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Renormalisation of g-boson Effects in an s–d–g Boson Model

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Abstract

A renormalisation method is developed in which g-boson coupling effects in a transformed Hamiltonian are minimised. The unitary transformation upon which this is based is that proposed by Otsuka and Ginocchio (1985). The conditions under which the g-boson coupling is eliminated are specified under various approximations and resulting spectra are compared with the exact ones obtained from diagonalisation of an exact s–d–g boson Hamiltonian.

1. Introduction

The interacting boson models have had considerable success when used phenomenologically to describe the properties of low lying collective states in even–even medium to heavy mass nuclei (Arima and Iachello 1981; Elliott 1985; and references therein). By and large, all such studies have been restricted to a model scheme of N, s and d bosons only. But there is an ever increasing amount of experimental data that is evidence of the need to extend the interacting boson approximation (IBA) beyond the simple s–d truncation (Wood 1984; Arima 1984) and such is particularly the case when the nuclei in question exhibit strongly deformed (rotational) characteristics. The most obvious extension is to incorporate g-boson effects in view of the important roles played by pairs of nucleons coupled to a J of 4 in the intrinsic states of strongly deformed nuclear systems (Otsuka 1981*a*, 1981*b*; Yoshinaga *et al.* 1984; Bes *et al.* 1982; van Egmond and Allaart 1984; Scholten 1983). Typically such G-pairs account for 10 to 15% of the absolute binding energy and intrinsic quadrupole moments of the systems (the S and D pairs accounting for virtually all of the remainder). However, the most marked effect is shown upon the moment of inertia of the ground state band. Removal of the G-pair effects reduced the 'exact' value by a factor of $\frac{3}{8}$. Similar effects of the G-pairs have been found from analyses of the intrinsic states of first excited 0^+ bands (Bohr and Mottelson 1982; Dieperink and Scholten 1983).

But g-boson effects have been incorporated in select phenomenological model studies. The SU(15) group theoretic model has been studied (Wu and Zhou 1984) and applied to ^{168}Er, improving upon the SU(6) prescriptions by providing the additional (and observed) 3_1^+, 0_4^+, 2_3^+ and 4_2^+ bands of states in the energy spectrum. Also, from a schematic model (van Isacker *et al.* 1982), g-boson effects were significant in the IBA assessment of the spectrum and electromagnetic transition rates of ^{156}Gd. However, it is not a trivial matter to extend the usual IBA studies to incorporate

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g-bosons. Indeed in IBA-1, the most general Hamiltonian involves nine parameters under the restriction to just s- and d-bosons (although only three or four are usually allowed to be variable). When the g-boson is included, one then has 32 parameters from which to choose. Clearly an approximation method is called for.

The most obvious approximation scheme is perturbation theory and such was done (Scholten 1983) in a microscopic model study of the IBA for nuclei of vibrational limit character. Thereby the macroscopic IBA parameter values could be specified from an underlying microscopic model of nuclear structure. But perturbation theory is not the appropriate method when rotational limit nuclei are considered and an alternative approach is required such as a renormalisation scheme as recently proposed (Otsuka and Ginocchio 1985). This method is studied herein and critique given of some of the inherent assumptions. Various approximations and applications of this renormalisation method are presented.

The transformation method and its application to individual boson operators are described in Section 2. In Section 3, the transformation is applied to the Hamiltonian, first by using the transformation upon individual boson operators and then by considering the transformation of operator products. Numerical evaluations have been made and the results are presented and discussed in Section 4.

2. Unitary Transformation Theory

Consider the simple, IBA-II, Hamiltonian that was used by Otsuka and Ginocchio (1985), namely

$$H^{\mathrm{sdg}} = \Sigma_\tau (\epsilon_{\mathrm{d}}^{(\tau)}\, n_{\mathrm{d}}^{(\tau)} + \epsilon_{\mathrm{g}}^{(\tau)}\, n_{\mathrm{g}}^{(\tau)}) - f\, Q^{(\pi)} \cdot Q^{(\nu)}\,, \tag{1}$$

in which the superscript in parentheses denotes proton (π) and neutron (ν) bosons of s(l=0), d(l=2) and g(l=4) type whose energies are ϵ_{d} and ϵ_{g} with respect to the s-boson energy. The interaction, which has a strength parameter f, is of proton quadrupole–neutron quadrupole form with components

$$\begin{aligned} Q_m^{(x)} = {} & q_1^{(x)}\{(d_{(x)}^\dagger\, \tilde{s}_{(x)})_m^{(2)} + (s_{(x)}^\dagger\, \tilde{d}_{(x)})_m^{(2)}\} + q_2^{(x)}\{(d_{(x)}^\dagger\, \tilde{d}_{(x)})_m^{(2)}\} \\ & + q_3^{(x)}\{(g_{(x)}^\dagger\, \tilde{d}_{(x)})_m^{(2)} + (d_{(x)}^\dagger\, \tilde{g}_{(x)})_m^{(2)}\} + q_4^{(x)}\{(g_{(x)}^\dagger\, \tilde{g}_{(x)})_m^{(2)}\}\,. \end{aligned} \tag{2}$$

The product parentheses denote standard angular momentum coupling with, for example,

$$(g_{(x)}^\dagger\, \tilde{d}_{(x)})_m^{(2)} = \Sigma_n \langle 4 2\; n\; m-n \,|\, 2\; m\rangle g_{(x)n}^\dagger\, \tilde{d}_{(x)m-n}\,, \tag{3}$$

and the tilde is used to specify the components of irreducible spherical tensor operators, *viz.*

$$\tilde{s} = s\,, \qquad \tilde{d}_a = (-)^a d_{-a}\,, \qquad \tilde{g}_a = (-)^a g_{-a}\,. \tag{4}$$

As alluded to in the Introduction, save for very special values of the component strengths $q_i^{(x)}$, this Hamiltonian is not readily diagonalised due to the inclusion of the g-boson terms, and we seek a unitary transformation

$$U = \exp(Z)\,, \tag{5}$$

with an appropriate choice for the antihermitian operator Z that enables us to define an approximate (transformed) Hamiltonian in which g-boson coupling is minimised (and then neglected). Thus we have

$$U H U^{-1} \rightarrow H^{\text{'sd'}}_{\text{effective}}, \tag{6}$$

and, with

$$U H U^{-1} = H + [Z, H] + \frac{1}{2!}[Z, [Z, H]] + \dots, \tag{7}$$

if

$$[Z, [Z, H]] \sim -\eta^2 H,$$

$$U H U^{-1} = H\cos(\eta) + (1/\eta)\sin(\eta)[Z, H], \tag{8}$$

gives a simple structure from which the effective Hamiltonian may be extracted. The latter condition transpires not to be the case for the Hamiltonian of (2) and the transformation now to be determined.

Following the Otsuka–Ginocchio method (referred to hereafter as the OG method) we use the antihermitian operator

$$Z = \theta_{(\nu)} Q^{(\pi)} . E^{(\nu)} + \theta_{(\pi)} Q^{(\nu)} . E^{(\pi)}, \tag{9}$$

wherein

$$E^{(x)} = \{(g^{\dagger}_{(x)} \tilde{d}_{(x)})^{(2)} - (d^{\dagger}_{(x)} \tilde{g}_{(x)})^{(2)}\}, \tag{10}$$

and $\theta_{(x)}$ are coupling angles to be chosen in each application to minimise the g–d boson coupling interaction terms in the transformed Hamiltonian.

Consider now the actions of this transformation upon individual boson operators. To develop the effects we require the commutators of Z with the individual boson operators. They are derived in Appendix A and approximate to the set

$$\begin{aligned}
[Z, s^{\dagger}] &= [Z, \tilde{s}] = 0, \\
[Z, d^{\dagger}_{(\nu)}] &= \theta_{(\nu)}(Q^{(\pi)} g^{\dagger}_{(\nu)})^{(2)}, \\
[Z, \tilde{d}_{(\nu)}] &= \theta_{(\nu)}(Q^{(\pi)} \tilde{g}_{(\nu)})^{(2)}, \\
[Z, g^{\dagger}_{(\nu)}] &= -\theta_{(\nu)} \sqrt{\tfrac{5}{9}}\,(Q^{(\pi)} d^{\dagger}_{(\nu)})^{(4)}, \\
[Z, \tilde{g}_{(\nu)}] &= -\theta_{(\nu)} \sqrt{\tfrac{5}{9}}\,(Q^{(\pi)} \tilde{d}_{(\nu)})^{(4)},
\end{aligned} \tag{11}$$

for neutrons and, by symmetry, to an identical set for protons. These results were obtained by OG under the assumption that the quadrupole operators $Q^{(x)}$ commuted with any operator as though they are C-numbers. This is a far more stringent condition than is necessary as all terms from the complete set of commutators that have been omitted involve components of the 'antiquadrupole' operator $E^{(x)}$.

It is sufficient, therefore, to restrict consideration to systems that resemble axially symmetric rotors, whence for any states from a single intrinsic determinant we have

$$\langle Q_a^{(x)} \rangle \gg \langle E_a^{(x)} \rangle, \tag{12}$$

and thus terms with $E_a^{(x)}$ can be neglected to give the commutators of (11).

The double commutators of Z may then be derived as specified in Appendix B to be

$$\begin{aligned}[Z,[Z, X_{(\nu)}]]_a &= \Sigma_\beta A_{\alpha\beta}^{(\nu)} X_{(\nu)\beta} \\ &= \Sigma_K C_K^{(\nu)} \{(Q^{(\pi)} Q^{(\pi)})^{(K)} X_{(\nu)}\}_a^{(2)},\end{aligned} \tag{13}$$

$$\begin{aligned}[Z,[Z, Y_{(\nu)}]]_a &= \Sigma_\beta B_{\alpha\beta}^{(\nu)} Y_{(\nu)\beta} \\ &= \Sigma_K D_K^{(\nu)} \{(Q^{(\pi)} Q^{(\pi)})^{(K)} Y_{(\nu)}\}_a^{(4)},\end{aligned} \tag{14}$$

for $X_{(\nu)}$ being $d_{(\nu)}^\dagger$ or $\tilde{d}_{(\nu)}$ and $Y_{(\nu)}$ being $g_{(\nu)}^\dagger$ or $\tilde{g}_{(\nu)}$. With a symmetry for proton boson results, the coefficients are

$$\begin{aligned}A_{\alpha\beta}^{(\nu)} &= -\theta_{(\nu)}^2 (-)^\beta \Sigma_m \langle 2\,4\,\alpha-\beta-m\,\beta+m|2\alpha\rangle \\ &\quad \times \langle 2\,4 -\beta\,\beta+m|2\,m\rangle Q_m^{(\pi)} Q_{\alpha-\beta-m}^{(\pi)},\end{aligned} \tag{15}$$

$$\begin{aligned}B_{\alpha\beta}^{(\nu)} &= -\sqrt{\tfrac{5}{9}}\, \theta_{(\nu)}^2 (-)^\beta \Sigma_m \langle 2\,2\,\alpha-\beta-m\,\beta+m|4\,\alpha\rangle \\ &\quad \times \langle 2\,4\,\beta+m -\beta|2\,m\rangle Q_m^{(\pi)} Q_{\alpha-\beta-m}^{(\pi)},\end{aligned} \tag{16}$$

$$C_K^{(\nu)} = -\theta_{(\nu)}^2 \{5(2K+1)\}^{\frac{1}{2}}\, W(24K2;\, 22), \tag{17}$$

$$D_K^{(\nu)} = -\sqrt{\tfrac{5}{9}}\, \theta_{(\nu)}^2 \{5(2K+1)\}^{\frac{1}{2}}\, W(42K2;\, 24). \tag{18}$$

The utility of the transformation scheme requires these double commutators to relate simply to the boson operator with which one started. OG achieved this by noting the dominance of the $K = 0$ Racah coefficient and so restricting the summation in (15) and (16). With $K = 0$ clearly we have

$$[Z,[Z, X_{(x)}]]_a \to C_0^{(\nu)} \langle\, | Q^{(\pi)} \cdot Q^{(\pi)} |\, \rangle X_{(\nu)a}, \tag{19}$$

as required. The nonzero K components are not negligible, however. They may be included within the desired residual form for the double commutators under the assumption that the nuclear states to be considered only have large expectation values for the zero projection components of the operator $(Q^{(\pi)} Q^{(\pi)})^{(K)}$, or equivalently there is diagonal dominance in expectation values of $A_{\alpha\beta}$ and $B_{\alpha\beta}$ so that

$$A_{\alpha\beta}^{(\nu)} \sim -\theta_{(\nu)}^2 \Sigma_m (-)^m W_{am} Q_m^{(\pi)} Q_{-m}^{(\pi)} \delta_{\alpha\beta}, \tag{20}$$

$$B_{\alpha\beta}^{(\nu)} \sim -\theta_{(\nu)}^2 \Sigma_m (-)^m W'_{am} Q_m^{(\pi)} Q_{-m}^{(\pi)} \delta_{\alpha\beta}. \tag{21}$$

Using the expansions (17) and (18) in the relevant expressions (13) and (14) for the double commutators with the zero projection restriction gives, alternatively,

$$[Z,[Z,X_{(\nu)}]]_a = -\theta^2_{(\nu)}\,\Sigma_{K\phi}\, G^{(K)}_{a\phi}(-)^{\phi}\, Q^{(\pi)}_{\phi}\, Q^{(\pi)}_{-\phi}\, X_{(\nu)a}\,, \tag{22}$$

$$[Z,[Z,Y_{(\nu)}]]_a = -\sqrt{\tfrac{5}{9}}\,\theta^2_{(\nu)}\,\Sigma_{K\phi}\, H^{(K)}_{a\phi}(-)^{\phi}\, Q^{(\pi)}_{\phi}\, Q^{(\pi)}_{-\phi}\, Y_{(\nu)a}\,. \tag{23}$$

Table 1. Weight coefficients W_{am}

m	a = 0	1	2
2	−0·11904	−0·27777	−0·55556
1	−0·23810	−0·31745	−0·27777
0	−0·28572	−0·23810	−0·11905
−1	−0·23810	−0·12700	−0·03968
−2	−0·11904	−0·03968	−0·00794
$\Sigma_m W_{am}$	−1·0	−1·0	−1·0

Table 2. Weight coefficients $G^{(K)}_{\mu\phi}$

μ	ϕ	$K=0$	$K=2$	$K=4$	$\Sigma_K G^{(K)}_{\mu\phi}$
0	±2	0·2	−0·0816	0·0007	0·1191
	±1	0·2	0·0408	−0·0027	0·2381
	0	0·2	0·0816	0·0040	0·2856
	$\Sigma_\phi G^{(K)}_{0\phi}$	1·0	0·0	0·0	$\Sigma_{K\phi} G^{(K)}_{0\phi} = 1{\cdot}0$
1	±2	0·2	−0·0408	−0·0005	0·1597
	±1	0·2	0·0204	0·0018	0·2222
	0	0·2	0·0408	−0·0026	0·2382
	$\Sigma_\phi G^{(K)}_{1\phi}$	1·0	0·0	0·0	$\Sigma_{K\phi} G^{(K)}_{1\phi} = 1{\cdot}0$
2	±2	0·2	0·0816	0·0001	0·2817
	±1	0·2	−0·0408	−0·0005	0·1587
	0	0·2	−0·0816	0·0008	0·1192
	$\Sigma_\phi G^{(K)}_{2\phi}$	1·0	0·0	0·0	$\Sigma_{K\phi} G^{(K)}_{2\phi} = 1{\cdot}0$

The weight functions W_{am} and $G^{(K)}_{\mu\phi}$ are given in Tables 1 and 2 respectively from which it is evident that no single entry dominates. The simplifying approximation used by OG therefore must relate to the expectation values of the double commutators. Indeed if

$$\langle\, | Q^{(\pi)}_0 | \,\rangle > \langle\, | Q^{(\pi)}_{m\neq 0} | \,\rangle$$

as assumed by OG, one also requires the $K=0$ approximations to give identical weight values for different μ in (22) and (23).

However, if instead we assume that

$$\langle Q^{(\pi)}_{\phi}\, Q^{(\pi)}_{-\phi}(-)^{\phi}\rangle = (\overline{Q^{(\pi)2}}) \qquad \text{for all } \phi\,, \tag{24}$$

then we may close the summations (over m for W_{am} or over K and ϕ for $G^{(K)}_{\mu\phi}$) to obtain

$$[Z,[Z,d^{\dagger}_{(\nu)}]] \sim -\theta^{2}_{(\nu)}(\overline{Q^{(\pi)2}})d^{\dagger}_{(\nu)}\,,$$

$$[Z,[Z,\tilde{g}_{(\nu)}]] \sim -\tfrac{5}{9}\,\theta^{2}_{(\nu)}(\overline{Q^{(\pi)2}})\tilde{g}_{(\nu)}\,, \tag{25}$$

with equivalent results for $\tilde{d}_{(\nu)}$ and $g^{\dagger}_{(\nu)}$ and for the proton boson operators.

From Table 2, it is true that the $K = 0$ values are the largest and that their sum (over ϕ) gives unity. Thereby the $K = 0$ approximation as used by OG may be reasonable, but only if the assumption as specified by (24) is used.

The single and double commutators of Z with the individual boson operators (11) and (25) then permit the transformation of the individual boson operators to be determined. As is developed in Appendix C, these are

$$U s^{\dagger} U^{-1} = s^{\dagger}\,, \tag{26}$$

$$U d^{\dagger}_{(\nu)a}\, U^{-1} \sim \cos(\eta^{(\nu)})\, d^{\dagger}_{(\nu)a} + (\overline{Q^{(\pi)2}})^{-\frac{1}{2}} \sin(\eta^{(\nu)})(Q^{(\pi)}\, g^{\dagger}_{(\nu)})^{(2)}_{a}\,, \tag{27}$$

$$U g^{\dagger}_{(\nu)a}\, U^{-1} \sim \cos(\eta^{(\nu)\prime})\, g^{\dagger}_{(\nu)a} - (\overline{Q^{(\pi)2}})^{-\frac{1}{2}} \sin(\eta^{(\nu)})(Q^{(\pi)}\, d^{\dagger}_{(\nu)})^{(4)}_{a}\,, \tag{28}$$

in which

$$\eta^{(\nu)\prime} = \surd\tfrac{5}{9}\,\eta^{(\nu)} = \surd\tfrac{5}{9}\,\theta_{(\nu)}(\overline{Q^{(\pi)2}})^{-\frac{1}{2}}\,, \tag{29}$$

and with similar results for the other and proton boson operators.

