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# Stability of mKdV–KdV Waves

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*Abstract*

We consider a combination of modified Korteweg–de Vries waves and ordinary Korteweg–de Vries waves, known as mKdV–KdV waves. The Infeld–Rowlands method is developed to study the instability of mKdV–KdV waves in the limit of long wavelength perturbations.

## 1. Introduction

It has been shown by several authors (Infeld *et al.* 1978; Jeffrey and Kakutani 1970, 1972; Benjamin 1972; Kadomtsev and Petviashvili 1970; Zakharov 1975) that the Korteweg–de Vries (KdV) waves are stable. Instabilities of the modified Korteweg–de Vries (mKdV) waves have been investigated by Fornberg and Whitlam (1978) and Murawski (1987). In this note we consider the combined form of these equations (e.g. Funakoshi 1985)

$$u_t - 6\alpha_1\, u^2 u_x - 3\alpha_2\, u u_x + u_{xxx} = 0\,. \tag{1}$$

The stability problem for the wave-like solutions of this equation is discussed in the next section, and we close this note by presenting numerical results.

## 2. Stability Problem

In order to find travelling wave solutions of equation (1) we make the transformation

$$u = B(\xi = x - ct)\,. \tag{2}$$

By integrating (1) twice, we transform it as follows:

$$B_\xi^2 = \alpha_1 B^4 + \alpha_2 B^3 + cB^2 + 2aB + l \equiv F(B)\,, \tag{3}$$

where $l$ and $a$ are integration constants. Two arbitrary parameters $a$ and $c$ have to be chosen for the existence of double roots of the equation $F(B) = 0$.

Suppose a nonlinear wave given by (3) is perturbed by a long wavelength linear wave with small amplitude $\delta u$ (Infeld and Rowlands 1979):

$$u = B(\xi) + \delta u(\xi)\, \exp\{\mathrm{i}(k\xi + wt)\}\,, \tag{4}$$

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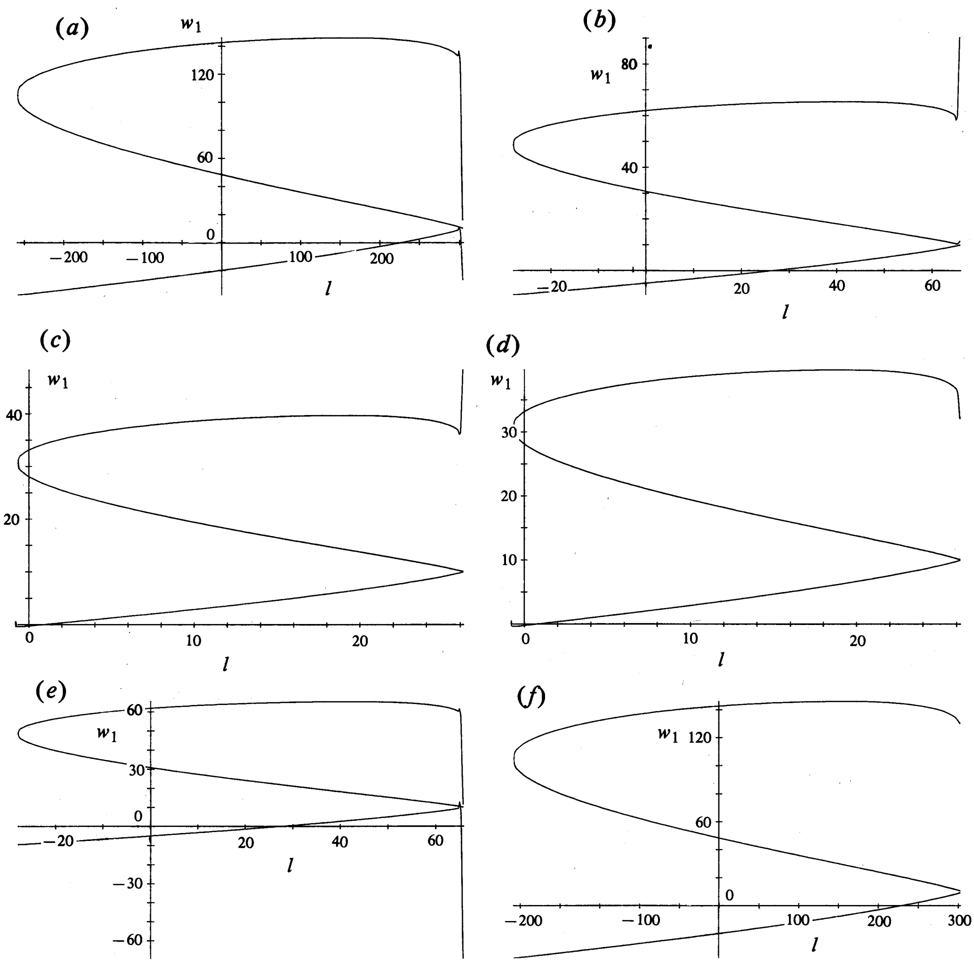


