-A_0\frac{R_1}{R_2}\right)A_{1\tau}+2\frac{R_1}{R_2}\left\{V_0+\frac{1}{\Psi V_0}\left(\frac{A_0 R_1}{qR_2}-1\right)\right\}A_1 A_{1\xi}$$
$$+qA_0 V_0\left(q\frac{R_2}{R_1}-A_0\right)p_{2\xi}=0. \tag{19}$$

Eliminating the terms in $p_{2\xi}$ we obtain the single equation for the first-order perturbed quantity p_1

$$p_{1\tau}+\beta p_1 p_{1\xi}+\alpha p_{1\xi\xi}=0. \tag{20}$$

The nonlinear β and dissipative α coefficients are described by

$$\beta=\frac{(1-\Psi V_0^2)(qR_2+A_0 R_1)+2A_0 R_1}{2qV_0\Psi(A_0 R_1-qR_2)}, \qquad \alpha=-\frac{\mu}{2q\Psi}. \tag{21}$$

4. Solution of Burgers' Equation

The travelling wave solution of Burgers' equation is well known and may be obtained, for instance, from the linear heat equation (Lamb 1980) to which Burgers' equation is transformed by the Cole–Hopf transformation (Hopf 1950). Here we look for a stationary wave solution of the form (see e.g. Bhatnagar 1979)

$$p_1=p_1(\zeta=\xi-c\tau). \tag{22}$$

In a new corodinate system moving with speed c, equation (20) may be rewritten as

$$-cp_{1\zeta}+\beta p_1 p_{1\zeta}+\alpha p_{1\zeta\zeta}=0. \tag{23}$$

Upon integration of (23) we get

$$\alpha p_{1\zeta}=-\tfrac{1}{2}\beta p_1^2+cp_1+\tfrac{1}{2}a_1, \tag{24}$$

where a_1 is a constant. Integrating this equation once more, we get

$$\operatorname{atanh}\left(\frac{c-\beta p_1}{\gamma}\right) = -\frac{\gamma}{2\alpha}\zeta + a_2, \qquad \gamma = (\beta a_1 + c^2)^{\frac{1}{2}}, \tag{25a, b}$$

where a_2 is another constant of integration. Finally, equation (25a) may be rewritten in the more convenient form

$$\beta p_1 = c - \gamma \tanh\left(-\frac{\gamma}{2\alpha}\zeta + a_2\right). \tag{26}$$

5. Shock-wave Velocity and Young's Modulus

Equation (26) contains two integration constants, a_1 and a_2, which should be determined by boundary conditions. In our model the pressure disturbance should be created by an electronic system (see Moodie *et al.* 1984) at, say, the left-hand side of the corn sample. Let us denote the amplitude of this disturbance by p_d. For other initial conditions, solutions may be found from the heat equation (Karpman 1975). At the right-hand side, the internal air pressure is equal to the outside pressure q. Let us assume that the sample length tends to infinity, with the ends at $x = 0$ and ∞. At these limits, $p_{1\xi} = 0$ and

$$p = p_d \quad \text{at} \quad x = 0 \qquad \text{and} \qquad p = q \quad \text{at} \quad x = \infty. \tag{27}$$

Using (8) these equations may be rewritten in the relevant form

$$q + \epsilon p_1 = p_d \quad \text{at} \quad x = 0 \qquad \text{and} \qquad p_1 = 0 \quad \text{at} \quad x = \infty. \tag{28}$$

Then from (24) it follows that

$$a_1 = 0, \qquad c = \frac{\beta}{2\epsilon}(p_d - q). \tag{29}$$

The constant a_2 may be calculated from initial conditions, but is not required here. We are now in a position to estimate the shock velocity c calculated in the laboratory frame. We have to write ζ in the original coordinates x and t as

$$\zeta = \xi - c\tau = \epsilon(x - V_0 t) - \epsilon^2 ct = \epsilon\{x - (V_0 + \epsilon c)t\}. \tag{30}$$

Substituting (29) into (30), we obtain the velocity of the shock wave

$$V = \left(\frac{Eh}{\Psi(Eh + 2aq)}\right)^{\frac{1}{2}} + \tfrac{1}{2}\beta(p_d - q). \tag{31}$$

This is the main result of this paper. Young's modulus E may be easily described by this formula. When the nonlinear effects are neglected, $\beta = 0$ and the wave velocity is equal to V_0. Then Young's modulus of the corn stalk may be described by

$$E = \frac{2aq\Psi V_0^2}{h(1 - \Psi V_0^2)}. \tag{32}$$

We stop at the first order of ϵ, although the next order of approximation of V may be estimated. In this case, however, higher order nonlinear differential equations appear, but with these next order terms causing diminishing changes to the measured velocity V.

6. Summary

This paper considers weakly nonlinear long pressure waves in an infinite tube with elastic compliant walls. Using the Taniuti–Wei method, Burgers' equation is derived to describe the waves. Although the shock-like solution of this equation is well known, we have repeated the calculations in order to obtain a formula which describes the velocity of pressure waves. This formula allows the calculation of Young's modulus in the circumferential direction of the corn stalk, given measurements of the required physical quantities including the wave velocity.

Describing Young's modulus both in the longitudinal direction of the corn stalk by different methods (Gawda and Haman 1983) and in the circumferential direction by the method presently proposed, one can find the anisotropic coefficient of the material of the stalk. It is equal to the quotient of these quantities.

Finally, it is worth noting that other models of the stalk may be applied and tested against experiment. Wave propagation in a medium with memory (Mooney–Rivilin material) has been discussed by Tait and Moodie (1984). Radial motion of a nonlinear viscoelastic tube has been studied by Tait *et al.* (1985). Two-dimensional analysis has been employed to study pulse propagation in thin-walled circularly cylindrical elastic tubes containing an inviscid and incompressible liquid (Barclay *et al.* 1984). The Korteweg–de Vries equation has been modified to include both the dissipative and dispersive effects of viscous boundary layers (Miles 1976). Also, model equations have been derived for thin wall tubes (Murawski 1986).

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