3. Transformation of the Hamiltonian

(a) Using the Transformed Single Boson Operators

The IBA-II Hamiltonian specified in (1) involves operators that are composites of the six s–d–g single boson operators and which one may transform using

$$U s^{\dagger}_{a}\, U^{-1} = s^{\dagger}_{a}\,,$$

$$U d^{\dagger}_{a}\, U^{-1} = \cos(\eta)\, d^{\dagger}_{a} + \sin(\eta)\, D^{\dagger}_{a}\,,$$

$$U g^{\dagger}_{a}\, U^{-1} = \cos(\eta')\, g^{\dagger}_{a} - \sin(\eta')\, G^{\dagger}_{a}\,, \tag{30}$$

with an equivalent set for the transformed annihilation operators and in which

$$D^{\dagger}_{(\pi)\alpha} = (\overline{Q^{(\nu)2}})^{-\frac{1}{2}}(Q^{(\nu)}\,g^{\dagger}_{(\pi)})^{(2)}_{\alpha}\,,$$

$$G^{\dagger}_{(\pi)\alpha} = (\overline{Q^{(\nu)2}})^{-\frac{1}{2}}(Q^{(\nu)}\,d^{\dagger}_{(\pi)})^{(4)}_{\alpha}\,. \tag{31}$$

The number operators are considered first and, with proton results being obtained simply by a complete interchange of the (π) and (ν) labels in the neutron results, we have

$$N'_{\rm d} = U(d^{\dagger}_{(\nu)}\cdot\tilde{d}_{(\nu)})\,U^{-1}$$

$$= \Sigma_m(-)^m\{\cos(\eta)\,d^{\dagger}_{(\nu)m} + \sin(\eta)\,D^{\dagger}_{(\nu)m}\}$$

$$\times\{\cos(\eta)\,\tilde{d}_{(\nu)-m} + \sin(\eta)\,\tilde{D}_{(\nu)-m}\}\,,$$

so that

$$N'_{\rm d} = \cos^2(\eta)\,d^{\dagger}_{(\nu)}\cdot\tilde{d}_{(\nu)} + \sin^2(\eta)\,D^{\dagger}_{(\nu)}\cdot\tilde{D}_{(\nu)}$$

$$+\cos(\eta)\sin(\eta)\{\,d^{\dagger}_{(\nu)}\cdot\tilde{D}_{(\nu)} + D^{\dagger}_{(\nu)}\cdot\tilde{d}_{(\nu)}\}\,. \tag{32}$$

Then, by expanding the $D_{(\nu)}$ operators and using the recoupling procedure as specified in Appendix D, one obtains

$$N'_{\rm d} = \cos^2(\eta)\,d^{\dagger}_{(\nu)}\cdot\tilde{d}_{(\nu)}$$

$$+\sin^2(\eta)(\overline{Q^{(\nu)2}})^{-1}\,\Sigma_K\,5\,W(2424;\,2K)(Q^{(\pi)}\,Q^{(\pi)})^{(K)}\cdot(g^{\dagger}_{(\nu)}\,\tilde{g}_{(\nu)})^{(K)}$$

$$+\sin(\eta)\cos(\eta)(\overline{Q^{(\nu)2}})^{-\frac{1}{2}}\{\,Q^{(\pi)}\cdot(d^{\dagger}_{(\nu)}\,\tilde{g}_{(\nu)})^{(2)}$$

$$+\,Q^{(\pi)}\cdot(g^{\dagger}_{(\nu)}\,\tilde{d}_{(\nu)})^{(2)}\}\,. \tag{33}$$

We assume that the third term gives zero condensed state matrix elements and in the OG limit ($K = 0$) this reduces to

$$N'_{\rm d} \sim \cos^2(\eta)\,d^{\dagger}_{(\nu)}\cdot\tilde{d}_{(\nu)} + \tfrac{1}{9}\sin^2(\eta)(g^{\dagger}_{(\nu)}\cdot\tilde{g}_{(\nu)})\,. \tag{34}$$

In like fashion the g-boson number operator transforms as

$$N'_{\rm g} = U g^{\dagger}_{(\nu)}\cdot\tilde{g}_{(\nu)}\,U^{-1} = \cos^2(\eta')\,g^{\dagger}_{(\nu)}\cdot\tilde{g}_{(\nu)}$$

$$+(\overline{Q^{(\pi)2}})^{-1}\sin^2(\eta')\,\Sigma_K\,9\,W(2222;\,4K)(Q^{(\pi)}\,Q^{(\pi)})^{(K)}\cdot(d^{\dagger}_{(\nu)}\,\tilde{d}_{(\nu)})^{(K)}$$

$$-(\overline{Q^{(\pi)2}})^{-\frac{1}{2}}\sin(\eta')\cos(\eta')\{\,Q^{(\pi)}\cdot(d^{\dagger}_{(\nu)}\,\tilde{g}_{(\nu)})^{(2)} + Q^{(\pi)}\cdot(g^{\dagger}_{(\nu)}\,\tilde{d}_{(\nu)})^{(2)}\}\,, \tag{35}$$

and which under the same constraints that led to the result given by (34) becomes

$$N'_{\rm g} \sim \cos^2(\eta')\,g^{\dagger}_{(\nu)}\cdot\tilde{g}_{(\nu)} + \tfrac{9}{25}\sin^2(\eta')\,d^{\dagger}_{(\nu)}\cdot\tilde{d}_{(\nu)}\,. \tag{36}$$

The interaction component of the Hamiltonian is

$$V = -f\, Q^{(\pi)} \cdot Q^{(\nu)},$$

and it transforms to

$$V' = -f\, U Q^{(\pi)} \cdot Q^{(\nu)} U^{-1} = -f\, \Sigma_m (-)^m (U Q_m^{(\pi)} U^{-1})(U Q_{-m}^{(\nu)} U^{-1}). \quad (37)$$

Using the transformation equations of the individual bosons one may obtain (see Appendix E)

$$\begin{aligned} U Q_\alpha^{(\pi)} U^{-1} = {} & Q_{(\mathrm{sd})\alpha}^{(\pi)} + q_4^{(\pi)} \cos^2(\eta')(g_{(\pi)}^\dagger \tilde{g}_{(\pi)})_\alpha^{(2)} + q_3^{(\pi)} \cos(\eta)\cos(\eta') P_{2,\alpha}^{(\pi)} \\ & + q_2^{(\pi)} \sin^2(\eta)(\overline{Q^{(\nu)2}})^{-1} \Sigma_{KK'\epsilon} (-)^{\epsilon+K'} 5(2K'+1)^{\frac{1}{2}} \begin{Bmatrix} 2 & 2 & K' \\ 2 & 2 & 2 \\ 4 & 4 & K \end{Bmatrix} \\ & \times \langle 2\, K'\, \alpha-\epsilon \,|\, K\, \alpha-\epsilon \rangle (Q^{(\nu)} Q^{(\nu)})_\epsilon^{(K')} (g_{(\pi)}^\dagger \tilde{g}_{(\pi)})_{\alpha-\epsilon}^{(K)} \\ & + q_4^{(\pi)} \sin^2(\eta')(\overline{Q^{(\nu)2}})^{-1} \Sigma_{KK'\epsilon} (-)^{\epsilon+K'} [5(2K'+1)]^{\frac{1}{2}} \begin{Bmatrix} 2 & 2 & K' \\ 4 & 4 & 2 \\ 2 & 2 & K \end{Bmatrix} \\ & \times \langle 2\, K'\, \alpha-\epsilon \,|\, K\, \alpha-\epsilon \rangle (Q^{(\nu)} Q^{(\nu)})_\epsilon^{(K')} (d_{(\pi)}^\dagger \tilde{d}_{(\pi)})_{\alpha-\epsilon}^{(K)} \\ & - q_3^{(\pi)} \sin(\eta)\sin(\eta')(\overline{Q^{(\nu)2}})^{-1} \Sigma_{KK'\epsilon} (-)^{\epsilon+K'} 15(2K'+1)^{\frac{1}{2}} \begin{Bmatrix} 2 & 2 & K' \\ 2 & 4 & 2 \\ 4 & 2 & K \end{Bmatrix} \\ & \times \langle 2\, K'\, \alpha-\epsilon \,|\, K\, \alpha-\epsilon \rangle (Q^{(\nu)} Q^{(\nu)})_\epsilon^{(K')} P_{K,\alpha-\epsilon}^{(\pi)}, \end{aligned} \quad (38)$$

in which

$$\begin{aligned} Q_{(\mathrm{sd})\alpha}^{(\pi)} = {} & q_1^{(\pi)} \cos(\eta)\{(s_{(\pi)}^\dagger \tilde{d}_{(\pi)})_\alpha^{(2)} + (d_{(\pi)}^\dagger s_{(\pi)})_\alpha^{(2)}\} \\ & + q_2^{(\pi)} \cos^2(\eta)(d_{(\pi)}^\dagger \tilde{d}_{(\pi)})_\alpha^{(2)}, \end{aligned} \quad (39)$$

$$P_{K,\gamma}^{(\pi)} = \{(g_{(\pi)}^\dagger \tilde{d}_{(\pi)})_\gamma^{(K)} + (d_{(\pi)}^\dagger \tilde{g}_{(\pi)})_\gamma^{(K)}\}. \quad (40)$$

For simplicity we apply the OG approximation to each transformed quadrupole operator in the interaction expression separately, that is to use $K' = 0$ in (38) so that

$$
\begin{aligned}
U Q_{a}^{(\pi)} U^{-1} \sim\; & Q_{(\mathrm{sd})a}^{(\pi)} + q_4^{(\pi)} \cos^2(\eta')(g_{(\pi)}^{\dagger}\, \tilde{g}_{(\pi)})_a^{(2)} + q_3^{(\pi)} \cos(\eta)\cos(\eta') P_{2,a}^{(\pi)} \\
& + q_2^{(\pi)} \sin^2(\eta) \sqrt{5} \begin{bmatrix} 4 & 4 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 0 \end{bmatrix} (g_{(\pi)}^{\dagger}\, \tilde{g}_{(\pi)})_a^{(2)} \\
& + q_4^{(\pi)} \sin^2(\eta') \begin{bmatrix} 2 & 2 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 0 \end{bmatrix} (d_{(\pi)}^{\dagger}\, \tilde{d}_{(\pi)})_a^{(2)} \\
& - q_3^{(\pi)} \sin(\eta')\sin(\eta)\, 3\sqrt{5} \begin{bmatrix} 2 & 4 & 2 \\ 4 & 2 & 2 \\ 2 & 2 & 0 \end{bmatrix} P_{2,a}^{(\pi)} \\
=\; & Q_{(\mathrm{sd})a}^{(\pi)} + q_4^{(\pi)} \sin^2(\eta') \tfrac{1}{5} W(2242;\, 22)(d_{(\pi)}^{\dagger}\, \tilde{d}_{(\pi)})_a^{(2)} \\
& + (g_{(\pi)}^{\dagger}\, \tilde{g}_{(\pi)})_a^{(2)} \{ q_4^{(\pi)} \cos^2(\eta') + q_2^{(\pi)} \sin^2(\eta) \sqrt{\tfrac{1}{5}}\, W(4422;\, 22)\} \\
& + P_{2,a}^{(\pi)}\, q_3^{(\pi)} \{\cos(\eta)\cos(\eta') - \sin(\eta)\sin(\eta')\, 3\sqrt{\tfrac{1}{5}}\, W(2442;\, 22)\}\,.
\end{aligned} \tag{41}
$$

The last term (d–g coupling) is eliminated if the angles (η, η') are such that

$$\cos(\eta)\cos(\eta') - \sin(\eta)\sin(\eta')\, 3\sqrt{\tfrac{1}{5}}\, W(2422;\, 22) = 0\,,$$

and as $\eta' = \sqrt{\frac{5}{9}}\,\eta$ this constant is equivalent to $\tan(\eta)\tan(\eta') = 94$, so that

$$\eta_{(x)} = \theta_{(x)}(\overline{Q^{(\pi)2}})^{\frac{1}{2}} \sim 90^{\circ}\,, \tag{42}$$

and with this choice

$$
\begin{aligned}
U Q_{a}^{(\pi)} U^{-1} \sim\; & q_1^{(\pi)} \cos(\eta)\{(d_{(\pi)}^{\dagger}\, s_{(\pi)})_a^{(2)} + (s_{(\pi)}^{\dagger}\, \tilde{d}_{(\pi)})_a^{(2)}\} \\
& (d_{(\pi)}^{\dagger}\, \tilde{d}_{(\pi)})_a^{(2)} \{ q_2^{(\pi)} \cos^2(\eta) + q_4^{(\pi)} \sin^2(\eta')\, W(2244;\, 22)\} \\
& + (g_{(\pi)}^{\dagger}\, \tilde{g}_{(\pi)})_a^{(2)} \{ q_4^{(\pi)} \cos^2(\eta') + q_2^{(\pi)} \sin^2(\eta) \sqrt{\tfrac{1}{5}}\, W(4422;\, 22)\} \\
=\; & (d_{(\pi)}^{\dagger}\, \tilde{d}_{(\pi)})_a^{(2)} (0{\cdot}094\, q_4^{(\pi)}) \\
& + (g_{(\pi)}^{\dagger}\, \tilde{g}_{(\pi)})_a^{(2)} \{ q_4^{(\pi)}(0{\cdot}152) + q_2^{(\pi)}(0{\cdot}05)\}\,.
\end{aligned} \tag{43}
$$

This phase angle then gives the number operator transformation from (34) and (36) of

$$N'_{\mathrm{d}} = \tfrac{1}{9}(g_{(\nu)}^{\dagger} \cdot g_{(\nu)})\,,$$

$$N'_{\mathrm{g}} = 0{\cdot}152(g_{(\nu)}^{\dagger} \cdot \tilde{g}_{(\nu)}) + 0{\cdot}305(d_{(\nu)}^{\dagger} \cdot \tilde{d}_{(\nu)})\,.$$

Thus the transformed Hamiltonian becomes

$$
\begin{aligned}
UHU^{-1} \sim\ & \Sigma_\tau \{0{\cdot}305\epsilon_{\rm g}^{(\tau)}(d^\dagger_{(\tau)} \cdot \tilde{d}_{(\tau)}) \\
& +(0{\cdot}111\epsilon_{\rm d}^{(\tau)}+0{\cdot}152\epsilon_{\rm g}^{(\tau)})(g^\dagger_{(\tau)} \cdot \tilde{g}_{(\tau)})\} \\
& -f(0{\cdot}0088\, q_4^{(\pi)}\, q_4^{(\nu)})(d^\dagger_{(\pi)}\, \tilde{d}_{(\pi)})^{(2)} \cdot (d^\dagger_{(\nu)}\, \tilde{d}_{(\nu)})^{(2)} \\
& -f(0{\cdot}05\, q_2^{(\pi)}+0{\cdot}152\, q_4^{(\pi)})(0{\cdot}05\, q_2^{(\nu)}+0{\cdot}152\, q_4^{(\nu)})(g^\dagger_{(\pi)}\, \tilde{g}_{(\pi)})^{(2)} \cdot (g^\dagger_{(\nu)}\, \tilde{g}_{(\nu)})^{(2)} .
\end{aligned} \tag{44}
$$

But this method is not favoured as every single boson operator is separately transformed and thus in product terms, and especially for the interaction specified in (37), the attendant errors of approximation are multiply compounded. Indeed, the number operators having a leading term in their transformation equations (equations 33 and 35) varying as the square of a cosine insures that the Hamiltonian cannot transform, obviously by this means, to appear as the form (8). Hence we consider instead the transformations of products of operators.

(b) Transformation of H via Its Commutators

As derived previously, the transformation of H is

$$UHU^{-1} = H+[Z,H]+\frac{1}{2!}[Z,[Z,H]]+\ldots, \tag{45}$$

so that we now seek the single and double commutators of the Hamiltonian (1) with Z (specified in equation 9).

The single commutators of all components of H and Z are derived in Appendix F, as are the single commutators of Z with select operator combinations that will be of later use. Using

$$[Z, d^\dagger_{(\nu)\beta}] = \theta_{(\nu)}(Q^{(\pi)}\, g^\dagger_{(\nu)})^{(2)}_\beta , \tag{46}$$

$$[Z, g^\dagger_{(\nu)\beta}] = -\surd\tfrac{5}{9}\,\theta_{(\nu)}(Q^{(\pi)}\, d^\dagger_{(\nu)})^{(4)}_\beta , \tag{47}$$

we obtain

$$[Z, d^\dagger_{(\nu)} \cdot \tilde{d}_{(\nu)}] = \theta_{(\nu)}(Q^{(\pi)} \cdot P^{(\nu)}), \qquad [Z, g^\dagger_{(\nu)} \cdot \tilde{g}_{(\nu)}] = -\theta_{(\nu)}(Q^{(\pi)} \cdot P^{(\nu)}), \tag{48a, b}$$

$$[Z, (s^\dagger_{(\nu)}\, \tilde{d}_{(\nu)})^{(2)}_\alpha] = \theta_{(\nu)}\{\, Q^{(\pi)}(s^\dagger_{(\nu)}\, \tilde{g}_{(\nu)})^{(4)}\}^{(2)}_\alpha , \tag{48c}$$

$$[Z, (d^\dagger_{(\nu)}\, \tilde{s}_{(\nu)})^{(2)}_\alpha] = \theta_{(\nu)}\{\, Q^{(\pi)}(g^\dagger_{(\nu)}\, \tilde{s}_{(\nu)})^{(4)}\}^{(2)}_\alpha , \tag{48d}$$

$$
\begin{aligned}
[Z, (d^\dagger_{(\nu)}\, \tilde{d}_{(\nu)})^{(\lambda)}_\alpha] = \theta_{(\nu)}\, & \Sigma_{K\mu}\, Q^{(\pi)}_\mu \langle K\, 2\, \alpha-\mu\, \mu | \lambda\, \alpha\rangle \\
& \times \{5(2K+1)\}^{\frac{1}{2}}\, W(22K2;\, 4\lambda)\tilde{P}^{(\nu)}_{K,\alpha-\mu}(\lambda),
\end{aligned} \tag{48e}
$$

$$
\begin{aligned}
[Z, (g^\dagger_{(\nu)}\, \tilde{g}_{(\nu)})^{(\lambda)}_\alpha] = -\theta_{(\nu)}\, & \Sigma_{K\mu}\, Q^{(\pi)}_\mu \langle K\, 2\, \alpha-\mu\, \mu | \lambda\, \alpha\rangle \{5(2K+1)\}^{\frac{1}{2}} \\
& \times (-)^{\lambda+K}\, W(24K4;\, 2\lambda)\tilde{P}^{(\nu)}_{K,\alpha-\mu}(\lambda),
\end{aligned} \tag{48f}
$$

$$[Z,(d^\dagger_{(\nu)}\,\tilde g_{(\nu)})^{(\lambda)}_\alpha] = \theta_{(\nu)}\,\Sigma_{K\mu}\,Q^{(\pi)}_\mu\langle K\,2\,\alpha-\mu\,\mu|\lambda\,\alpha\rangle\{5(2K+1)\}^{\frac12}$$

$$\times\{\,W(22K4;\,4\lambda)(g^\dagger_{(\nu)}\,\tilde g_{(\nu)})^{(K)}_{\alpha-\mu}-(-)^{\lambda+K}\,W(24K2;\,2\lambda)(d^\dagger_{(\nu)}\,\tilde d_{(\nu)})^{(K)}_{\alpha-\mu}\}\,, \quad (48g)$$

$$[Z,(g^\dagger_{(\nu)}\,\tilde d_{(\nu)})^{(\lambda)}_\alpha] = \theta_{(\nu)}\,\Sigma_{K\mu}\,Q^{(\pi)}_\mu\langle K\,2\,\alpha-\mu\,\mu|\lambda\,\alpha\rangle\{5(2K+1)\}^{\frac12}$$

$$\times(-)^{\lambda+K}\{\,W(22K4;\,4\lambda)(g^\dagger_{(\nu)}\,\tilde g_{(\nu)})^{(K)}_{\alpha-\mu}$$

$$-(-)^{\lambda+K}\,W(24K2;\,2\lambda)(d^\dagger_{(\nu)}\,\tilde d_{(\nu)})^{(K)}_{\alpha-\mu}\}\,, \quad (48h)$$

$$[Z,\,Q^{(\nu)}_\alpha] = \theta_{(\nu)}\,\Sigma_{K\mu}\,Q^{(\pi)}_\mu\langle K\,2\,\alpha-\mu\,\mu|2\,\alpha\rangle\{5(2K+1)\}^{\frac12}$$

$$\times[q^{(\nu)}_1\sqrt{\tfrac{1}{45}}\,\delta_{K4}\{(s^\dagger_{(\nu)}\,\tilde g_{(\nu)})^{(4)}_{\alpha-\mu}+(g^\dagger_{(\nu)}\,s_{(\nu)})^{(4)}_{\alpha-\mu}\}$$

$$+q^{(\nu)}_2\,W(22K2;\,42)\tilde P^{(\nu)}_{K,\alpha-\mu}(2)$$

$$+q^{(\nu)}_3\{1+(-)^K\}\{\,W(22K4;\,42)(g^\dagger_{(\nu)}\,\tilde g_{(\nu)})^{(K)}_{\alpha-\mu}$$

$$-(-)^K\,W(24K2;\,22)(d^\dagger_{(\nu)}\,\tilde d_{(\nu)})^{(K)}_{\alpha-\mu}\}$$

$$-q^{(\nu)}_4(-)^K\,W(24K4;\,22)\tilde P^{(\nu)}_{K,\alpha-\mu}(2)] \quad (48i)$$

in which, with $P^{(\nu)}$ as given by (40),

$$\tilde P^{(\nu)}_{K,\gamma}(\lambda) = (g^\dagger_{(\nu)}\,\tilde d_{(\nu)})^{(K)}_\gamma+(-)^{K+\lambda}(d^\dagger_{(\nu)}\,\tilde g_{(\nu)})^{(K)}_\gamma\,. \quad (49)$$