**Fig. 1.** Real roots of equation (17) versus the parameter $l$. This case corresponds to periodic waves which are bounded both by linear and shock waves, and where $\alpha_1 = 1$ and $c = -10$:

(*a*) $\alpha_2 = -10$, $a = 87{\cdot}5$;
(*b*) $\alpha_2 = -5$, $a = 20{\cdot}3125$;
(*c*) $\alpha_2 = -1$, $a = 2{\cdot}5625$;
(*d*) $\alpha_2 = 1$, $a = -2{\cdot}5625$;
(*e*) $\alpha_2 = 5$, $a = -20{\cdot}3125$;
(*f*) $\alpha_2 = 10$, $a = -87{\cdot}5$.

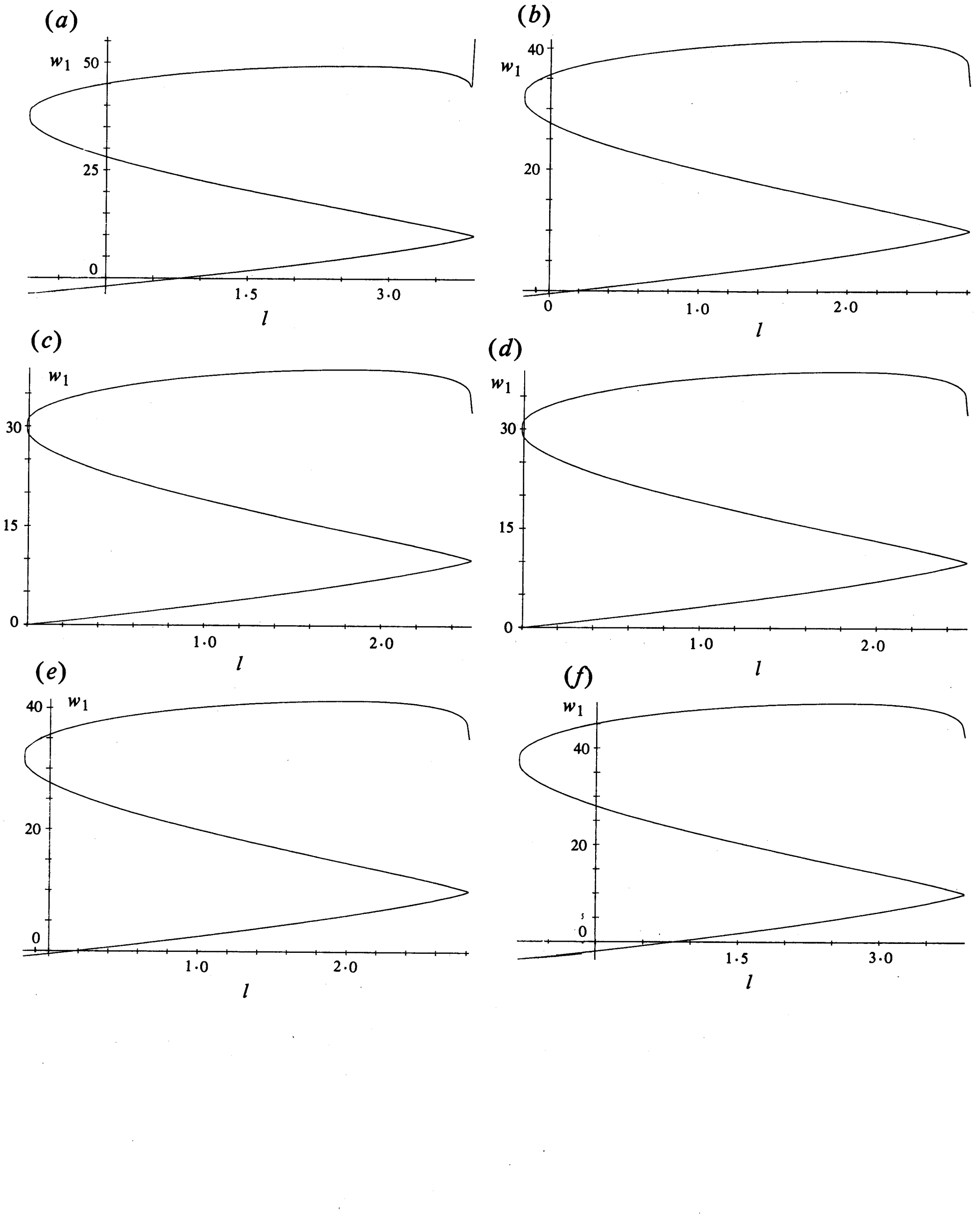


**Fig. 2.** Real roots of equation (17) versus the parameter $l$. This case corresponds to periodic waves which are bounded both by linear and shock waves, and where $\alpha_1 = 10$ and $c = -10$:

(*a*) $\alpha_2 = -10,\ a = 3{\cdot}125$;
(*b*) $\alpha_2 = -5,\ a = 1{\cdot}328125$;
(*c*) $\alpha_2 = -1,\ a = 0{\cdot}250625$;
(*d*) $\alpha_2 = 1,\ a = -0{\cdot}250625$;
(*e*) $\alpha_2 = 5,\ a = -1{\cdot}328125$;
(*f*) $\alpha_2 = 10,\ a = -3{\cdot}125$.