These commutators then allow the specification of the commutator of H with Z since

$$[Z,H] = \Sigma_\tau\{\epsilon^{(\tau)}_{\rm d}[Z_{(\tau)},\,d^\dagger_{(\tau)}\cdot\tilde d_{(\tau)}]$$

$$+\epsilon^{(\tau)}_{\rm g}[Z_{(\tau)},\,g^\dagger_{(\tau)}\cdot\tilde g_{(\tau)}]-f\,\Sigma_m(-)^m Q^{(-\tau)}_{-m}[Z_{(\tau)},\,Q^{(\tau)}_m]\}\,, \quad (50)$$

and using (48a), (48b) and (48i) we get

$$[Z,H] = \Sigma_\tau\,\theta_{(\tau)}(\epsilon^{(\tau)}_{\rm d}-\epsilon^{(\tau)}_{\rm g})Q^{(-\tau)}\cdot P^{(\tau)}-f\,\Sigma_\tau\,\theta_{(\tau)}\,\Sigma_{m\mu}(-)^m$$

$$\times\,Q^{(-\tau)}_{-m}\,Q^{(-\tau)}_\mu\,\Sigma_K\langle K\,2\,m-\mu\,\mu|2\,m\rangle\{5(2K+1)\}^{\frac12}$$

$$\times[q^{(\tau)}_1\,\delta_{K4}\sqrt{\tfrac{1}{45}}\,\{(s^\dagger_{(\nu)}\,\tilde g_{(\nu)})^{(4)}_{m-\mu}+(g^\dagger_{(\nu)}\,s_{(\nu)})^{(4)}_{m-\mu}\}$$

$$+\{\,q^{(\tau)}_2\,W(22K2;\,42)-q^{(\tau)}_4(-)^K\,W(24K4;\,22)\}\,\tilde P^{(\tau)}_{K,m-\mu}(2)$$

$$+q^{(\tau)}_3\{1+(-)^K\}\{\,W(22K4;\,42)(g^\dagger_{(\tau)}\,\tilde g_{(\tau)})^{(K)}_{m-\mu}$$

$$-(-)^K\,W(24K2;\,22)(d^\dagger_{(\tau)}\,\tilde d_{(\tau)})^{(K)}_{m-\mu}\}]\,. \quad (51)$$

If we define the (τ) dependent terms collectively as $\Omega^{(\tau)}_{K,m-\mu}$ in the above, angular momentum coupling gives

$$[Z,H] = \Sigma_\tau\, \theta_{(\tau)}(\epsilon^{(\tau)}_{\mathrm{d}} - \epsilon^{(\tau)}_{\mathrm{g}})Q^{(-\tau)}\,.\,P^{(\tau)}$$
$$-f\,\Sigma_\tau\, \theta_{(\tau)}\,\Sigma_K\, 5(Q^{(-\tau)}Q^{(-\tau)})^{(K)}\,.\,\Omega^{(\tau)}_K\,, \tag{52}$$

from which, in the $K = 0$ OG limit, one readily obtains

$$[Z,H] \rightarrow \Sigma_\tau\, \theta_{(\tau)}\{(\epsilon^{(\tau)}_{\mathrm{d}} - \epsilon^{(\tau)}_{\mathrm{g}})Q^{(-\tau)}\,.\,P^{(\tau)} - q^{(\tau)}_3\,\theta_{(\tau)}\,2\sqrt{5}\,f$$
$$\times\,(\overline{Q^{(-\tau)2}})(\tfrac{1}{9}g^\dagger_{(\tau)}\cdot\tilde{g}_{(\tau)} - \tfrac{1}{5}d^\dagger_{(\tau)}\cdot\tilde{d}_{(\tau)})\}$$
$$= \Sigma_\tau\theta_{(\tau)}\{\tfrac{1}{5}A_{(\tau)}\,d^\dagger_{(\tau)}\cdot\tilde{d}_{(\tau)} - \tfrac{1}{9}A_{(\tau)}\,g^\dagger_{(\tau)}\cdot\tilde{g}_{(\tau)}$$
$$+(\epsilon^{(\tau)}_{\mathrm{d}} - \epsilon^{(\tau)}_{\mathrm{g}})Q^{(-\tau)}\,.\,P^{(\tau)}\}\,, \tag{53}$$

in which

$$A_{(\tau)} = q^{(\tau)}_3\,2\sqrt{5}\,f(\overline{Q^{(-\tau)2}}) = 5\alpha_{(\tau)}\,. \tag{54}$$

This reduced form of the single commutator of H and Z is particularly convenient for use in deriving the forms of the multiple commutators of H with Z, as it enables the series expansion of UHU^{-1} to be condensed.

The double commutator is

$$[Z,[Z,H]] \sim \Sigma_\tau\, \theta_{(\tau)}\{(\epsilon^{(\tau)}_{\mathrm{d}} - \epsilon^{(\tau)}_{\mathrm{g}})[Z, Q^{(-\tau)}\,.\,P^{(\tau)}] - f\,q^{(\tau)}_3$$
$$\times\,2\sqrt{5}\,(\overline{Q^{(-\tau)2}})(\tfrac{1}{9}[Z, g^\dagger_{(\tau)}\cdot\tilde{g}_{(\tau)}] - \tfrac{1}{5}[Z, d^\dagger_{(\tau)}\cdot\tilde{d}_{(\tau)}])\}\,, \tag{55}$$

and using the results derived in Appendix F, in the $K = 0$ OG limit we get

$$[Z,[Z,H]] \sim \Sigma_\tau\, \theta_{(\tau)}\,\delta\epsilon^{(\tau)}\{\theta_{(\tau)}\,\tfrac{2}{5}(\overline{Q^{(-\tau)2}})(\tfrac{5}{9}\,g^\dagger_{(\tau)}\cdot\tilde{g}_{(\tau)} - d^\dagger_{(\tau)}\cdot\tilde{d}_{(\tau)})$$
$$+\theta_{(-\tau)}\,\tfrac{2}{5}q^{(-\tau)}_3(\overline{Q^{(\tau)}P^{(\tau)}})(\tfrac{5}{9}\,g^\dagger_{(-\tau)}\cdot\tilde{g}_{(-\tau)} - d^\dagger_{(-\tau)}\cdot\tilde{d}_{(-\tau)})\}$$
$$+\Sigma_\tau\, \theta_{(\tau)}\,5\alpha_{(\tau)}(\tfrac{14}{45}\,Q^{(-\tau)}\,.\,P^{(\tau)})\theta_{(\tau)}\,,$$

and which by using $\tau \rightarrow -\tau$ in the sum over the $\theta_{(-\tau)}$ term above gives

$$[Z,[Z,H]] = \Sigma_\tau\,\{\theta^2_{(\tau)}\,\delta\epsilon^{(\tau)}\tfrac{2}{5}(Q^{(-\tau)2})(\tfrac{5}{9}\,g^\dagger_{(\tau)}\cdot\tilde{g}_{(\tau)} - d^\dagger_{(\tau)}\cdot\tilde{d}_{(\tau)})$$
$$+\theta_{(-\tau)}\,\theta_{(\tau)}\,\delta\epsilon^{(-\tau)}\tfrac{2}{5}\,q^{(\tau)}_3(\overline{Q^{(-\tau)}P^{(-\tau)}})(\tfrac{5}{9}\,g^\dagger_{(\tau)}\cdot\tilde{g}_{(\tau)} - d^\dagger_{(\tau)}\cdot\tilde{d}_{(\tau)})$$
$$+\theta^2_{(\tau)}\,\alpha_{(\tau)}\tfrac{14}{9}\,Q^{(-\tau)}\,.\,P^{(\tau)}\}$$
$$= -\Sigma_\tau\,\theta^2_{(\tau)}\,\delta\epsilon^{(\tau)}\beta_{(\tau)}\,d^\dagger_{(\tau)}\cdot\tilde{d}_{(\tau)} + \Sigma_\tau\,\theta^2_{(\tau)}\,\delta\epsilon^{(\tau)}\tfrac{5}{9}\,\beta_{(\tau)}\,g^\dagger_{(\tau)}\cdot\tilde{g}_{(\tau)}$$
$$+\Sigma_\tau\,\theta^2_{(\tau)}\,\tfrac{14}{9}\,\alpha^{(\tau)}Q^{(-\tau)}\,.\,P^{(\tau)}\,, \tag{56}$$

in which, with $\alpha_{(\tau)}$ as defined in (54), we have

$$\delta\epsilon^{(\tau)} = \epsilon_{\rm d}^{(\tau)} - \epsilon_{\rm g}^{(\tau)},$$

$$\beta_{(\tau)} = \tfrac{2}{5}(\overline{Q^{(-\tau)2}}) + \tfrac{2}{5}\, q_3^{(\tau)}(\overline{Q^{(-\tau)}P^{(-\tau)}})\, \frac{\theta_{(-\tau)}}{\theta_{(\tau)}}\, \frac{\delta\epsilon^{(-\tau)}}{\delta\epsilon^{(\tau)}}. \tag{57}$$

Consider first the circumstances in which $(\overline{Q^{(-\tau)}P^{(-\tau)}})$ is negligible. In such cases the transformation angles decouple so that

$$\beta_{(\tau)} \to b_{(\tau)} = \tfrac{2}{5}(\overline{Q^{(-\tau)2}}) = \alpha_{(\tau)} f q_3^{(\tau)}/\sqrt{5}, \tag{58}$$

and we get

$$[Z, H] \sim \Sigma_\tau\, \theta_{(\tau)}(\alpha_{(\tau)}\, d^\dagger_{(\tau)} \cdot \tilde{d}_{(\tau)} - \tfrac{5}{9}\alpha_{(\tau)}\, g^\dagger_{(\tau)} \cdot \tilde{g}_{(\tau)} + \delta\epsilon^{(\tau)} Q^{(-\tau)} . P^{(\tau)}), \tag{59}$$

$$\begin{aligned}[Z,[Z,H]] \sim \Sigma_\tau\, \theta^2_{(\tau)}(&- b_{(\tau)}\, \delta\epsilon^{(\tau)} d^\dagger_{(\tau)} \cdot \tilde{d}_{(\tau)} + \tfrac{5}{9} b_{(\tau)}\, \delta\epsilon^{(\tau)} g^\dagger_{(\tau)} \cdot \tilde{g}_{(\tau)} \\ &+ \tfrac{14}{9}\alpha_{(\tau)}\, Q^{(-\tau)} . P^{(\tau)}).\end{aligned} \tag{60}$$

The nth commutator, defined by

$$[Z,[Z,[Z,\dots[Z,H]\dots]]] = [\quad]_{(n)}, \tag{61}$$

can then be found by induction [subscripts (τ) are omitted whenever superfluous for simplicity] as

$$\begin{aligned}[\quad]_{(3)} &\sim \Sigma_\tau\, \theta^3_{(\tau)}\{-\tfrac{14}{9}\alpha b d^\dagger . \tilde{d} + \tfrac{5}{9}(\tfrac{14}{9}\alpha) b g^\dagger . \tilde{g} - \tfrac{14}{9} b \delta\epsilon\, Q^{(-\tau)} . P\}, \\ [\quad]_{(4)} &\sim \Sigma_\tau\, \theta^4_{(\tau)}\{\tfrac{14}{9} b^2 \delta\epsilon\, d^\dagger . \tilde{d} - \tfrac{5}{9}(\tfrac{14}{9} b^2)\delta\epsilon\, g^\dagger . \tilde{g} \\ &\qquad - (\tfrac{14}{9}\alpha)(\tfrac{14}{9} b) Q^{(-\tau)} . P^{(\tau)}\}, \qquad \text{etc.}\end{aligned} \tag{62}$$

Using these commutator results in the series expansion of the transformed Hamiltonian (45) gives

$$\begin{aligned} UHU^{-1} \sim\ & \Sigma_\tau\, d^\dagger_{(\tau)} \cdot \tilde{d}_{(\tau)}\{\epsilon_{\rm d}^{(\tau)} + \alpha_{(\tau)}(9/14 b_{(\tau)})^{\frac{1}{2}} \sin \Theta_{(\tau)} \\ &+ (9\delta\epsilon^{(\tau)}/14)(\cos \Theta_{(\tau)} - 1)\} \\ &+ \Sigma_\tau\, g^\dagger_{(\tau)} \cdot \tilde{g}_{(\tau)}\{\epsilon_{\rm g}^{(\tau)} - (5\alpha_{(\tau)}/9)(9/14 b_{(\tau)})^{\frac{1}{2}} \sin \Theta_{(\tau)} \\ &- (5\delta\epsilon^{(\tau)}/14)(\cos \Theta_{(\tau)} - 1)\} \\ &+ \Sigma_\tau\, Q^{(-\tau)} . P^{(\tau)}\{-f/2 + \delta\epsilon^{(\tau)}(9/14 b_{(\tau)})^{\frac{1}{2}} \sin \Theta_{(\tau)} \\ &- (\alpha_{(\tau)}/b_{(\tau)})(\cos \Theta_{(\tau)} - 1)\} + \Sigma_\tau (f/2) Q^{(-\tau)} . (P^{(\tau)} - Q^{(\tau)}), \end{aligned} \tag{63}$$

in which

$$\Theta_{(\tau)} = (14 b_{(\tau)}/9)^{\frac{1}{2}} \theta_{(\tau)} . \tag{64}$$

If we then assume that the residual interaction is

$$- Q^{(-\tau)} . (P^{(\tau)} - Q^{(\tau)}) \sim Q^{(-\tau)}_{(\mathrm{sd})} \cdot Q^{(\tau)}_{(\mathrm{sd})} + q_4^{(-\tau)} q_4^{(\tau)} (g^\dagger_{(-\tau)} \tilde{g}_{(-\tau)})^{(2)} \cdot (g^\dagger_{(\tau)} \tilde{g}_{(\tau)})^{(2)} , \tag{65}$$

in which $Q^{(x)}_{(\mathrm{sd})}$ is an s–d boson space quadrupole operator then, under the condition on the coupling angle $\Theta_{(\tau)}$ that

$$\delta\epsilon^{(\tau)} (9 b_{(\tau)}/14)^{\frac{1}{2}} \sin \Theta_{(\tau)} - \alpha_{(\tau)} \cos \Theta_{(\tau)} + \alpha_{(\tau)} - \tfrac{1}{2} f b_{(\tau)} = 0 , \tag{66}$$

we have

$$\begin{aligned} U H U^{-1} \to {} & \Sigma_\tau (\hat{\epsilon}^{(\tau)}_{\mathrm{d}}\, d^\dagger_{(\tau)} \cdot \tilde{d}_{(\tau)} - \tfrac{1}{2} f Q^{(-\tau)}_{(\mathrm{sd})} \cdot Q^{(\tau)}_{(\mathrm{sd})}) \\ & + \Sigma_\tau \{ \hat{\epsilon}^{(\tau)}_{\mathrm{g}}\, g^\dagger_{(\tau)} \cdot \tilde{g}_{(\tau)} - \tfrac{1}{2} f q_4^{(\tau)} q_4^{(-\tau)} \\ & \times (g^\dagger_{(-\tau)} \tilde{g}_{(-\tau)})^{(2)} \cdot (g^\dagger_{(\tau)} \tilde{g}_{(\tau)})^{(2)} \} = H_{(\mathrm{sd})} + H_{(\mathrm{g})} , \end{aligned} \tag{67}$$

in which the 'new' single boson energies are

$$\hat{\epsilon}^{(\tau)}_{\mathrm{d}} = \epsilon^{(\tau)}_{\mathrm{d}} + \alpha_{(\tau)} (9/14 b_{(\tau)})^{\frac{1}{2}} \sin \Theta_{(\tau)} + (9 \delta\epsilon^{(\tau)}/14)(\cos \Theta_{(\tau)} - 1) , \tag{68}$$

$$\hat{\epsilon}^{(\tau)}_{\mathrm{g}} = \epsilon^{(\tau)}_{\mathrm{g}} - \tfrac{5}{9} \alpha_{(\tau)} (9/14 b_{(\tau)})^{\frac{1}{2}} \sin \Theta_{(\tau)} - \tfrac{5}{9} (9 \delta\epsilon^{(\tau)}/14)(\cos \Theta_{(\tau)} - 1) , \tag{69}$$

having a separation

$$\begin{aligned} \delta\hat{\epsilon}^{(\tau)} &= \delta\epsilon^{(\tau)} + \tfrac{14}{9} \alpha_{(\tau)} (9/14 b_{(\tau)})^{\frac{1}{2}} \sin \Theta_{(\tau)} + \delta\epsilon^{(\tau)} (\cos \Theta_{(\tau)} - 1) \\ &= (14 \alpha^2_{(\tau)}/9 b_{(\tau)})^{\frac{1}{2}} \sin \Theta_{(\tau)} + \delta\epsilon^{(\tau)} \cos \Theta_{(\tau)} . \end{aligned} \tag{70}$$