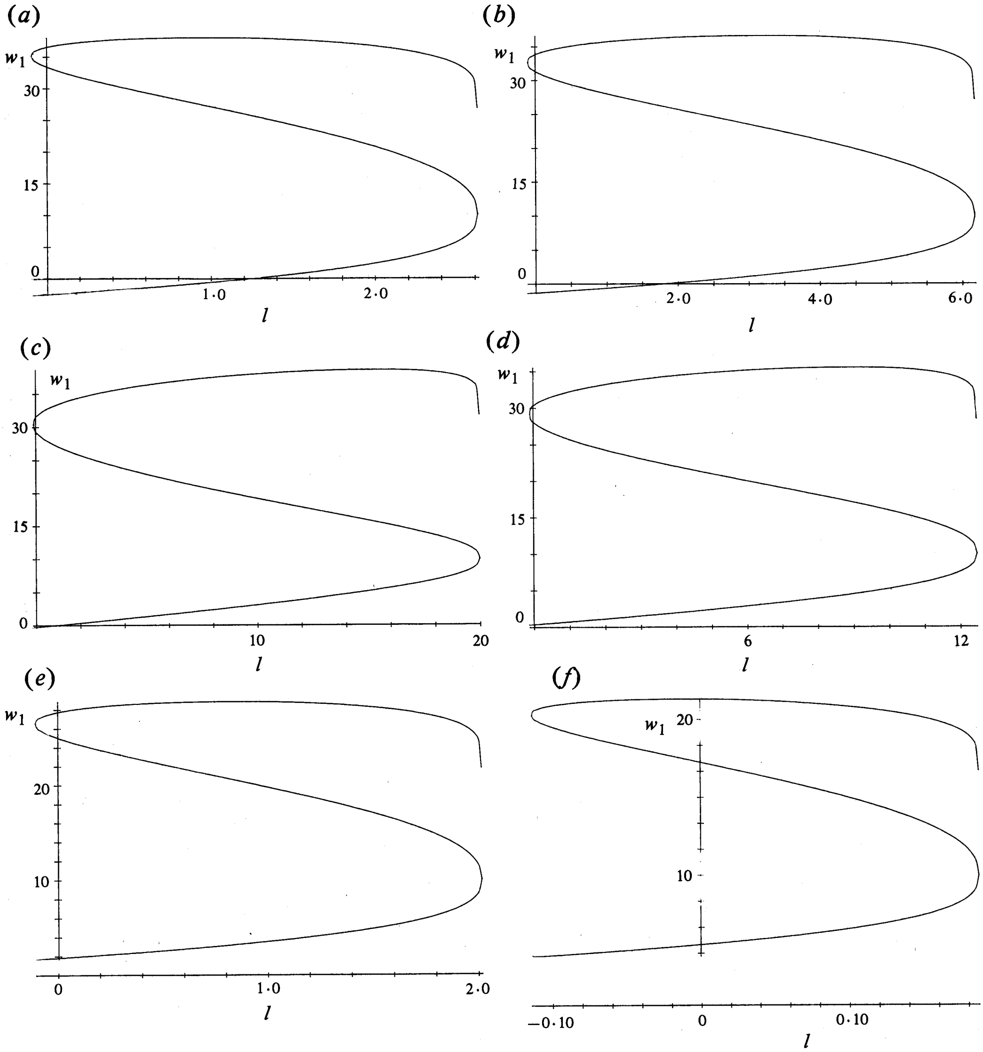


**Fig. 3.** Real roots of equation (17) versus the parameter $l$. This case corresponds to periodic waves which are bounded both by a linear wave and a soliton, and where $\alpha_1 = 1$, $a = 1$ and $c = -10$:

(*a*) $\alpha_2 = -10$;
(*b*) $\alpha_2 = -5$;
(*c*) $\alpha_2 = -1$;
(*d*) $\alpha_2 = 1$;
(*e*) $\alpha_2 = 5$;
(*f*) $\alpha_2 = 10$.