Consider now circumstances in which $(\overline{Q^{(x)} P^{(x)}})$ is treated to first order. With results obtained previously but given again for convenience, namely

$$H = \Sigma_\tau (\epsilon^{(\tau)}_{\mathrm{d}} d^\dagger_{(\tau)} \cdot \tilde{d}_{(\tau)} + \epsilon^{(\tau)}_{\mathrm{g}} g^\dagger_{(\tau)} \cdot \tilde{g}_{(\tau)} - \tfrac{1}{2} f Q^{(-\tau)} . Q^{(\tau)}) ,$$

$$[Z, H] = \Sigma_\tau (\theta_{(\tau)} \alpha_{(\tau)} d^\dagger_{(\tau)} \cdot \tilde{d}_{(\tau)} - \theta_{(\tau)} \tfrac{5}{9} \alpha_{(\tau)} g^\dagger_{(\tau)} \cdot \tilde{g}_{(\tau)} + \theta_{(\tau)} \delta\epsilon^{(\tau)} Q^{(-\tau)} . P^{(\tau)}) ,$$

$$[Z, d^\dagger_{(\tau)} \cdot \tilde{d}_{(\tau)}] = \theta_{(\tau)} Q^{(-\tau)} . P^{(\tau)} , \qquad [Z, g^\dagger_{(\tau)} \cdot \tilde{g}_{(\tau)}] = -\theta_{(\tau)} Q^{(-\tau)} . P^{(\tau)} , \tag{71}$$

and, from Appendix F, as

$$[Z, \Sigma_\tau C_{(\tau)} Q^{(-\tau)} . P^{(\tau)}] = \Sigma_\tau \theta_{(\tau)} (\tfrac{5}{9} g^\dagger_{(\tau)} \cdot \tilde{g}_{(\tau)} - d^\dagger_{(\tau)} \cdot \tilde{d}_{(\tau)}) \times (C_{(\tau)} b_{(\tau)} + C_{(-\tau)} \gamma_{(-\tau)}) , \tag{72}$$

Table 3. Expansion functions $C_{\mathrm{d}}^{(n)}$ $(=-\frac{9}{5}C_{\mathrm{g}}^{(n)})$
Subscripts (τ) are omitted wherever possible and $(-)$ signifies $(-\tau)$

n	$(\gamma_{(x)})^0$ terms	$(\gamma_{(x)})^1$ terms	Higher order terms
1	θa	—	—
2	$-\theta^2 b\delta\epsilon$	$-\theta\theta_{(-)}\gamma_{(-)}\delta\epsilon^{(-)}$	—
3	$-\theta^3(\tfrac{14}{9}b)a$	$-\theta\theta_{(-)}^2(\tfrac{14}{9}\gamma_{(-)})a_{(-)}$	—
4	$\theta^4(\tfrac{14}{9}b)b\delta\epsilon$	$\theta^3\theta_{(-)}(\tfrac{14}{9}b)\gamma_{(-)}\delta\epsilon^{(-)}$ $+\theta\theta_{(-)}^3(\tfrac{14}{9}b_{(-)})\gamma_{(-)}\delta\epsilon^{(-)}$	$\theta^2\theta_{(-)}^2(\tfrac{14}{9})\gamma\gamma_{(-)}\delta\epsilon$
5	$\theta^5(\tfrac{14}{9}b)^2 a$	$\theta^3\theta_{(-)}^2(\tfrac{14}{9}b)(\tfrac{14}{9}\gamma_{(-)})a_{(-)}$ $+\theta\theta_{(-)}^4(\tfrac{14}{9}b_{(-)})(\tfrac{14}{9}\gamma_{(-)})a_{(-)}$	$\theta^3\theta_{(-)}^2\,(\tfrac{14}{9})^2\gamma\gamma_{(-)}a$
6	$-\theta^6(\tfrac{14}{9}b)^2 b\delta\epsilon$	$-\theta^5\theta_{(-)}(\tfrac{14}{9}b)(\tfrac{14}{9}\gamma_{(-)})b\delta\epsilon^{(-)}$ $-\theta^3\theta_{(-)}^3(\tfrac{14}{9}b_{(-)})(\tfrac{14}{9}\gamma_{(-)})b\delta\epsilon^{(-)}$ $-\theta\theta_{(-)}^5(\tfrac{14}{9}b_{(-)})^2\gamma_{(-)}\delta\epsilon^{(-)}$	$-\theta^4\theta_{(-)}^2(\tfrac{14}{9}\gamma)(\tfrac{14}{9}\gamma_{(-)})b\delta\epsilon$ $-\theta^2\theta_{(-)}^4(\tfrac{14}{9}b_{(-)})(\tfrac{14}{9}\gamma)\gamma_{(-)}\delta\epsilon$ $-\theta^4\theta_{(-)}^2(\tfrac{14}{9}b)(\tfrac{14}{9}\gamma)\gamma_{(-)}\delta\epsilon$ $\theta^3\theta_{(-)}^3(\tfrac{14}{9}\gamma)(\tfrac{14}{9}\gamma_{(-)})\,\gamma_{(-)}\delta\epsilon^{(-)}$
7	$-\theta^7(\tfrac{14}{9}b)^3 a$	$-\theta^5\theta_{(-)}^2(\tfrac{14}{9}b)^2\,(\tfrac{14}{9}\gamma_{(-)})a_{(-)}$ $-\theta^3\theta_{(-)}^4(\tfrac{14}{9}b)(\tfrac{14}{9}b_{(-)})\,(\tfrac{14}{9}\gamma_{(-)})a_{(-)}$ $-\theta\theta_{(-)}^6(\tfrac{14}{9}b_{(-)})^2(\tfrac{14}{9}\gamma_{(-)})a_{(-)}$	$-\theta^5\theta_{(-)}^2\,(\tfrac{14}{9}b)(\tfrac{14}{9}\gamma)(\tfrac{14}{9}\gamma_{(-)})a$ $-\theta^3\theta_{(-)}^4\,(\tfrac{14}{9}b_{(-)})(\tfrac{14}{9}\gamma)(\tfrac{14}{9}\gamma_{(-)})a$ $-\theta^5\theta_{(-)}^2(\tfrac{14}{9}b)\,(\tfrac{14}{9}\gamma)(\tfrac{14}{9}\gamma_{(-)})a$ $-\theta^3\theta_{(-)}^4(\tfrac{14}{9}\gamma)(\tfrac{14}{9}\gamma_{(-)})^2 a_{(-)}$
8	$\theta^8(\tfrac{14}{9}b)^3 b\delta\epsilon$	$\theta^7\theta_{(-)}(\tfrac{14}{9}b)^2\,(\tfrac{14}{9}\gamma_{(-)})b\delta\epsilon^{(-)}$ $+\theta^5\theta_{(-)}^3(\tfrac{14}{9}b)(\tfrac{14}{9}\gamma_{(-)})b\delta\epsilon$ $+\theta^3\theta_{(-)}^5(\tfrac{14}{9}b_{(-)})^2(\tfrac{14}{9}\gamma_{(-)})b\delta\epsilon^{(-)}$ $+\theta\theta_{(-)}^7(\tfrac{14}{9}b_{(-)})^3\gamma_{(-)}\delta\epsilon^{(-)}$	11 terms

Table 4. Expansion functions $V^{(n)}$
Subscripts (τ) are omitted wherever possible and $(-)$ signifies $(-\tau)$

n	$(\gamma_{(x)})^0$ terms	$(\gamma_{(x)})^1$ terms	Higher order terms
1	$\theta\delta\epsilon$	—	—
2	$\theta^2(\tfrac{14}{9}a)$	—	—
3	$-\theta^3(\tfrac{14}{9}b)\delta\epsilon$	$-\theta^2\theta_{(-)}^2(\tfrac{14}{9}\gamma_{(-)})\delta\epsilon^{(-)}$	—
4	$-\theta^4(\tfrac{14}{9}b)(\tfrac{14}{9}a)$	$-\theta^2\theta_{(-)}^2(\tfrac{14}{9}\gamma_{(-)})(\tfrac{14}{9}a_{(-)})$	—
5	$\theta^5(\tfrac{14}{9}b)^2\delta\epsilon$	$\theta^4\theta_{(-)}(\tfrac{14}{9}b)(\tfrac{14}{9}\gamma_{(-)})\,\delta\epsilon^{(-)}$ $+\theta^2\theta_{(-)}^3(\tfrac{14}{9}b_{(-)})(\tfrac{14}{9}\gamma_{(-)})\delta\epsilon^{(-)}$	$\theta^3\theta_{(-)}^2(\tfrac{14}{9}\gamma)(\tfrac{14}{9}\gamma_{(-)})\delta\epsilon$
6	$\theta^6(\tfrac{14}{9}b)^2(\tfrac{14}{9}a)$	$\theta^4\theta_{(-)}^2(\tfrac{14}{9}b)(\tfrac{14}{9}\gamma_{\ (-)})(\tfrac{14}{9}a_{(-)})$ $+\theta^2\theta_{(-)}^4(\tfrac{14}{9}b_{(-)})(\tfrac{14}{9}\gamma_{(-)})(\tfrac{14}{9}a_{(-)})$	$\theta^4\theta_{(-)}^2(\tfrac{14}{9}\gamma)\,(\tfrac{14}{9}\gamma_{(-)})(\tfrac{14}{9}a)$
7	$-\theta^7(\tfrac{14}{9}b)^3\delta\epsilon$	$-\theta^6\theta_{(-)}(\tfrac{14}{9}b)^2\,(\tfrac{14}{9}\gamma_{(-)})\delta\epsilon^{(-)}$ $-\theta^4\theta_{(-)}^3(\tfrac{14}{9}b)(\tfrac{14}{9}b_{(-)})\,(\tfrac{14}{9}\gamma_{(-)})\delta\epsilon^{(-)}$ $-\theta^2\theta_{(-)}^5(\tfrac{14}{9}b_{(-)})^2(\tfrac{14}{9}\gamma_{(-)})\,\delta\epsilon^{(-)}$	$-\theta^5\theta_{(-)}^2(\tfrac{14}{9}b)\,(\tfrac{14}{9}\gamma)(\tfrac{14}{9}\gamma_{(-)})\delta\epsilon$ $-\theta^3\theta_{(-)}^4\,(\tfrac{14}{9}b_{(-)})(\tfrac{14}{9}\gamma)(\tfrac{14}{9}\gamma_{(-)})\delta\epsilon$ $-\theta^4\theta_{(-)}^3(\tfrac{14}{9}\gamma)(\tfrac{14}{9}\gamma_{(-)})(\tfrac{14}{9}b)\delta\epsilon$ $-\theta^5\theta_{(-)}^2(\tfrac{14}{9}b_{(-)})(\tfrac{14}{9}\gamma)(\tfrac{14}{9}\gamma_{(-)})\delta\epsilon$
8	$-\theta^8(\tfrac{14}{9}b)^3(\tfrac{14}{9}a)$	$-\theta^6\theta_{(-)}^2(\tfrac{14}{9}b)^{\,2}(\tfrac{14}{9}\gamma_{(-)})(\tfrac{14}{9}a_{(-)})$ $-\theta^2\theta_{(-)}^6(\tfrac{14}{9}b_{(-)})^2(\tfrac{14}{9}\gamma_{(-)})\,(\tfrac{14}{9}a_{(-)})$ $-\theta^4\theta_{(-)}^4(\tfrac{14}{9}b)(\tfrac{14}{9}b_{(-)})\,(\tfrac{14}{9}\gamma_{(-)})(\tfrac{14}{9}a_{(-)})$	4 terms

Hamiltonian, namely as given by (67), but where now the 'new' boson energies are

$$\begin{aligned}\hat{\epsilon}_{\mathrm{d}}^{(\tau)} = {} & \epsilon_{\mathrm{d}}^{(\tau)} + (9\alpha_{(\tau)}^2/14 b_{(\tau)})^{\frac{1}{2}} \sin\phi_{(\tau)} + (9\delta\epsilon^{(\tau)}/14)(\cos\phi_{(\tau)} - 1) \\ & + \gamma_{(-\tau)}\, \delta\epsilon^{(-\tau)}\, \tfrac{9}{14}(b_{(\tau)}\, b_{(-\tau)})^{-\frac{1}{2}} \{\phi_{(\tau)}\,\phi_{(-\tau)}/(\phi_{(\tau)}^2 - \phi_{(-\tau)}^2)\}(\cos\phi_{(\tau)} - \cos\phi_{(-\tau)}) \\ & + \gamma_{(-\tau)}\, \alpha_{(-\tau)} \tfrac{9}{14}(b_{(\tau)}\, b_{(-\tau)}^2)^{-\frac{1}{2}} \{\phi_{(-\tau)}/(\phi_{(\tau)}^2 - \phi_{(-\tau)}^2)\} \\ & \times(\phi_{(-\tau)} \sin\phi_{(\tau)} - \phi_{(\tau)} \sin\phi_{(-\tau)}),\end{aligned} \tag{80}$$

and $\hat{\epsilon}_{\mathrm{g}}^{(\tau)}$ is shifted from $\epsilon_{\mathrm{g}}^{(\tau)}$ by $-\frac{5}{9}$ of the above corrections.

The coupling angles to eliminate any $Q^{(-\tau)} \cdot P^{(\tau)}$ component in the transformed Hamiltonian are now determined as solutions of the coupled algebraic equations

$$\begin{aligned}& -\tfrac{1}{2}f + \alpha_{(\tau)}/b_{(\tau)} - (\alpha_{(\tau)}/b_{(\tau)})\cos\phi_{(\tau)} + (9/14 b_{(\tau)})^{\frac{1}{2}}\delta\epsilon^{(\tau)} \sin\phi_{(\tau)} \\ & + \gamma_{(-\tau)}\,\delta\epsilon^{(-)}(9/14 b_{(\tau)}^2\, b_{(-\tau)})^{\frac{1}{2}}\{\phi_{(\tau)}/(\phi_{(\tau)}^2 - \phi_{(-\tau)}^2)\}\{\phi_{(-\tau)}\sin\phi_{(\tau)} - \phi_{(\tau)}/(\phi_{(\tau)}^2 - \phi_{(-\tau)}^2)\} \\ & + \gamma_{(-\tau)}\,\alpha_{(-\tau)}/\{b_{(\tau)}\, b_{(-\tau)}(\phi_{(\tau)}^2 - \phi_{(-\tau)}^2)\}\{\phi_{(-\tau)}^2(1 - \cos\phi_{(\tau)}) - \phi_{(\tau)}^2(1 - \cos\phi_{(\tau)})\} = 0.\end{aligned} \tag{81}$$

With weak coupling, $\phi_{(-)} \sim \phi_{(+)} - \Delta$, and in the limit $\Delta \to 0$, the foregoing reduce to

$$\begin{aligned}\hat{\epsilon}_{\mathrm{d}} = {} & \epsilon_{\mathrm{d}} - \tfrac{9}{14}\delta\epsilon \\ & + \sin\phi\{(9/14 b)^{\frac{1}{2}}\alpha - \gamma\delta\epsilon(9/14 b)\tfrac{1}{2}\phi - (9/14 b)^{\frac{1}{2}}\gamma\alpha/2b\} \\ & + \cos\phi\{\tfrac{9}{14}\delta\epsilon + (9/14 b)^{\frac{1}{2}}\gamma\alpha\phi/2b\},\end{aligned} \tag{82}$$

$$\begin{aligned}& -\tfrac{1}{2}f + \alpha/b - \gamma\alpha/b^2 + \sin\phi\{(9/14 b)^{\frac{1}{2}}\delta\epsilon - (9/14 b)^{\frac{1}{2}}\gamma\delta\epsilon/2b + \gamma\alpha\phi/2b^2\} \\ & + \cos\phi\{-\alpha/b + \gamma\alpha/b^2 + (9/14 b)^{\frac{1}{2}}\gamma\delta\epsilon\phi/2b\} = 0,\end{aligned} \tag{83}$$

wherein the (τ) labels have been omitted as they are then irrelevant.

The structure of the transformed Hamiltonian in either approximation (zero or first order in γ) is, as given in (67),

$$\begin{aligned}UHU^{-1} & = H_{(\mathrm{sd})} + H_{(\mathrm{g})} \\ & \sim \Sigma_\tau\, \hat{\epsilon}_{\mathrm{d}}^{(\tau)}\, d_{(\tau)}^{\dagger} \cdot \tilde{d}_{(\tau)} - f\, Q_{(\mathrm{sd})}^{(\pi)} \cdot Q_{(\mathrm{sd})}^{(\nu)},\end{aligned} \tag{84}$$

if the single boson splittings are large (whence $H_{(\mathrm{g})}$ can be neglected in a study of low excitation spectra).

A crucial factor in the above scheme is that the residual interaction, the s–d quadrupole–quadrupole force, has not been altered in strengths (f, q_1, q_2) from that s–d–g form with which we started. All that has been obtained with these approximations to a transformation scheme is a splitting of d- and g-boson kinetic energies which, while minimising g-boson content in low excitation states, does not substantially influence the moments of intertia. This effect is described in detail in the next subsection and, as a result, a better approximation is needed if the method is to achieve the desired result of a reasonable spectrum comparison. Thus, we consider:

(c) An Extended Model—I

In the preceding subsection, select parts of the interaction in the Hamiltonian were treated as a perturbation, and in particular, the s–d interaction term (involving $q_1^{(\tau)}$). This is inadequate in view of the important role such terms play in defining the moments of inertia associated with the exact Hamiltonian spectrum.

Noting that the previous results (59)–(63) for $q_1 = 0$ are unchanged, a better approximation is to consider

$$U H U^{-1} = U H(q_1 = 0)\, U^{-1} + U\{ H(q_1) - H(q_1 = 0)\}\, U^{-1}; \tag{85}$$

the latter term now providing a better treatment of the s–d coupling under transformation. With

$$\begin{aligned} T &= H(q_1) - H(q_1 = 0) \\ &\sim -f q_1^{(\pi)} q_1^{(\nu)} \{ (s_{(\pi)}^{\dagger}\, \tilde{d}_{(\pi)})^{(2)} + (d_{(\pi)}^{\dagger}\, s_{(\pi)})^{(2)} \} . \{ (s_{(\nu)}^{\dagger}\, \tilde{d}_{(\nu)})^{(2)} + (d_{(\nu)}^{\dagger}\, s_{(\nu)})^{(2)} \}, \end{aligned} \tag{86}$$

the transformation follows simply since s and $s^{\dagger}$ both commute with Z (and so U), so that we have the further simplification

$$U d_{(x)\mu}^{\dagger}\, U^{-1} \sim \cos \eta_{(x)}\, d_{(x)\mu}^{\dagger}, \tag{87}$$

$$\begin{aligned} U T U^{-1} \sim\; & -f (q_1^{(\pi)} \cos \eta_{(\pi)})(q_1^{(\nu)} \cos \eta_{(\nu)}) \\ & \times \{ (s_{(\pi)}^{\dagger}\, \tilde{d}_{(\pi)})^{(2)} + (d_{(\pi)}^{\dagger}\, s_{(\pi)})^{(2)} \} . \{ (s_{(\nu)}^{\dagger}\, \tilde{d}_{(\nu)})^{(2)} + (d_{(\nu)}^{\dagger}\, s_{(\nu)})^{(2)} \}. \end{aligned} \tag{88}$$

Thus, ignoring the g-boson Hamiltonian of (67), we get

$$U H U^{-1} \sim \Sigma_{\tau} (\hat{\epsilon}_{\mathrm{d}}^{(\tau)}\, d_{(\tau)}^{\dagger} . \tilde{d}_{(\tau)} - \tfrac{1}{2} f\, Q_{(\mathrm{sd})}^{(-\tau)\prime} . Q_{(\mathrm{sd})}^{(\tau)\prime}), \tag{89}$$

in which

$$Q_{(\mathrm{sd})}^{(\tau)\prime} = q_1^{(\tau)} \cos \eta_{(\tau)} \{ (s_{(\tau)}^{\dagger}\, \tilde{d}_{(\tau)})^{(2)} + (d_{(\tau)}^{\dagger}\, s_{(\tau)})^{(2)} \} + q_2^{(\tau)} (d_{(\tau)}^{\dagger}\, \tilde{d}_{(\tau)})^{(2)}, \tag{90}$$

and with

$$\eta_{(\tau)} = \theta_{(\tau)} (\overline{Q^{(\tau)2}})^{\frac{1}{2}},$$

determined by the critical coupling angles $\theta_{(\tau)}$, as given by the condition (81).