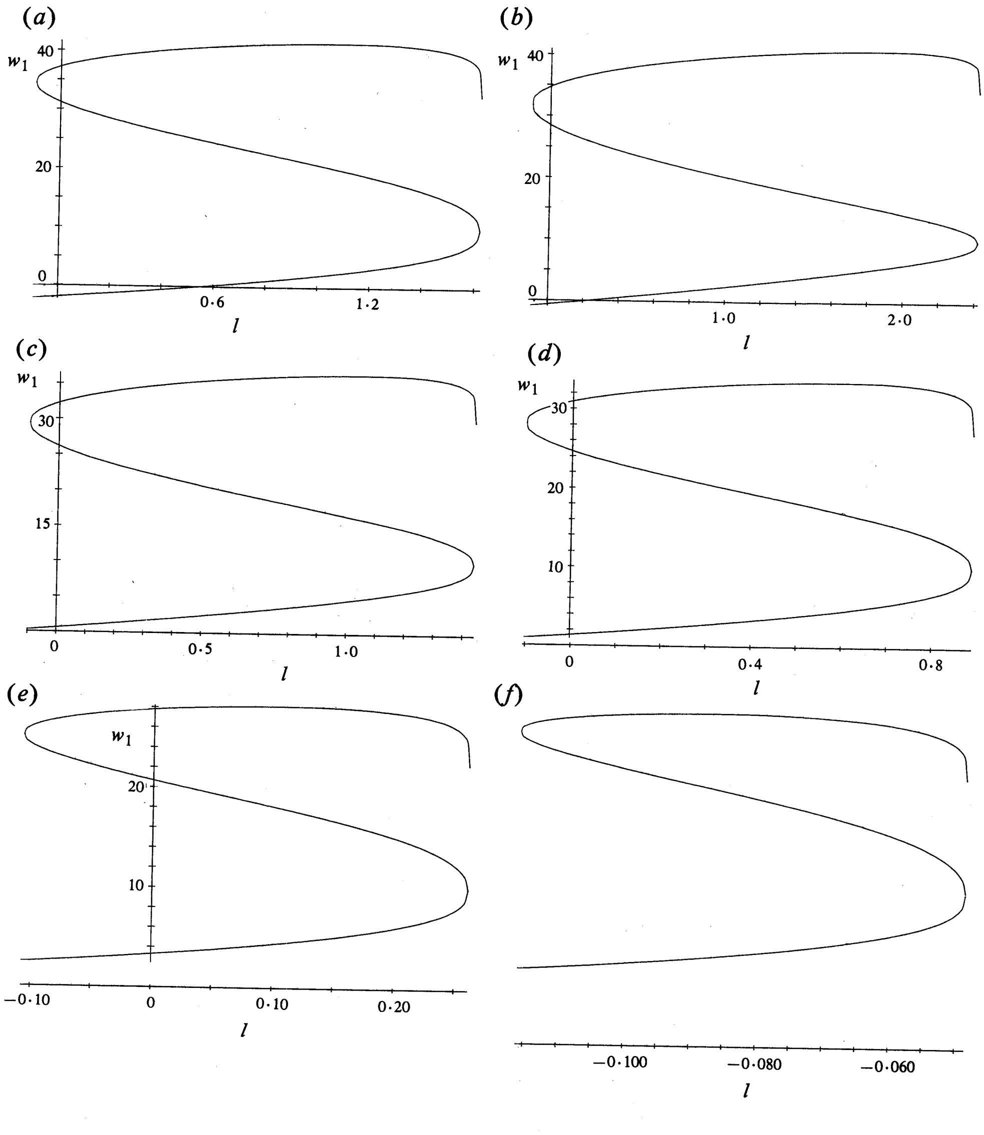


**Fig. 4.** Real roots of equation (17) versus the parameter $l$ corresponding to periodic waves which are bounded both by a linear wave and a soliton, and where $\alpha_1 = 10$, $a = 1$ and $c = -10$:

(*a*) $\alpha_2 = -10$;
(*b*) $\alpha_2 = -5$;
(*c*) $\alpha_2 = -1$;
(*d*) $\alpha_2 = 1$;
(*e*) $\alpha_2 = 5$;
(*f*) $\alpha_2 = 10$.