Herein we have made use of the (approximate) transformation of the individual d-boson operator on a select part of the residual interaction. But one may consider a different approximation scheme with the transformation of the complete residual interactions:

(d) An Extended Model—II

The previous results (59)–(63), under the OG approximation, were obtained by treating the residual interaction

$$\Delta = -\tfrac{1}{2}f\,\Sigma_\tau\, Q^{(-\tau)}.(Q^{(\tau)}-P^{(\tau)}) = -\tfrac{1}{2}f\,\Sigma_\tau\, Q^{(-\tau)}.\,V^{(\tau)} \tag{91}$$

as a perturbation, i.e. with $H = H'+\Delta$,

$$UHU^{-1} = UH'U^{-1}+U\Delta U^{-1} \sim UH'U^{-1}+\Delta, \tag{92}$$

But as

$$U\Delta U^{-1} = \Delta+[Z,\Delta]+\frac{1}{2!}[Z,[Z,\Delta]]+\dots,$$

an approximation scheme to express the commutators of Δ with Z in a convenient form is possible. Such is developed in Appendix G giving

$$\begin{aligned}[Z,\Delta] &\approx -\tfrac{1}{2}f\,\Sigma_\tau\, Q^{(\tau)}.\,V^{(\tau)}(-\theta_{(-\tau)}\,X^{(-\tau)}),\\ [Z,[Z,\Delta]] &\approx -\tfrac{1}{2}f\,\Sigma_\tau\, Q^{(-\tau)}.\,V^{(\tau)}(\theta_{(-\tau)}\,X^{(-\tau)})(\theta_{(\tau)}\,X^{(\tau)}),\end{aligned} \tag{93}$$

from which all higher order commutators can be readily deduced in terms of operators $Q^{(a)}.\,V^{(a)}$ and $Q^{(-a)}.\,V^{(a)}$ and the scale factors involving the d- and g-boson numbers,

$$X^{(a)} = \tfrac{2}{5}q_3^{(a)}(n_{\mathrm{d}}^{(a)}-\tfrac{5}{9}n_{\mathrm{g}}^{(a)}). \tag{94}$$

The transformation series then has the form

$$\begin{aligned}U\Delta U^{-1} \sim & -\tfrac{1}{2}f\,\Sigma_\tau\, Q^{(-\tau)}.\,V^{(\tau)}\Big(1+\frac{1}{2!}(\theta_{(\tau)}\,X^{(\tau)})(\theta_{(-\tau)}\,X^{(-\tau)})\\ &+\frac{1}{4!}(\theta_{(\tau)}\,X^{(\tau)})(\theta_{(-\tau)}\,X^{(-\tau)})^2+\dots\Big)\\ &-\tfrac{1}{2}f\,\Sigma_\tau\, Q^{(\tau)}.\,V^{(\tau)}\Big((-\theta_{(-\tau)}\,X^{(-\tau)})+\frac{1}{3!}(-\theta_{(-\tau)}\,X^{(-\tau)})^2(-\theta_{(\tau)}\,X^{(\tau)})\\ &+\frac{1}{5!}(-\theta_{(-\tau)}\,X^{(-\tau)})^3(-\theta_{(\tau)}\,X^{(\tau)})^2+\dots\Big),\end{aligned} \tag{95}$$

and which, with

$$x_{(\tau)}^2 = \theta_{(\tau)}\,X^{(\tau)}, \tag{96}$$

is

$$U\Delta U^{-1} = -\tfrac{1}{2}f\,\Sigma_\tau\, Q^{(-\tau)}\,.\,V^{(\tau)}\Big(1+\frac{1}{2!}\,x_{(\tau)}^2\,x_{(-\tau)}^2+\frac{1}{4!}\,x_{(\tau)}^4\,x_{(-\tau)}^4+\dots\Big)$$

$$+\tfrac{1}{2}f\,\Sigma_\tau\, Q^{(\tau)}\,.\,V^{(\tau)}\Big(x_{(\tau)}^2+\frac{1}{3!}\,x_{(-\tau)}^4\,x_{(\tau)}^2+\frac{1}{5!}\,x_{(-\tau)}^6\,x_{(\tau)}^4+\dots\Big)$$

$$= -\tfrac{1}{2}f\,\Sigma_\tau\, Q^{(-\tau)}\,.\,V^{(\tau)}\cosh(x_{(\tau)}\,x_{(-\tau)})$$

$$+\tfrac{1}{2}f\,\Sigma_\tau\, Q^{(\tau)}\,.\,V^{(\tau)}(x_{(-\tau)}/x_{(\tau)})\sinh(x_{(\tau)}\,x_{(-\tau)})\,. \tag{97}$$

Using the critical angles $\theta_{(\pm\tau)}$ as determined by the condition of (81) (and assuming that $\theta_{(\tau)}\,X_{(\tau)}$ have the same overall sign for both values of τ), the transformation now takes the form

$$UHU^{-1} \sim \Sigma_\tau\,\{\hat{\epsilon}_{\rm d}^{(\tau)}\,d_{(\tau)}^\dagger\cdot\tilde{d}_{(\tau)}+\hat{\epsilon}_{\rm g}^{(\tau)}\,g_{(\tau)}^\dagger\cdot\tilde{g}_{(\tau)}$$

$$-\tfrac{1}{2}f\,Q^{(-\tau)}\,.\,V^{(\tau)}\cosh(\theta_{(\tau)}\,\theta_{(-\tau)}\,X^{(\tau)}X^{(-\tau)})^{\frac{1}{2}} \tag{98}$$

$$+\tfrac{1}{2}f\,Q^{(\tau)}\,.\,V^{(\tau)}(\theta_{(-\tau)}\,X^{(-\tau)}/\theta_{(\tau)}\,X^{(\tau)})^{\frac{1}{2}}\sinh(\theta_{(\tau)}\,\theta_{(-\tau)}\,X^{(\tau)}X^{(-\tau)})^{\frac{1}{2}}\}\,.$$

If the transformation raises the individual g-boson energies $\hat{\epsilon}_{\rm g}$ sufficiently, the low lying spectrum will have little dependence upon the (now uncoupled) g-boson Hamiltonian, so that

$$UHU^{-1} \sim \Sigma_\tau\,\{\hat{\epsilon}_{\rm d}^{(\tau)}\,d_{(\tau)}^\dagger\cdot\tilde{d}_{(\tau)}-\tfrac{1}{2}f\cosh(\phi^{(\tau)})\,Q_{\rm (sd)}^{(-\tau)}\cdot Q_{\rm (sd)}^{(\tau)}$$

$$+\tfrac{1}{2}f\sinh(\phi^{(\tau)})\,Q_{\rm (sd)}^{(\tau)}\cdot Q_{\rm (sd)}^{(\tau)}\}\,. \tag{99}$$

Reorganising (98) gives for the equal neutron–proton boson system ($\phi^{(\tau)}=\phi$)

$$UHU^{-1} = \Sigma_\tau\,\hat{\epsilon}_{\rm d}^{(\tau)}(d_{(\tau)}^\dagger\cdot d_{(\tau)})-\tfrac{1}{2}f\cosh(\phi)\;Q\,.\,Q$$

$$+\tfrac{1}{2}f\exp\phi(Q^{(\pi)}\,.\,Q^{(\pi)}+Q^{(\nu)}\,.\,Q^{(\nu)})\,. \tag{100}$$

4. A Test Case

An exact s–d–g boson spectrum has been evaluated for the specific Hamiltonian

$$H = -\,Q^{(\pi)}\,.\,Q^{(\nu)}\,, \tag{101}$$

where $Q^{(x)}$ is the SU(3) model s–d–g boson quadrupole operator for which

$$q_1^{(x)} = 2(7/5)^{\frac{1}{2}} \approx 2\cdot 37\,,\qquad q_2^{(x)} = -11/(14)^{\frac{1}{2}} \approx -2\cdot 94\,,$$

$$q_3^{(x)} = 18/(35)^{\frac{1}{2}} \approx 3\cdot 04\,,\qquad q_4^{(x)} = -3(11/7)^{\frac{1}{2}} \approx -3\cdot 76\,. \tag{102}$$

The resulting excitation spectrum of a four boson (two proton–two neutron) system up to 30 MeV excitation is shown in the first (left-hand) column of Fig. 1.

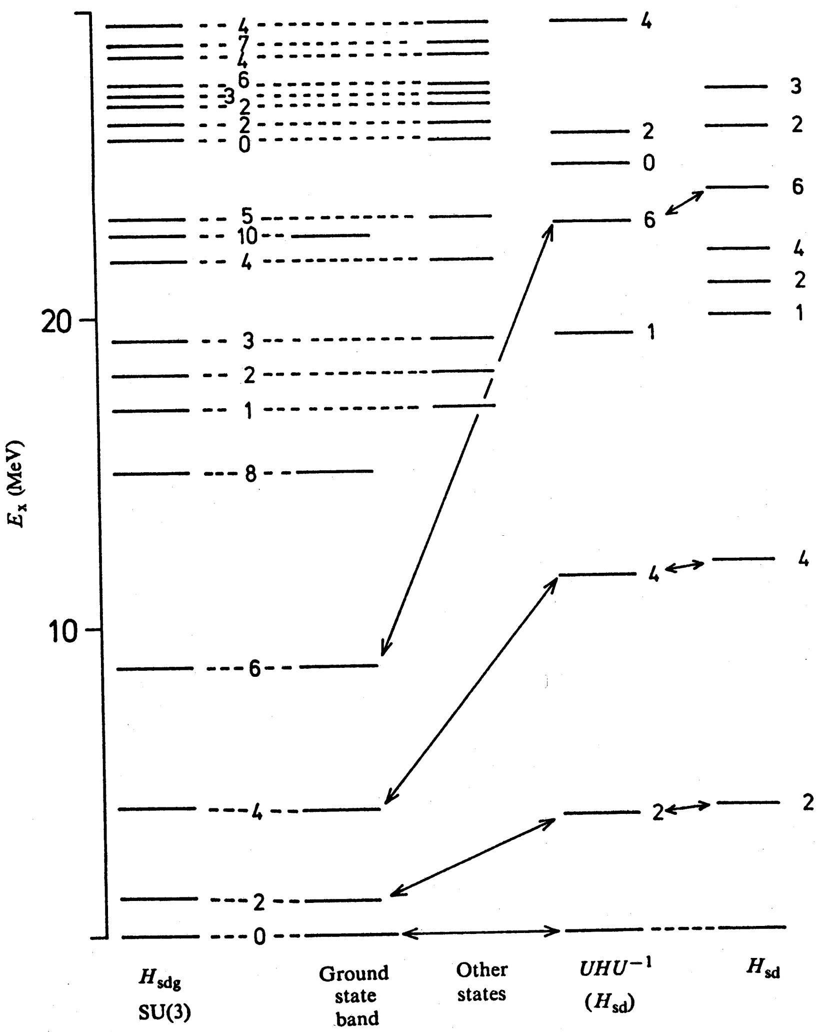

Fig. 1. Exact spectrum from the s–d–g test case Hamiltonian (column 1), its ground state (column 2) and the other low excitation states (column 3) compared with those obtained by using the simplest approximation to the transformed Hamiltonian and by simply omitting all g-bosons components in the exact calculation (columns 4 and 5 respectively).

For the ground state band, $\langle\overline{Q^2}\rangle$ and $\langle\overline{QP}\rangle$ are approximately 14 and 1·4 respectively and the overall binding energy is 143 MeV. The ground state band from this SU(3) model s–d–g Hamiltonian is displayed by itself in the second column of Fig. 1 with the other states shown in column 3. These other bands start with the $L = 1$ component of the band for which f-spin is approximately its maximum value minus one.

A second reference spectrum is shown in the rightmost column of Fig. 1 (and designated by the label H_{sd}). This was obtained by simply setting $q_3^{(\tau)}$ and $q_4^{(\tau)}$ to zero in the test case model Hamiltonian (101) and obtaining the associated exact spectrum. The binding energy in this case reduces to 108 MeV and clearly the moment of inertia is smaller than that associated with the s–d–g model results (to about $\frac{3}{8}$ of the s–d–g model value in fact). This is exactly the result obtained from analyses of the contributions of S, D and G fermion pairs in the intrinsic states of deformed nuclei (Bes *et al.* 1982; Yoshinaga *et al.* 1984; van Egmond and Allaart 1984).

Using the weak coupling conditions for eliminating d–g boson coupling in the ground state band (equation 83), we obtain the constraint upon the transformation angle as

$$-\tfrac{1}{2}+\alpha/b-\gamma\alpha/b^2+(\gamma\alpha\phi/2b^2)\sin\phi+(\alpha/b)(\gamma/b-1)\cos\phi = 0\,, \qquad (103)$$

and which is $4\cdot2-4\cdot7\cos\phi+\phi\sin\phi = 0$; a solution of which is $-22\cdot5°$. Thus, as $\phi_{(\tau)} = (14b_{(\tau)}/9)^{\frac{1}{2}}\theta_{(\tau)}$, we find that $\theta_{(\tau)} = -7\cdot6°$. This solution is almost independent of the exact value of γ/b (<1) and, as $\gamma/b \sim 0\cdot3$ for $\langle\overline{QP}\rangle \sim 0\cdot1\langle\overline{Q^2}\rangle$, the solution is independent therefore of the exact choice of $\langle\overline{QP}\rangle$. This 'stable' solution* then determines the transformed single boson energies to be

$$\hat{\epsilon}_{\rm d}^{(\tau)} \sim -4\cdot9\ \mathrm{MeV}; \qquad \hat{\epsilon}_{\rm g}^{(\tau)} \sim 2\cdot7\ \mathrm{MeV}, \qquad (104)$$

giving the desired splitting between single boson energies that enables us to neglect the g-boson term (to first order) in the transformed Hamiltonian, so that

$$UHU^{-1} \sim -4\cdot9\,\Sigma_\tau\, d^\dagger_{(\tau)}\cdot\tilde{d}_{(\tau)} - Q^{(\pi)}_{(\rm sd)}\cdot Q^{(\nu)}_{(\rm sd)}\,.$$

This is simply the s–d boson Hamiltonian obtained by setting q_3 and q_4 to zero in (101) to which is added a d-boson kinetic energy term. It is not surprising, therefore, that the transformed Hamiltonian spectrum is very similar to the pure s–d boson model spectrum, as is evident when the results are compared as in Fig. 1. There is a marked improvement in the absolute binding energy (to a value of 122 MeV) but the moment of inertia of the ground state band is essentially unchanged and very much smaller than that given by the s–d–g boson model calculations.

The most crucial factor in determining the splittings of the ground states in the s–d model calculations is the value of $q_1^{(\tau)}$ and, in the OG study, a sizeable reduction of these values is quoted to give a match to their 'exact' s–d–g boson spectrum. In so doing, however, the other bands are pushed even lower in the resulting spectrum. But, in view of the assumptions made with this transformation model, it is really only pertinent to discuss the ground state band.

Thus one must look more carefully at the OG study since both their and our (weak coupling model) calculations produce a splitting of the d- and g-boson energies. We do note, however, that our complete development of their approach (leading to equation 44) gives a splitting that depends upon the original single boson energies $\epsilon_{\rm d}^{(\tau)}$ and $\epsilon_{\rm g}^{(\tau)}$. With our test case (degenerate zero energies) no splitting then results. But it is the interaction that is primarily responsible for the state splittings (moment of inertia). OG selected only one leading term and used the approximate transformations of individual boson operators, whence they obtained an effective s–d boson $Q^{(\pi)}\cdot Q^{(\nu)}$ interaction with component coefficients $\tilde{q}_1^{(\tau)}$ scaling as $\cos^2$, from the original $q_1^{(\tau)}$. As

* An alternative solution for ϕ is $-265°$ when γ/b is $0\cdot1$, but it is unstable as it varies markedly with the exact choice of γ/b.

we shall see, this has a significant effect on the ground state band moment of inertia but there are very many comparable terms in the full expansion of the transformed interaction that have been omitted.

The extended model I has a resultant s–d boson Hamiltonian as given by (89) in which the quadrupole operator coefficients $q_i^{(\tau)}$ are varied from those values given in (102), with

$$q_1^{(\tau)} \rightarrow q_1^{(\tau)} \cos(\langle \overline{Q^{(\tau)2}} \rangle^{\frac{1}{2}} \theta_{(\tau)}\,.$$

For the test case data this gives $q_1^{(\tau)} \rightarrow 0{\cdot}88\, q_1^{(\tau)}$. The resulting spectrum from this approximation to the transformed s–d–g Hamiltonian is shown in Fig. 2 by the middle column, designated by $H_{sd}^{(1)}$. The exact spectrum, and the previously discussed first order perturbation approximation results, are given in the first and second columns respectively. Clearly this extended model I has slightly improved the ground state band moment of inertia, but not enough to give a satisfactory approximation to that exact spectrum. The lower f-spin band is reasonably placed but the binding energy is now just 106 MeV (compared with the exact result of 143 MeV).

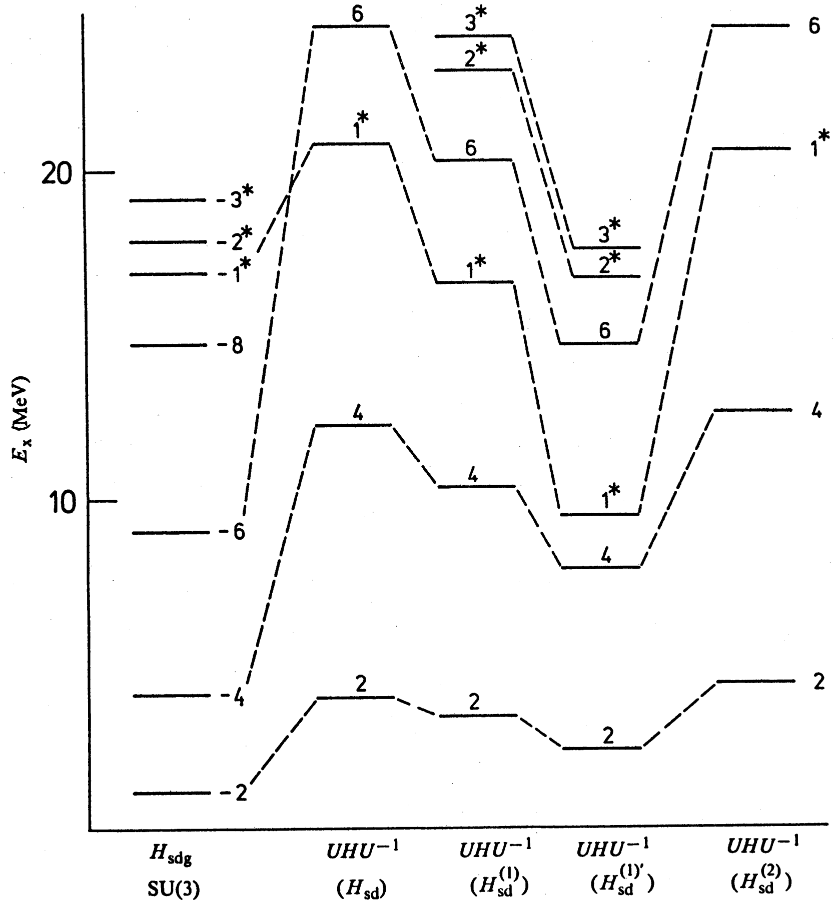

Fig. 2. Spectra of the various approximate forms of the transformed, test case s–d–g boson Hamiltonian compared with the exact results. Details of the entries are specified in the text.

It is obvious, however, that the s–d coupling parameter in the effective Hamiltonian is an important factor in the definition of the moment of inertia for the ground state band. The spectrum given in the column labelled $H_{\rm sd}^{(1)'}$ in Fig. 2 results when one uses in extended model I calculations $q_1^{(\tau)} \rightarrow 0\cdot 7 q_1^{(\tau)}$. This *ad hoc* adjustment obviously improves the ground state band comparison, but at the expense of a binding energy of just 85 MeV and an overdrastic depression of the lower f-spin band. Neither of these latter factors were considered in the OG study, whence a further reduction of $q_1^{(\tau)}$ would indeed give a match to the exact ground state band spectrum.

The extended model II offers new possibilities as the transformed Hamiltonian in this case takes the form

$$UHU^{-1} \rightarrow \Sigma_\tau \{\hat{\epsilon}_{\rm d}^{(\tau)} d_{(\tau)}^\dagger \cdot \tilde{d}_{(\tau)} - \tfrac{1}{2}\cosh(\phi_{(\tau)})(Q_{(\rm sd)}^{(\tau)} \cdot Q_{(\rm sd)}^{(\tau)}) + \tfrac{1}{2}\sinh(\phi_{(\tau)}) Q_{(\rm sd)}^{(\tau)} \cdot Q_{(\rm sd)}^{(\tau)}\}\,, \qquad (105)$$

in which, with $\hat{\epsilon}_{\rm d}^{(\tau)}$ again being $-4\cdot 9$ MeV, $\phi_{(\tau)} \approx -\frac{2}{5} q_3^{(\tau)} N_{\rm d}^{(\tau)} \theta_{(\tau)}$ depends upon the d-boson number. With $\theta_{(\tau)}$ being $-7\cdot 6°$ and with $N_{\rm d}^{(\tau)} = 2$,

$$\begin{aligned} UHU^{-1} &\rightarrow -4\cdot 9\,\Sigma_\tau\, d_{(\tau)}^\dagger \cdot \tilde{d}_{(\tau)} - 1\cdot 05\, Q_{(\rm sd)}^{(\pi)} \cdot Q_{(\rm sd)}^{(\nu)} \\ &\quad + 0\cdot 16(Q_{(\rm sd)}^{(\pi)} \cdot Q_{(\rm sd)}^{(\pi)} + Q_{(\rm sd)}^{(\nu)} \cdot Q_{(\rm sd)}^{(\nu)}) \\ &\sim H'_{(\rm sd)} + 0\cdot 16\,\Sigma_\tau (Q_{(\rm sd)}^{(\tau)})^2\,, \end{aligned} \qquad (106)$$

where $H'_{(\rm sd)}$ is essentially the Hamiltonian that gave the results listed in column 4 of Fig. 1.

The resulting spectrum is given as the last column, labelled $H_{\rm sd}^{(2)}$, in Fig. 2. The associated binding energy is 104 MeV whence this extended model II has actually made agreement worse. This is a somewhat disheartening result since the extended model II accounts for more terms in the infinite series resulting from the transformation than any of the other approximations. At least this is further convincing evidence that g-boson effects should not be treated by any perturbation scheme or other weak coupling approach, if the system under study shows rotational band characteristics.

Given the need to retain g-boson interactions, we then considered restricting the exact boson model calculations by limiting the number of g-bosons in all state prescriptions. The results are displayed in Fig. 3. From left to right the spectra are those of the unrestricted (four boson) calculation of the test s–d–g Hamiltonian, a calculation in which only one g-boson (of proton and/or neutron type) is permitted, and the s–d limit (no g-bosons) respectively. Binding energies from these calculations are, in order, 143, 138 and 108 MeV. Clearly, the one g-boson limit calculation gives the best approximation to the exact spectrum of all the approximation calculations reported here.

5. Conclusions

The (sdg) model of Otsuka and Ginocchio (1985) has been investigated in detail. The simple double commutator approximation used to close the renormalisation series has been extended to allow exact summation to all orders but still only within the $K = 0$ OG restriction. Clearly the diverse schemes by which we have been able to

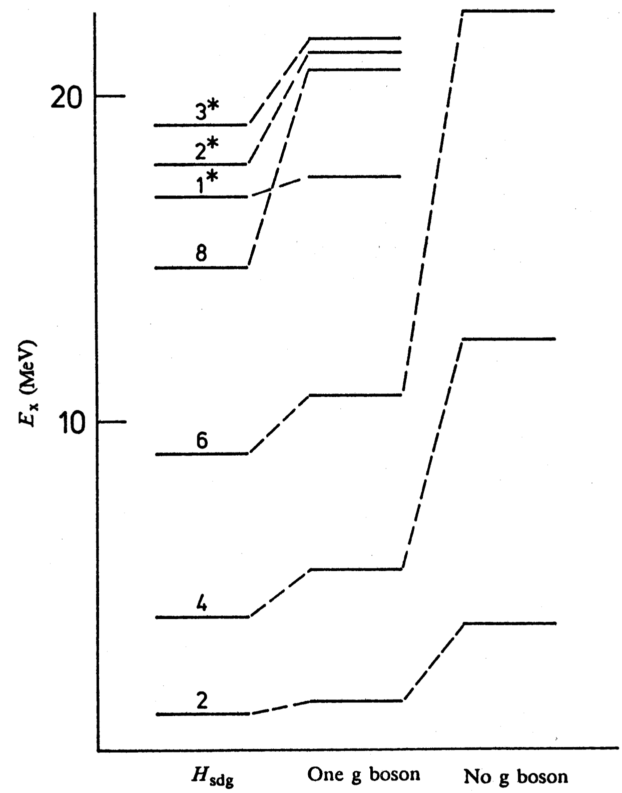

Fig. 3. Spectra from the many boson, shell model calculations of the test s–d–g boson Hamiltonian. Details are specified in the text.

sum (in simple form) selective parts of the transformed residual interaction all fail to include those contributions that are essential to map the exact spectrum.

We confirm the result that the effect of the renormalisation on the single particle energy spectrum is to lower $\hat{\epsilon}_d$ and raise $\hat{\epsilon}_g$. But a proper treatment of the two body interaction is much more difficult. We have shown that it is possible to select one or few terms in the series to approximate the correct moment of inertia in the ground state band, but then the positions of the excited bands are very doubtful. The renormalisation model provides a qualitative guide to gross structure of the s–d–g boson system, but as yet cannot be used as a quantitative tool.

In spite of earlier promise, a consistent, systematic and accurate summation (approximation) strategy is difficult to achieve. The explicit inclusion of a few g-bosons in any shell model calculation seems less fraught with uncertainties.

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Appendix A: Commutators of Boson Operators

The s-, d- and g-boson operators (for a given particle type that subscript is omitted) satisfy the commutator relations

$$[d_\mu, d_\nu^\dagger] = [g_\mu, g_\nu^\dagger] = \delta_{\mu\nu}, \tag{A1.1}$$

$$[\tilde{d}_\mu, d_\nu^\dagger] = [\tilde{g}_\mu, g_\nu^\dagger] = (-)^\mu \delta_{-\mu\nu}, \tag{A1.2}$$

$$[s, s^\dagger] = [\tilde{s}, s^\dagger] = 1, \tag{A1.3}$$

from which it is straightforward to ascertain that the mth component of the quadrupole operator (of the same particle type) satisfies the set

$$[Q_m, s^\dagger] = -q_1\, \tilde{d}_m, \tag{A2.1}$$

$$[Q_m, s^\dagger] = q_1\, d_m^\dagger, \tag{A2.2}$$

$$[Q_m, \tilde{d}_n] = (-)^{n+1}(q_1\, \tilde{s}\delta_{m,-n} + q_2\langle 2\,2-n\; m+n|2\; m\rangle \tilde{d}_{m+n} + q_3\langle 2\,4-n\; m+n|2\; m\rangle \tilde{g}_{m+n}), \tag{A2.3}$$

$$[q_m, d_n^\dagger] = (-)^n(q_1\, s^\dagger\delta_{-mn} + q_2\langle 2\,2-n\; m+n|2\; m\rangle d_{m+n}^\dagger + q_3\langle 2\,4-n\; m+n|2\; m\rangle g_{m+n}^\dagger), \tag{A2.4}$$

$$[Q_m, \tilde{g}_n] = (-)^{n+1}(q_3\langle 4\,2-n\; m+n|2\; m\rangle \tilde{d}_{m+n} + q_4\langle 4\,4-n\; m+n|2\; m\rangle \tilde{g}_{m+n}), \tag{A2.5}$$

$$[Q_m, g_n^\dagger] = (-)^n(q_3\langle 4\,2 - n\; m+n|2\; m\rangle d_{m+n}^\dagger$$
$$+ q_4\langle 4\,4 - n\; m+n|2\; m\rangle g_{m+n}^\dagger). \tag{A2.6}$$

Clearly Q_m never commute with the boson operators (of the same type) and thus they do not commute with any operator as if they are C-numbers which was assumed by Otsuka and Ginocchio (OG) (1985).

To evaluate the commutators of Z (given by equation 11), one also needs the product operator commutators

$$[(g^\dagger \tilde{d})^{(2)}_{-\mu}, d_\alpha^\dagger] = (-)^\mu\langle 4\,2\,\mu-\alpha\,\alpha|2\,\mu\rangle g_{\alpha-\mu}^\dagger, \tag{A3.1}$$

$$[(g^\dagger \tilde{d})^{(2)}_{-\mu}, \tilde{g}_\alpha] = (-)^{\mu+1}\langle 4\,2\,\alpha\,\mu-\alpha|2\,\mu\rangle \tilde{d}_{\alpha-\mu}, \tag{A3.2}$$

$$[(d^\dagger \tilde{g})^{(2)}_{-\mu}, \tilde{d}_\alpha] = (-)^{\alpha+1}\langle 2\,4\alpha\,\mu-\alpha|2\,\mu\rangle \tilde{g}_{\alpha-\mu}, \tag{A3.3}$$

$$[(d^\dagger \tilde{g})^{(2)}_{-\mu}, g_\alpha^\dagger] = (-)^{\alpha}\langle 2\,4\,\mu-\alpha\,\alpha|2\,\mu\rangle d_{\alpha-\mu}^\dagger. \tag{A3.4}$$

Consider the action of Z upon the individual neutrons boson operators (proton and neutron cases have symmetry):

$$[Z, Q_{(\nu)\alpha}] = \Sigma_\mu (-)^\mu\{\theta_{(\nu)}\, Q_{(\pi)\mu}[E_{(\nu)-\mu}\, Q_{(\nu)\alpha}]$$
$$+\theta_{(\pi)}[Q_{(\nu)\mu}\cdot Q_{(\nu)\alpha}]E_{(\nu)-\mu}\}, \tag{A4}$$

where

$$E_\beta = (g^\dagger \tilde{d})^{(2)}_\beta - (d^\dagger \tilde{g})^{(2)}_\beta, \tag{A5}$$

and by using (A1), (A2) and (A3) one obtains

$$[Z, \tilde{s}_{(\nu)}] = -\theta_{(\pi)} q_{1(\nu)}(\tilde{d}_{(\nu)}\cdot E_{(\pi)}), \tag{A6.1}$$

$$[Z, s_{(\nu)}^\dagger] = \theta_{(\pi)}\, q_{1(\nu)}(d_{(\nu)}^\dagger\cdot E_{(\pi)}), \tag{A6.2}$$

$$[Z, \tilde{d}_{(\nu)\alpha}] = \theta_{(\nu)}(Q_{(\pi)}\,\tilde{g}_{(\nu)})^{(2)}_\alpha - (\tilde{L}_{(\nu)} E_{(\pi)})^{(2)}_\alpha, \tag{A6.3}$$

$$[Z, d_{(\nu)\alpha}^\dagger] = \theta_{(\nu)}(Q_{(\pi)}\, g_{(\nu)}^\dagger)^{(2)}_\alpha + (L_{(\nu)}^\dagger\, E_{(\pi)})^{(2)}_\alpha, \tag{A6.4}$$

$$[Z, \tilde{g}_{(\nu)\alpha}] = -\sqrt{\tfrac{5}{9}}\,\theta_{(\nu)}(Q_{(\nu)}\,\tilde{d}_{(\pi)})^{(4)}_\alpha$$
$$-\sqrt{\tfrac{5}{9}}\,[(q_3\tilde{d}_{(\nu)} + q_4\,\tilde{g}_{(\nu)})E_{(\pi)}]^{(4)}_\alpha, \tag{A6.5}$$

$$[Z, g_{(\nu)\alpha}^\dagger] = -\sqrt{\tfrac{5}{9}}\,\theta_{(\nu)}(Q_{(\pi)}\, d_{(\nu)}^\dagger)^{(4)}_\alpha$$
$$+\sqrt{\tfrac{5}{9}}\,[(q_3 d_{(\nu)}^\dagger + q_4\, g_{(\nu)}^\dagger)E_{(\pi)}]^{(4)}_\alpha, \tag{A6.6}$$

where

$$L_{(\nu)}^\dagger = q_1\, s_{(\nu)}^\dagger + q_2\, d_{(\nu)}^\dagger + q_3\, g_{(\nu)}^\dagger, \tag{A7}$$

and similarly for $\tilde{L}_{(\nu)}$.

Appendix B: Derivation of the Double Commutators

Using the approximate commutators of Z with any boson operator that is specified in the text (equation 13), it is evident that

$$[Z,[Z,\tilde{s}]]=[Z,[Z,s^{\dagger}]]=0, \tag{B1}$$

and that for other (neutron) operators (x, y)

$$[Z,[Z,x_{(\nu)}]]\propto[Z,(Q^{(\pi)}y_{(\nu)})^{(L)}]. \tag{B2}$$

The d-boson Operator Commutators

We consider

$$X_a^{(2)}=[Z,[Z,\tilde{d}_{(\nu)a}]]=\theta_{(\nu)}[Z,(Q^{(\pi)}\tilde{g}_{(\nu)})_a^{(2)}], \tag{B3}$$

and, using the definition of Z (equations 11 and 12), we get

$$\begin{aligned} X_a^{(2)} &= \theta_{(\nu)}\{\theta_{(\nu)}[Q^{(\pi)}\,.\,E^{(\nu)},(Q^{(\pi)}\tilde{g}_{(\nu)})_a^{(2)}] \\ &\quad +\theta_{(\pi)}[Q^{(\nu)}\,.\,E^{(\pi)},(Q^{(\pi)}\tilde{g}_{(\nu)})_a^{(2)}]\} \\ &= \Sigma_{mm'}(-)^m\langle 2\,4\,m'\,a-m'|2\,a\rangle\{\theta_{(\nu)}^2[Q_m^{(\pi)}\,E_{-m}^{(\nu)},\,Q_{m'}^{(\pi)}\,\tilde{g}_{(\nu)a-m'}] \\ &\quad +\theta_{(\nu)}\,\theta_{(\pi)}[Q_m^{(\nu)}\,E_{-m}^{(\pi)},\,Q_{m'}^{(\pi)}\,\tilde{g}_{(\nu)a-m'}]\}\,. \end{aligned} \tag{B4}$$

Using the identity

$$[AB,\,CD]=A[B,\,C]D+AC[B,\,D]+[A,\,C]DB+C[A,\,D]B, \tag{B5}$$

we have

$$\begin{aligned} X_a^{(2)} &\equiv \Sigma_{mm''}(-)^m\langle 2\,4\,m''\,a-m''|1\,a\rangle \\ &\quad\times\{\theta_{(\nu)}^2\,Q_m^{(\pi)}[E_{-m}^{(\nu)},\,Q_{m''}^{(\pi)}]\tilde{g}_{(\nu)a-m''}+\theta_{(\nu)}^2\,Q_m^{(\pi)}\,Q_{m''}^{(\pi)}[E_{-m}^{(\nu)},\,\tilde{g}_{(\nu)a-m''}] \\ &\quad+\theta_{(\nu)}^2[Q_m^{(\pi)},\,Q_{m''}^{(\pi)}]\tilde{g}_{(\nu)a-m''}\,E_{-m}^{(\nu)}+\theta_{(\nu)}^2\,Q_{m''}^{(\pi)}[Q_m^{(\pi)},\,\tilde{g}_{(\nu)a-m''}]E_{-m}^{(\nu)} \\ &\quad+\theta_{(\nu)}\,\theta_{(\pi)}\,Q_m^{(\nu)}[E_{-m}^{(\pi)},\,Q_{m''}^{(\pi)}]\tilde{g}_{(\nu)a-m''}+\theta_{(\nu)}\,\theta_{(\pi)}\,Q_m^{(\nu)}\,Q_{m''}^{(\pi)}[E_{-m}^{(\pi)},\,\tilde{g}_{(\nu)a-m''}] \\ &\quad+\theta_{(\nu)}\,\theta_{(\pi)}[Q_m^{(\nu)},\,Q_{m''}^{(\pi)}]\tilde{g}_{(\nu)a-m''}\,E_{-m}^{(\pi)}+\theta_{(\nu)}\,\theta_{(\pi)}\,Q_{m''}^{(\pi)}[Q_m^{(\nu)},\,\tilde{g}_{(\nu)a-m''}]E_{-m}^{(\pi)}\}\,. \end{aligned} \tag{B6}$$

Of these eight components, the commutators of entries 1, 3, 4, 6 and 7 are identically zero, so that

$$\begin{aligned} X_a^{(2)} &= \Sigma_{mm''}(-)^m\langle 2\,4\,m''\,a-m''|2\,a\rangle\{\theta_{(\nu)}^2\,Q_m^{(\pi)}\,Q_{m''}^{(\pi)}[E_{-m}^{(\nu)},\,\tilde{g}_{(\nu)a-m''}] \\ &\quad+\theta_{(\nu)}\,\theta_{(\pi)}\,Q_m^{(\nu)}[E_{-m}^{(\pi)},\,Q_{m''}^{(\pi)}]\tilde{g}_{(\nu)a-m''}+\theta_{(\nu)}\,\theta_{(\pi)}\,Q_{m''}^{(\pi)}[Q_m^{(\nu)},\,\tilde{g}_{(\nu)a-m''}]E_{-m}^{(\pi)}\}\,, \end{aligned} \tag{B7}$$

the latter two terms of which, by our assumptions on the intrinsic state, are very much less in expectation than the first. Thus, with the approximation by which the single commutators of Z were determined, we have

$$X_{\alpha}^{(2)} = [Z, [Z, \tilde{d}_{(\nu)\alpha}]] = \theta_{(\nu)}^{2} \Sigma_{mm''} \langle 2\,4\, m''\, \alpha - m'' | 2\, \alpha \rangle \times Q_{m}^{(\pi)} Q_{m''}^{(\pi)} [E_{-m}^{(\nu)}, \tilde{g}_{(\nu)\alpha - m''}]. \tag{B8}$$

The definition of $E_{-m}^{(\nu)}$ and the relevant commutation relations then give the result

$$X_{\alpha}^{(2)} = [Z, [Z, \tilde{d}_{(\nu)\alpha}]] = \theta_{(\nu)}^{2} \Sigma_{mm'm''} \langle 2\,4\, m''\, \alpha - m'' | 2\, \alpha \rangle$$

$$\times (-)^{m} \langle 4\,2 - m'' - m\; m'' | 2 - m \rangle [g_{(\nu) - m - m''}^{\dagger}\, \tilde{d}_{(\nu) m''}, \tilde{g}_{(\nu)\alpha - m''}] Q_{m}^{(\pi)} Q_{m''}^{(\pi)},$$

that contracts to

$$X_{\alpha}^{(2)} = -\theta_{(\nu)}^{2} \Sigma_{mm'} (-)^{m''} \langle 2\,4\, \alpha - m - m''\; m'' + m | 2\, \alpha \rangle \times \langle 4\,2 - m' - m\; m' | 2 - m \rangle Q_{m}^{(\pi)} Q_{\alpha - m - m''}^{(\pi)} \tilde{d}_{(\nu) m''}, \tag{B9}$$

which, using standard angular momentum algebra, is also given by

$$[Z, [Z, \tilde{d}_{(\nu)\alpha}]] = -\theta_{(\nu)}^{2} \Sigma_{K} [5(2K+1)]^{\frac{1}{2}} \times W(24K2;\, 22) [(Q^{(\pi)} Q^{(\pi)})^{(K)} \tilde{d}_{(\nu)}]_{\alpha}^{(2)}. \tag{B10}$$

A similar reduction for $d_{(\nu)\alpha}^{\dagger}$ gives

$$[Z, [Z, d_{(\nu)\alpha}^{\dagger}]] = -\theta_{(\nu)}^{2} \Sigma_{mm'} (-)^{m'} \langle 2\,4\, \alpha - m'\; m' + m | 2\, \alpha \rangle \tag{B11}$$

$$\times \langle 2\,4 - m'\; m + m' | 2\, m \rangle Q_{m}^{(\pi)} Q_{\alpha - m' - m}^{(\pi)} d_{(\nu) m'}^{\dagger}$$

$$= -\theta_{(\nu)}^{2} \Sigma_{K} [5(2K+1)]^{\frac{1}{2}} W(24K2;\, 22) [(Q^{(\pi)} Q^{(\pi)})^{(K)} d_{(\nu)}^{\dagger}]_{\alpha}^{(2)}.$$

Symmetry gives the results for the double commutators of the d-boson proton operators.