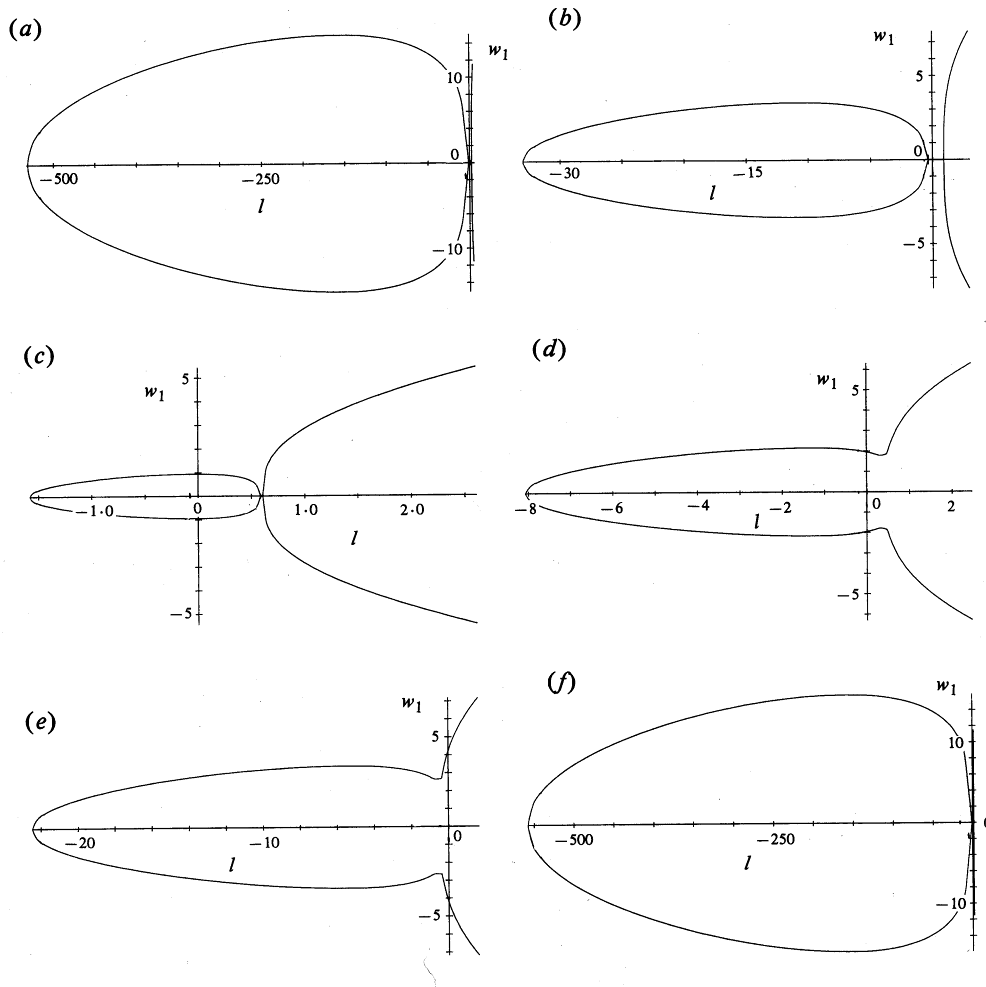


**Fig. 5.** Imaginary parts of the roots of equation (17) versus $l$ for $\alpha_1 = -1$ and $a = 1$:

(*a*) $\alpha_2 = -10$, $c = -10$;
(*b*) $\alpha_2 = -5$, $c = -2$;
(*c*) $\alpha_2 = -1$, $c = 1{\cdot}4$;
(*d*) $\alpha_2 = 1$, $c = 2{\cdot}8$;
(*e*) $\alpha_2 = 5$, $c = -4{\cdot}2$;
(*f*) $\alpha_2 = 10$, $c = -10$.