The g-boson Operator Commutators

We consider

$$X_{\alpha}^{(4)} = [Z, [Z, \tilde{g}_{(\nu)\alpha}]] = -\theta_{(\nu)} \surd \tfrac{5}{9} [Z, (Q^{(\pi)} \tilde{d}_{(\nu)})_{\alpha}^{(4)}]. \tag{B12}$$

Identical arguments to those given in the d-boson commutator reductions give

$$X_{\alpha}^{(4)} = -\surd \tfrac{5}{9}\, \theta_{(\nu)}^{2} \Sigma_{mm'} (-)^{m'} \langle 4\,2\, m' - m - m' | 2 - m \rangle \times \langle 2\,2\, \alpha - m - m\; m + m' | 4\, \alpha \rangle Q_{m}^{(\pi)} Q_{\alpha - m - m'}^{(\pi)} \tilde{g}_{(\nu) m'}, \tag{B13}$$

which reduces further to

$$X_a^{(4)} = -\sqrt{\tfrac{5}{9}}\,\theta_{(\nu)}^2\,\Sigma_K[5(2K+1)]^{\frac{1}{2}}\,W(42K2;\,24)(Q^{(\pi)}Q^{(\pi)})^{(K)}(\tilde{g}_{(\nu)})_a^{(4)}, \quad \text{(B14)}$$

and with similar results for $g_{(\nu)a}^{\dagger}$ and the proton g-boson.

The $K = 0$ Otsuka–Ginocchio Limits

The $K = 0$ limit approximation greatly simplifies the foregoing results with

$$\begin{aligned} X_a^{(2)} &\sim -\theta_{(\nu)}^2\sqrt{5}\,W(2222;\,02)(Q^{(\pi)}Q^{(\pi)})^{(0)}\tilde{d}_{(\nu)a} \\ &= -\theta_{(\nu)}^2\sqrt{\tfrac{1}{5}}\,(Q^{(\pi)}Q^{(\pi)})^{(0)}\tilde{d}_{(\nu)a}, \end{aligned} \quad \text{(B15)}$$

and, by averaging over the proton space,

$$\begin{aligned} \langle X_a^{(2)}\rangle &= -\theta_{(\nu)}^2\,\tfrac{1}{5}\,\Sigma_m\langle(-)^m Q_m^{(\pi)}\,Q_{-m}^{(\pi)}\rangle\tilde{d}_{(\nu)a} \\ &\sim -\theta_{(\nu)}^2(\overline{Q^{(\pi)2}})\tilde{d}_{(\nu)a}, \end{aligned} \quad \text{(B16)}$$

where, assuming a zero projection scheme,

$$(\overline{Q^{(\pi)2}}) = \langle Q_0^{(\pi)}\,Q_0^{(\pi)}\rangle. \quad \text{(B17)}$$

Likewise, we have

$$\begin{aligned} X_a^{(4)} &\sim -\sqrt{\tfrac{5}{9}}\,\theta_{(\nu)}^2\sqrt{5}\,W(4422;\,02)(Q^{(\pi)}Q^{(\pi)})^{(0)}\tilde{g}_{(\nu)a} \\ &= -\sqrt{\tfrac{5}{9}}\,\theta_{(\nu)}^2\sqrt{\tfrac{5}{9}}\,\Sigma_m(-)^m\,Q_m^{(\pi)}Q_{-m}^{(\pi)}\,\tilde{g}_{(\nu)a}, \end{aligned} \quad \text{(B18)}$$

so that, under the same approximations as above, we have

$$\langle X_a^{(4)}\rangle = -\tfrac{5}{9}\theta_{(\nu)}^2\,(\overline{Q^{(\pi)2}})\tilde{g}_{(\nu)a}. \quad \text{(B19)}$$

Zero Projection Model

With the zero projection assumption, the $K \neq 0$ terms in the expressions for $\langle X_a^{(2)}\rangle$ may be summed or equivalently, using (B9),

$$\begin{aligned} \langle X_a^{(2)}\rangle &\sim -\theta_{(\nu)}^2\,\Sigma_m\langle(-)^m Q_m^{(\pi)}\,Q_{-m}^{(\pi)}\rangle\tilde{d}_{(\nu)a}(\tfrac{5}{9}\langle 2\,2\,m\,a|4\,m+a\rangle^2) \\ &= -\theta_{(\nu)}^2(\overline{Q^{(\pi)2}})\tilde{d}_{(\nu)a}, \end{aligned} \quad \text{(B20)}$$

if

$$\langle(-)^m Q_m^{(\pi)}\,Q_{-m}^{(\pi)}\rangle = \tfrac{1}{5}\langle Q_0^{(\pi)}\,Q_0^{(\pi)}\rangle \qquad \text{for all } m. \quad \text{(B21)}$$

Likewise from (B13) we have

$$\begin{aligned} \langle X_a^{(4)}\rangle &\sim -\theta_{(\nu)}^2\sqrt{\tfrac{5}{9}}\,\Sigma_m\langle(-)^m Q_m^{(\pi)}\,Q_{-m}^{(\pi)}\rangle\tilde{g}_{(\nu)a}(\sqrt{\tfrac{5}{9}}\,\langle 2\,2\,m\,a-m|4\,a\rangle^2) \\ &= -\tfrac{5}{9}\,\theta_{(\nu)}^2(\overline{Q^{(\pi)2}})\tilde{g}_{(\nu)a}, \end{aligned} \quad \text{(B22)}$$

under the same approximation (equation B21).

Appendix C: Transformed Boson Operators

For any single boson operator x, its transformed version is

$$x' = UxU^{-1}$$
$$= x+[Z,x]+\frac{1}{2!}[Z,[Z,x]]+\frac{1}{3!}[Z,[Z,[Z,x]]]\ldots, \tag{C1}$$

and, given that

$$[Z,[Z,x]] = -\gamma^2 x, \tag{C2}$$

then

$$x' = x-\frac{1}{2!}\gamma^2 x+\frac{1}{4!}(-\gamma^2)^2 x+\ldots$$
$$+[Z,x]-\frac{1}{3!}\gamma^2[Z,x]+\frac{1}{5!}(-\gamma^2)^2[Z,x]+\ldots$$
$$= x\cos\gamma+[Z,x]\frac{1}{\gamma}\sin\gamma. \tag{C3}$$

For the d-boson operator $d^\dagger_{(\nu)}$, our development has given (equations 11 and 25)

$$[Z, d^\dagger_{(\nu)}] = \theta_{(\nu)}(Q^{(\pi)}g^\dagger_{(\nu)})^{(2)},$$
$$[Z,[Z,d^\dagger_{(\nu)}]] = -(\theta^2_{(\nu)}\,\overline{Q^{(\pi)2}})d^\dagger_{(\nu)},$$

so that with γ being $(\overline{Q^{(\pi)2}})^{\frac{1}{2}}\theta_{(\nu)}$,

$$(d^\dagger_{(\nu)a})' = d^\dagger_{(\nu)a}\cos[(\overline{Q^{(\pi)2}})^{\frac{1}{2}}\theta_{(\nu)}]$$
$$+(\overline{Q^{(\pi)2}})^{-\frac{1}{2}}\sin[(\overline{Q^{(\pi)2}})^{\frac{1}{2}}\theta_{(\nu)}](Q^{(\pi)}g^\dagger_{(\nu)})^{(2)}_a. \tag{C4}$$

Similarly the g-boson commutators are

$$[Z, g^\dagger_{(\nu)}] = -\sqrt{\tfrac{5}{9}}\,\theta_{(\nu)}(Q^{(\pi)}d^\dagger_{(\nu)})^{(4)},$$
$$[Z,[Z,g^\dagger_{(\nu)}]] = -\tfrac{5}{9}\theta^2_{(\nu)}(\overline{Q^{(\pi)2}})g^\dagger_{(\nu)},$$

with

$$\gamma = \sqrt{\tfrac{5}{9}}\,\theta_{(\nu)}(\overline{Q^{(\pi)2}})^{\frac{1}{2}},$$

$$(g^\dagger_{(\nu)a})' = g^\dagger_{(\nu)a}\cos[\sqrt{\tfrac{5}{9}}\,\theta_{(\nu)}(\overline{Q^{(\pi)2}})^{\frac{1}{2}}]$$
$$-(\overline{Q^{(\pi)2}})^{-\frac{1}{2}}\sin[\sqrt{\tfrac{5}{9}}\,(\overline{Q^{(\pi)2}})^{\frac{1}{2}}\theta_{(\nu)}](Q^{(\pi)}d^\dagger_{(\nu)})^{(4)}_a. \tag{C5}$$

Appendix D: Angular Momentum Recoupling Terms

The d-boson number operator in the transformed Hamiltonian contains two terms that can be recoupled as

$$\begin{aligned}\text{Term } 1 &= \Sigma_{m\epsilon\epsilon'}\langle 2\,4\,\epsilon\; m-\epsilon|2\; m\rangle(-)^m\langle 2\,4\,-\epsilon'\,-m+\epsilon'|2\,-m\rangle \\ &\quad\times Q^{(\pi)}_{\epsilon}\,Q^{(\pi)}_{-\epsilon}\,g^{\dagger}_{(\nu)m-\epsilon}\,\tilde{g}_{(\nu)-m+\epsilon'} \\ &= \Sigma_{m\epsilon\epsilon' KK'}(-)^m\langle 2\,4\,\epsilon\; m-\epsilon|2\; m\rangle\langle 2\,4\,-\epsilon'\,-m+\epsilon'|2\,-m\rangle \\ &\quad\times\langle 2\,2\,\epsilon-\epsilon'|K\,\epsilon-\epsilon'\rangle\langle 4\,4\; m-\epsilon\,\epsilon'-m|K'\,\epsilon'-\epsilon\rangle \\ &\quad\times(Q^{(\pi)}Q^{(\pi)})^{(K)}_{\epsilon-\epsilon'}(g^{\dagger}_{(\nu)}\,\tilde{g}_{(\nu)})^{(K')}_{\epsilon'-\epsilon} \\ &= \Sigma_K\,5\,W(2424;\,2K)(Q^{(\pi)}Q^{(\pi)})^{(K)}\cdot(g^{\dagger}_{(\nu)}\,\tilde{g}_{(\nu)})^{(K)}, \end{aligned} \tag{D1}$$

$$\begin{aligned}\text{Term } 2 &= d^{\dagger}_{(\nu)}\cdot(Q^{(\pi)}\tilde{g}_{(\nu)})^{(2)} \\ &= \Sigma_{m\epsilon}(-)^m\langle 2\,4\,\epsilon\; m-\epsilon|2\; m\rangle d^{\dagger}_{(\nu)-m}\,Q^{(\pi)}_{\epsilon}\,\tilde{g}_{(\nu)m-\epsilon} \\ &= Q^{(\pi)}\cdot(d^{\dagger}_{(\nu)}\,\tilde{g}_{(\nu)})^{(2)}. \end{aligned} \tag{D2}$$

The g-boson number operator contains two similar terms which upon recoupling are

$$\begin{aligned}\text{Term } 1 &= \Sigma_{m\epsilon\epsilon'}(-)^m\langle 2\,2\,\epsilon\; m-\epsilon|4\; m\rangle\langle 2\,2\,-\epsilon'\,\epsilon'-m|4\,-m\rangle \\ &\quad\times\langle 2\,2\,\epsilon\,-\epsilon'|K\,\epsilon-\epsilon'\rangle\langle 2\,2\; m-\epsilon\,\epsilon'-m|K'\,\epsilon'-\epsilon\rangle \\ &\quad\times(Q^{(\pi)}Q^{(\pi)})^{(K)}_{\epsilon-\epsilon'}(d^{\dagger}_{(\nu)}\,\tilde{d}_{(\nu)})^{(K')}_{\epsilon'-\epsilon} \\ &= 9\,W(2222;\,4K)(Q^{(\pi)}Q^{(\pi)})^{(K)}\cdot(d^{\dagger}_{(\nu)}\,\tilde{d}_{(\nu)})^{(K)}, \end{aligned} \tag{D3}$$

$$\text{Term } 2 = (Q^{(\pi)}d^{\dagger}_{(\nu)})^{(4)}\cdot\tilde{g}_{(\nu)} = Q^{(\pi)}\cdot(d^{\dagger}_{(\nu)}\,\tilde{g}_{(\nu)})^{(2)}. \tag{D4}$$

Appendix E: Transformation of the Operators $Q^{(\pi)}_m$

Using the single boson operator transformation relations with

$$\eta' = \surd\tfrac{5}{9}\,\eta = \surd\tfrac{5}{9}\,[\theta_{(\pi)}(\overline{Q^{(\pi)2}})^{\frac{1}{2}}],$$

and

$$U d^{\dagger}_{\alpha}\,U^{-1} = \cos\eta\; d^{\dagger}_{\alpha} + \sin\eta\; D^{\dagger}_{\alpha}, \qquad U g^{\dagger}_{\alpha}\,U^{-1} = \cos\eta'\; g^{\dagger}_{\alpha} - \sin\eta'\; G^{\dagger}_{\alpha},$$

$$D^{\dagger}_{\alpha} = (\overline{Q^{(\nu)2}})^{-\frac{1}{2}}(Q^{(\nu)}g^{\dagger}_{(\pi)})^{(2)}_{\alpha}, \qquad G^{\dagger}_{\alpha} = (\overline{Q^{(\nu)2}})^{-\frac{1}{2}}(Q^{(\nu)}d^{\dagger}_{(\pi)})^{(4)}_{\alpha}, \tag{E1}$$

we obtain from the definition of $Q_m^{(\pi)}$

$$
\begin{aligned}
U Q_m^{(\pi)} U^{-1} = {} & q_1^{(\pi)} \cos\eta[(d_{(\pi)}^\dagger s_{(\pi)})_m^{(2)} + (s_{(\pi)}^\dagger \tilde{d}_{(\pi)})_m^{(2)}] \\
& + q_1^{(\pi)} \sin\eta[(D_{(\pi)}^\dagger s_{(\pi)})_m^{(2)} + (s_{(\pi)}^\dagger \tilde{D}_{(\pi)})_m^{(2)}] \\
& + q_2^{(\pi)} \cos^2\eta (d_{(\pi)}^\dagger \tilde{d}_{(\pi)})_m^{(2)} + q_2^{(\pi)} \sin^2\eta (D_{(\pi)}^\dagger \tilde{D}_{(\pi)})_m^{(2)} \\
& + q_2^{(\pi)} \sin\eta\cos\eta[(d_{(\pi)}^\dagger \tilde{D}_{(\pi)})_m^{(2)} + (D_{(\pi)}^\dagger \tilde{d}_{(\pi)})_m^{(2)}] \\
& + q_3^{(\pi)} \cos\eta\cos\eta'[(g_{(\pi)}^\dagger \tilde{d}_{(\pi)})_m^{(2)} + (d_{(\pi)}^\dagger \tilde{g}_{(\pi)})_m^{(2)}] \\
& - q_3^{(\pi)} \sin\eta\sin\eta'[(G_{(\pi)}^\dagger \tilde{D}_{(\pi)})_m^{(2)} + (D_{(\pi)}^\dagger \tilde{G}_{(\pi)})_m^{(2)}] \\
& - q_3^{(\pi)} \cos\eta\sin\eta'[(G_{(\pi)}^\dagger \tilde{d}_{(\pi)})_m^{(2)} + (d_{(\pi)}^\dagger \tilde{G}_{(\pi)})_m^{(2)}] \\
& + q_3^{(\pi)} \sin\eta\cos\eta'[(g_{(\pi)}^\dagger \tilde{D}_{(\pi)})_m^{(2)} + (D_{(\pi)}^\dagger \tilde{g}_{(\pi)})_m^{(2)}] \\
& + q_4^{(\pi)} \cos^2\eta' (g_{(\pi)}^\dagger \tilde{g}_{(\pi)})_m^{(2)} + q_4^{(\pi)} \sin^2\eta' (G_{(\pi)}^\dagger \tilde{G}_{(\pi)})_m^{(2)} \\
& - q_4^{(\pi)} \cos\eta' \sin\eta'[(g_{(\pi)}^\dagger \tilde{G}_{(\pi)})_m^{(2)} + (G_{(\pi)}^\dagger \tilde{g}_{(\pi)})_m^{(2)}]\,. \qquad \text{(E2)}
\end{aligned}
$$

All terms that involve only *one* capital designated operator will be ignored, since we have an interaction of the form $V = -f\, Q^{(\pi)} \,.\, Q^{(\nu)}$, whence the approximation

$$\langle (-)^\gamma Q_\gamma^{(x)} Q_\epsilon^{(y)} \rangle = (\overline{Q^{(x)2}})\delta_{xy}\delta_{\epsilon-\gamma} \qquad \text{all } \gamma \qquad \text{(E3)}$$

to be used in the development will ensure the single operator (D or G) terms vanish.