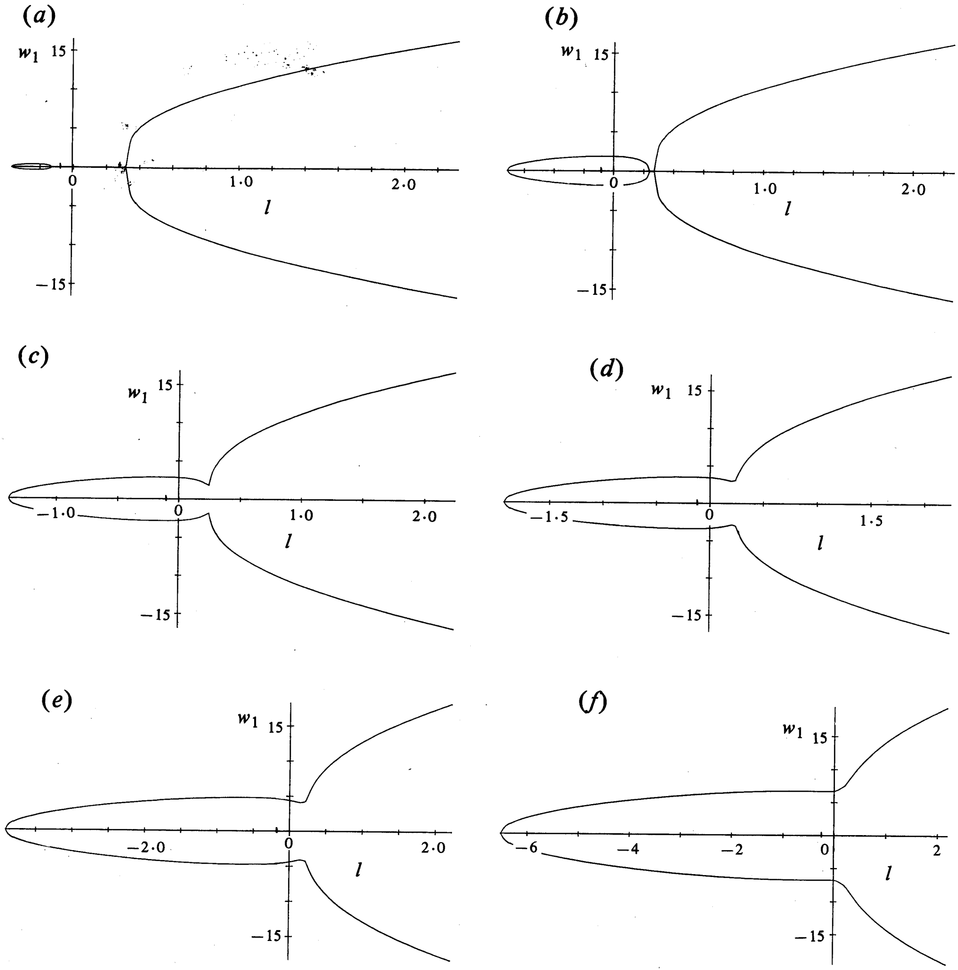


**Fig. 6.** Imaginary parts of the roots of equation (17) versus $l$ for $a_1 = -10$ and $a = 1$:

(*a*) $a_2 = -10,\ c = 1;$
(*b*) $a_2 = -5,\ c = 3;$
(*c*) $a_2 = -1,\ c = 4{\cdot}5;$
(*d*) $a_2 = 1,\ c = 5;$
(*e*) $a_2 = 5,\ c = 6;$
(*f*) $a_2 = 10,\ c = 4{\cdot}5.$

where the stretched coordinates are introduced

$$\xi = x - ct, \qquad t = t, \tag{5}$$

and

$$w = w_1 k + w_2 k^2 + \ldots, \qquad \delta u = \delta u_0 + k\delta u_1 + \ldots . \tag{6}$$

Writing equation (1) in the moving frame coordinates $\xi$ and $t$, introducing (4), and dropping terms proportional to $k^3$ and $\delta u^2$, we find that

$$(L\delta u)_\xi = -\mathrm{i}\, \overline{w}\delta u + 3\mathrm{i}\, kB\delta u(2\alpha_1 B + \alpha_2) - 3\mathrm{i}\, k\delta u_{\xi\xi} + 3k^2\delta u_\xi , \tag{7}$$

where

$$L \equiv \partial_\xi^2 - 6\alpha_1 B^2 - 3\alpha_2 B - c, \qquad \overline{w} = w - kc. \tag{8}$$

From zeroth and first orders of $k$, after substitution of (6) into (7), we get

$$\delta u_0 = B_\xi , \tag{9}$$

$$\delta u_1 = DQ_0 + \mathrm{i}\,\bar{R}Q_1 - \tfrac{1}{2}\mathrm{i}(\overline{w}_1 + c)Q_2 , \tag{10}$$

$$B_\xi \int \frac{B^p \, \mathrm{d}\xi}{B_\xi^2} = S(p)\xi B_\xi + Q_p(\xi),$$

$$p = 0, 1, 2, \quad S(0) = \beta, \quad S(1) = \gamma, \quad S(2) = \kappa, \tag{11}$$

$$D = \frac{\mathrm{i}}{2\beta}\{\beta l + (\overline{w}_1 + c)\kappa - 2\bar{R}\gamma + 2\} - \tfrac{1}{2}\mathrm{i}\, l, \tag{12}$$

$$\bar{R} \equiv -(a + \mathrm{i}\, R). \tag{13}$$

Here, $R$ is an integration constant and $Q_p$ are periodic functions with the same period as $B$, i.e. $\lambda$.

Equation (7) leads in second order of $k$ to

$$\overline{w}_1\langle \delta u_1\rangle - 6\alpha_1\langle B^2\delta u_1\rangle - 3\alpha_2\langle B\delta u_1\rangle = 0, \tag{14}$$

$$\begin{aligned} 3\langle BB_{\xi\xi}\rangle - 3\mathrm{i}\langle B\delta u_{1\xi\xi}\rangle - \mathrm{i}\,\overline{w}_1\langle B\delta u_1\rangle & \\ + 6\mathrm{i}\,\alpha_1\langle B^3\delta u_1\rangle + 3\mathrm{i}\,\alpha_2\langle B^2\delta u_1\rangle &= 0, \end{aligned} \tag{15}$$

where

$$\langle f\rangle = \frac{1}{\lambda}\int_0^\lambda f \, \mathrm{d}\xi . \tag{16}$$

Finally, from equations (10), (12), (14) and (15), a cubic equation in $w_1$ may be obtained:

$$w_1^3 + b_2\, w_1^2 + c_2\, w_1 + d_2 = 0\,. \tag{17}$$

The coefficients $b_2$, $c_2$ and $d_2$ depend on $a$, $c$, $l$, $\beta$, $\gamma$, $\kappa$, $\langle B\rangle$ and $\langle B^2\rangle$.

## 3. Numerical Results

Let us first investigate the case of periodic waves which are limited both by linear and shock waves. In this case the polynomial on the right-hand side of (3) should be symmetric around the axis $x = \nu$ and satisfy the following equation:

$$F(\nu + B) = F(\nu - B)\,. \tag{18}$$

Here, $\nu$ is a function of $\alpha_1$, $\alpha_2$, $c$ and $a$. A numerically obtained plot of $w_1$ versus the parameter $l$ is shown in Figs 1 and 2. Three real roots of equation (17) support the statement that the nonlinear waves are stable. In Fig. 1 we have $\alpha_1 = 1$ and $c = -10$, with $\alpha_2$ equal to $-10$, $-5$, $-1$, 1, 5 and 10. The case $\alpha_2 = 0$ corresponds to the KdV equation examined previously (Murawski 1987) and for this reason is not discussed here. The parameter $a$ is chosen to satisfy the symmetry condition (18). The case $\alpha_1 = 10$ is considered in Fig. 2.

Secondly, we consider a set of waves bounded both by a soliton and a linear wave. We set $a = 1$ and $c = -10$, and $\alpha_2$ follows the same range of values as in the case above. Three real roots of (17) are shown versus the parameter $l$ for $\alpha_1 = 1$ and 10 in Figs 3 and 4 respectively. The waves are stable.

The case $\alpha_1 < 0$ is more complicated. We can keep $a = 1$, but $c$ has to be chosen for the existence of double roots for the equation $F(B) = 0$, and we distinguish three regions determined by the three parameters $l_{min} < l_{av} < l_{max}$ corresponding to double roots of $B_\xi^2 = 0$. In the region $l_{min} < l < l_{av}$ waves are unstable for all values of the parameter $l$, but in the region $l_{av} < l < l_{max}$ all waves are stable. Imaginary parts of $w_1$ for $l_{max} < l < l_{max}+2$ are presented in Figs 5 and 6 for $\alpha_1 = -1$ and $-10$ respectively. For some values of the parameters the region $l_{av} < l < l_{max}$ disappears, but the waves are still unstable.

In conclusion we note that for $\alpha_1 > 0$ all waves are stable, but for $\alpha_1 < 0$ waves are unstable for both $l < l_{av}$ and $l > l_{max}$. Nevertheless, the waves for $l_{av} < l < l_{max}$ are stable.

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