Thus, with

$$
\begin{aligned}
\hat{Q}_{(\mathrm{sd})m}^{(\pi)} \equiv {} & q_1^{(\pi)} \cos\eta[(d_{(\pi)}^\dagger s_{(\pi)})_m^{(2)} + (s_{(\pi)}^\dagger \tilde{d}_{(\pi)})_m^{(2)}] \\
& + q_2^{(\pi)} \cos^2\eta (d_{(\pi)}^\dagger \tilde{d}_{(\pi)})_m^{(2)}\,, \qquad \text{(E4)}
\end{aligned}
$$

we have

$$
\begin{aligned}
U Q_m^{(\pi)} U^{-1} \sim {} & Q_{(\mathrm{sd})m}^{(\pi)} + Q_{(\mathrm{sD})m}^{(\pi)} \\
& + q_3^{(\pi)} \cos\eta\cos\eta'[(g_{(\pi)}^\dagger \tilde{d}_{(\pi)})_m^{(2)} + (d_{(\pi)}^\dagger \tilde{g}_{(\pi)})_m^{(2)}] \\
& - q_3^{(\pi)} \sin\eta\sin\eta'[(G_{(\pi)}^\dagger \tilde{D}_{(\pi)})_m^{(2)} + (D_{(\pi)}^\dagger \tilde{G}_{(\pi)})_m^{(2)}] \\
& + q_4^{(\pi)} \sin^2\eta' (G_{(\pi)}^\dagger \tilde{G}_{(\pi)})_m^{(2)} + q_4^{(\pi)} \cos^2\eta' (g_{(\pi)}^\dagger \tilde{g}_{(\pi)})_m^{(2)}\,, \qquad \text{(E5)}
\end{aligned}
$$

in which the only term of $Q_{(\mathrm{sD})m}^{(\pi)}$ to be retained initially is

$$Q_{(\mathrm{sD})m}^{(\pi)} \sim q_2^{(\pi)} \sin^2\eta (D_{(\pi)}^\dagger \tilde{D}_{(\pi)})_m^{(2)}\,. \qquad \text{(E6)}$$

The $(D^\dagger \tilde{D})^{(2)}$ *Terms*

These are terms of the form

$$F = q_2^{(\pi)} \sin^2 \eta (\overline{Q^{(\nu)2}})^{-1} [(Q^{(\nu)} g^\dagger_{(\pi)})^{(2)} (Q^{(\nu)} \tilde{g}_{(\pi)})^{(2)}]^{(2)}_m ,$$

which recouple to

$$F = q_2^{(\pi)} \sin^2 \eta (\overline{Q^{(\nu)2}})^{-1} \Sigma_{KK'\alpha\beta} (-)^{\alpha+\beta} (-)^{K'}$$

$$\times [5(2K'+1)]^{\frac{1}{2}} \langle 2\, K'\; m\; -\alpha-\beta \,|\, K\; m-\alpha-\beta \rangle$$

$$\times \begin{bmatrix} 2 & 2 & K' \\ 2 & 2 & 2 \\ 4 & 4 & K \end{bmatrix} (Q^{(\nu)} Q^{(\nu)})^{(K')}_{\alpha+\beta} (g^\dagger_{(\pi)}\, \tilde{g}_{(\pi)})^{(K)}_{m-\alpha-\beta} . \qquad \text{(E7)}$$

The $(G^\dagger \tilde{G})^{(2)}$ *Terms*

This term has the form

$$\text{Term} = q_4^{(\pi)} \sin^2 \eta' [(Q^{(\nu)} d^\dagger_{(\pi)})^{(4)} (Q^{(\nu)} \tilde{d}_{(\pi)})^{(4)}]^{(2)}_m$$

$$= q_4^{(\pi)} \sin^2 \eta' (\overline{Q^{(\nu)2}})^{-1} \Sigma_{\epsilon\alpha\beta KK'} \langle 4\, 4\, \epsilon\; m-\epsilon \,|\, 2\; m \rangle$$

$$\times \langle 2\, 2\, \alpha\; \epsilon-\alpha \,|\, 4\, \epsilon \rangle \langle 2\, 2\, \beta\; m-\epsilon-\beta \,|\, 4\; m-\epsilon \rangle \langle 2\, 2\, \epsilon-\alpha\; m-\epsilon-\beta \,|\, K\; m-\alpha-\beta \rangle$$

$$\times \langle 2\, 2\, \alpha\, \beta \,|\, K'\; \alpha+\beta \rangle (Q^{(\nu)} Q^{(\nu)})^{(K')}_{\alpha+\beta} (d^\dagger_{(\pi)}\, \tilde{d}_{(\pi)})^{(K)}_{m-\alpha-\beta} ,$$

which recouples to be

$$\text{Term} = q_4^{(\pi)} \sin^2 \eta' (\overline{Q^{(\nu)2}})^{-1} \Sigma_{KK'\alpha\beta} (-)^{\alpha+\beta} (-)^{K'}$$

$$\times [5(2K'+1)]^{\frac{1}{2}} \langle 2\, K'\; m\; -\alpha-\beta \,|\, K\; m-\alpha-\beta \rangle$$

$$\times \begin{bmatrix} 2 & 2 & K' \\ 4 & 4 & 2 \\ 2 & 2 & K \end{bmatrix} (Q^{(\nu)} Q^{(\nu)})^{(K')}_{\alpha+\beta} (d^\dagger_{(\pi)}\, \tilde{d}_{(\pi)})^{(K)}_{m-\alpha-\beta} . \qquad \text{(E8)}$$

The $(G^\dagger \tilde{D})^{(2)}$ *and* $(D^\dagger \tilde{G})^{(2)}$ *Terms*

These involve components

$$(G^\dagger_{(\pi)}\, \tilde{D}_{(\pi)})^{(2)}_m = (\overline{Q^{(\nu)2}})^{-1} \Sigma_{\alpha\beta\epsilon} \langle 2\, 4\, \epsilon\; m-\epsilon \,|\, 2\; m \rangle$$

$$\times \langle 2\, 4\, \alpha\; \epsilon-\alpha \,|\, 2\, \epsilon \rangle \langle 2\, 2\, \beta\; m-\epsilon-\beta \,|\, 4\; m-\epsilon \rangle$$

$$\times Q^{(\nu)}_\alpha\, Q^{(\nu)}_\beta\, g^\dagger_{(\pi)\epsilon-\alpha}\, \tilde{d}_{(\pi)m-\epsilon-\beta} ,$$

which recouple to

$$(G^{\dagger}_{(\pi)}\,\tilde{D}_{(\pi)})^{(2)}_{m} = (\overline{Q^{(\nu)2}})^{-1}15\,\Sigma_{KK'\alpha\beta}\,(-)^{\alpha+\beta}(-)^{K'}(2K'+1)^{\frac{1}{2}}$$

$$\times\langle 2\,K'\;m-\alpha-\beta\,|\,K\;m-\alpha-\beta\rangle\begin{bmatrix}2&2&K'\\2&4&2\\4&2&K\end{bmatrix}$$

$$\times\,(Q^{(\nu)}\,Q^{(\nu)})^{(K')}_{\alpha+\beta}(d^{\dagger}_{(\pi)}\,\tilde{g}_{(\pi)})^{(K)}_{m-\alpha-\beta}\,. \qquad \text{(E9)}$$

Likewise, recoupling gives

$$(D^{\dagger}_{(\pi)}\,\tilde{G}_{(\pi)})^{(2)}_{m} = (\overline{Q^{(\nu)2}})^{-1}15\,\Sigma_{KK'\alpha\beta}\,(-)^{\alpha+\beta}(-)^{K'}(2K+1)^{\frac{1}{2}}$$

$$\times\langle 2\,K'\;m-\alpha-\beta\,|\,K\;m-\alpha-\beta\rangle\begin{bmatrix}2&2&K'\\4&2&2\\2&4&K\end{bmatrix}$$

$$\times\,(Q^{(\nu)}\,Q^{(\nu)})^{(K')}_{\alpha+\beta}(g^{\dagger}_{(\pi)}\,\tilde{d}_{(\pi)})^{(K)}_{m-\alpha-\beta}\,. \qquad \text{(E10)}$$

Thus, by using the results specified by (E7), (E8), (E9) and (E10) in the expansion of $U\,Q^{(\pi)}_{m}\,U^{-1}$ (as given in E5), we obtain the transformed components of the quadrupole operator:

$$U\,Q^{(\pi)}_{m}\,U^{-1} = Q^{(\pi)}_{(\mathrm{sd})\,m} + q^{(\pi)}_{4}\cos^{2}\eta'(g^{\dagger}_{(\pi)}\,g_{(\pi)})^{(2)}_{m}$$

$$+\,q^{(\pi)}_{2}\sin^{2}\eta(\overline{Q^{(\nu)2}})^{-1}\,\Sigma_{KK'\alpha\beta}\,(-)^{\alpha+\beta}(-)^{(K')}5(2K'+1)^{\frac{1}{2}}$$

$$\times\langle 2\,K'\;m-\alpha-\beta\,|\,K\;m-\alpha-\beta\rangle\begin{bmatrix}2&2&K'\\2&2&2\\4&4&K\end{bmatrix}(Q^{(\nu)}\,Q^{(\nu)})^{(K')}_{\alpha+\beta}(g^{\dagger}_{(\pi)}\,g_{(\pi)})^{(K)}_{m-\alpha-\beta}$$

$$+\,q^{(\pi)}_{4}\sin^{2}\eta'(\overline{Q^{(\nu)2}})^{-1}\,\Sigma_{KK'\alpha\beta}\,(-)^{\alpha+\beta}(-)^{K'}[5(2K'+1)]^{\frac{1}{2}}$$

$$\times\langle 2\,K'\;m-\alpha-\beta\,|\,K\;m-\alpha-\beta\rangle\begin{bmatrix}2&2&K'\\4&4&2\\2&2&K\end{bmatrix}(Q^{(\nu)}\,Q^{(\nu)})^{(K')}_{\alpha+\beta}(d^{\dagger}_{(\pi)}\,\tilde{d}_{(\pi)})^{(K)}_{m-\alpha-\beta}$$

$$+\,q^{(\pi)}_{3}\cos\eta\cos\eta'[(g^{\dagger}_{(\pi)}\,\tilde{d}_{(\pi)})^{(2)}_{m}+(d^{\dagger}_{(\pi)}\,\tilde{g}_{(\pi)})^{(2)}_{m}]$$

$$-\,q^{(\pi)}_{3}\sin\eta\sin\eta'(\overline{Q^{(\nu)2}})^{-1}\,\Sigma_{KK'\alpha\beta}\,(-)^{\alpha+\beta}(-)^{K'}14(2K'+1)^{\frac{1}{2}}$$

$$\times \langle 2\, K'\, m - \alpha - \beta | K\, m - \alpha - \beta \rangle \begin{bmatrix} 2 & 2 & K' \\ 2 & 4 & 2 \\ 4 & 2 & K \end{bmatrix}$$

$$\times (Q^{(\nu)} Q^{(\nu)})^{(K')}_{\alpha+\beta} [(g^{\dagger}_{(\pi)}\, \tilde{d}_{(\pi)})^{(K)}_{m-\alpha-\beta} + (d^{\dagger}_{(\pi)}\, \tilde{g}_{(\pi)})^{(K)}_{m-\alpha-\beta}], \tag{E11}$$

in which $Q^{(\pi)}_{(\mathrm{sd})}$ is the s–d boson quadrupole operator.

In the OG limit we get

$$\begin{aligned} U Q^{(\pi)}_m U^{-1} \sim\ & Q^{(\pi)}_{(\mathrm{sd})m} + q_4^{(\pi)} \cos^2 \eta' (g^{\dagger}_{(\pi)}\, \tilde{g}_{(\pi)})^{(2)}_m \\ & + q_2^{(\pi)} \sin^2 \eta (g^{\dagger}_{(\pi)}\, \tilde{g}_{(\pi)})^{(2)}_m \surd \tfrac{1}{5}\, W(2244;\, 22) \\ & + q_4^{(\pi)} \sin^2 \eta' (d^{\dagger}_{(\pi)}\, \tilde{d}_{(\pi)})^{(2)}_m\, \tfrac{1}{5}\, W(2244;\, 22) \\ & + q_3^{(\pi)} \cos \eta \cos \eta' [(g^{\dagger}_{(\pi)}\, \tilde{d}_{(\pi)})^{(2)}_m + (d^{\dagger}_{(\pi)}\, \tilde{g}_{(\pi)})^{(2)}_m] \\ & - q_3^{(\pi)} \sin \eta \sin \eta' [(g^{\dagger}_{(\pi)}\, \tilde{d}_{(\pi)})^{(2)}_m + (d^{\dagger}_{(\pi)}\, \tilde{g}_{(\pi)})^{(2)}_m]\, 3\surd \tfrac{1}{5}\, W(2442;\, 22) \\ =\ & Q^{(\pi)}_{(\mathrm{sd})m} + q_4^{(\pi)} \sin^2 \eta'\, \tfrac{1}{5}\, W(2244;\, 22) (d^{\dagger}_{(\pi)}\, \tilde{d}_{(\pi)})^{(2)}_m \\ & + [q_4^{(\pi)} \cos^2 \eta' + q_2^{(\pi)} \sin^2 \eta \surd \tfrac{1}{5}\, W(2244;\, 22)] (g^{\dagger}_{(\pi)}\, \tilde{g}_{(\pi)})^{(2)}_m \\ & + q_3^{(\pi)} [\cos \eta \cos \eta' - 3\surd \tfrac{1}{5}\, W(2442;\, 22) \sin \eta \sin \eta'] \\ & \times [(g^{\dagger}_{(\pi)}\, \tilde{d}_{(\pi)})^{(2)}_m + (d^{\dagger}_{(\pi)}\, \tilde{g}_{(\pi)})^{(2)}_m]. \end{aligned} \tag{E12}$$

Appendix F: Single Commutators of Z with Composite Operators

Neutron boson operator products will be considered only herein, it being understood that proton results can simply be obtained by symmetry. In the development, the commutators of single boson operators, *viz.*

$$[Z_{(\nu)}, s^{\dagger}_{(\nu)}] = 0, \tag{F1.1}$$

$$[Z_{(\nu)}, d^{\dagger}_{(\nu)\alpha}] = \theta_{(\nu)} (Q^{(\pi)} g^{\dagger}_{(\nu)})^{(2)}_{\alpha}, \tag{F1.2}$$

$$[Z_{(\nu)}, g^{\dagger}_{(\nu)\alpha}] = -\surd \tfrac{5}{9}\, \theta_{(\nu)} (Q^{(\pi)} d^{\dagger}_{(\nu)})^{(4)}_{\alpha}, \tag{F1.3}$$

with a comparable set for the annihilation operators will be used.

Operators of the Form $(d^{\dagger}_{(\nu)}\, \tilde{d}_{(\nu)})^{(\lambda)}$

This commutator expands as

$$\begin{aligned} [Z_{(\nu)}, (d^{\dagger}_{(\nu)}\, \tilde{d}_{(\nu)})^{(\lambda)}_{\alpha}] =\ & \Sigma_{\epsilon} \langle 2\, 2\, \epsilon\, \alpha - \epsilon | \lambda\, \alpha \rangle \\ & \times \{[Z_{(\nu)}, d^{\dagger}_{(\nu)\epsilon}] \tilde{d}_{(\nu)\alpha-\epsilon} + d^{\dagger}_{(\nu)\epsilon} [Z_{(\nu)}, \tilde{d}_{(\nu)\alpha-\epsilon}]\}, \end{aligned} \tag{F2}$$

and by using equation (F1.2) and standard angular momentum coupling we get

$$[Z_{(\nu)}, (d^\dagger_{(\nu)}\, \tilde{d}_{(\nu)})^{(\lambda)}_\alpha] = \theta_{(\nu)} \Sigma_{K\mu}\, Q^{(\pi)}_\mu \langle K\, 2\, \alpha-\mu\, \mu | \lambda\, \alpha\rangle$$
$$\times [5(2K+1)]^{\frac{1}{2}}\, W(22K2;\, 4\lambda)\, \tilde{P}^{(\nu)}_{K,\alpha-\mu}(\lambda), \tag{F3}$$

in which

$$\tilde{P}^{(\nu)}_{K,\gamma}(\lambda) = (g^\dagger_{(\nu)}\, \tilde{d}_{(\nu)})^{(K)}_\gamma + (-)^{K+\lambda}(d^\dagger_{(\nu)}\, \tilde{g}_{(\nu)})^{(K)}_\gamma\,. \tag{F4}$$

The number operator is a special case since for $\lambda = 0$

$$(d^\dagger_{(\nu)}\, d_{(\nu)})^{(0)}_0 = \Sigma_\epsilon (-)^\epsilon d^\dagger_{(\nu)\epsilon}\, \tilde{d}_{(\nu)-\epsilon}/\sqrt{5}$$
$$= (d^\dagger_{(\nu)} \cdot \tilde{d}_{(\nu)})/\sqrt{5}\,. \tag{F5}$$

Thus, as $K = 2$ for $\lambda = 0$ in (F3), we have

$$[Z_{(\nu)}, d^\dagger_{(\nu)} \cdot \tilde{d}_{(\nu)}] = \theta_{(\nu)}\, Q^{(\pi)} \,.\, P^{(\nu)}, \tag{F6}$$

in which

$$P^{(\nu)}_\gamma = P^{(\nu)}_{2\gamma}(0) = [(g^\dagger_{(\nu)}\, \tilde{d}_{(\nu)}) + (d^\dagger_{(\nu)}\, \tilde{g}_{(\nu)})]^{(2)}_\gamma\,. \tag{F7}$$

Operators of the Form $(g^\dagger_{(\nu)}\, g_{(\nu)})^{(\lambda)}$

As with the foregoing, the commutator of any such operator with $Z_{(\nu)}$ expands as

$$[Z_{(\nu)}, (g^\dagger_{(\nu)}\, \tilde{g}_{(\nu)})^{(\lambda)}_\alpha] = \Sigma_\epsilon \langle 4\, 4\, \epsilon\, \alpha-\epsilon | \lambda\, \alpha\rangle$$
$$\times \{[Z_{(\nu)}, g^\dagger_{(\nu)\epsilon}]\tilde{g}_{(\nu)\alpha-\epsilon} + g^\dagger_{(\nu)\epsilon}[Z_{(\nu)} \cdot \tilde{g}_{(\nu)\alpha-\epsilon}]\}, \tag{F8}$$

and, by using equation (F1.3) with standard angular momentum coupling, we have

$$[Z_{(\nu)}, (g^\dagger_{(\nu)}\, \tilde{g}_{(\nu)})^{(\lambda)}_\alpha] = -\theta_{(\nu)} \Sigma_{K\mu}\, Q^{(\pi)}_\mu \langle K\, 2\, \alpha-\mu\, \mu | \lambda\, \alpha\rangle$$
$$\times (-)^{\lambda+K}[5(2K+1)]^{\frac{1}{2}}\, W(24K4;\, 2\lambda)\, \tilde{P}^{(\nu)}_{K,\alpha-\mu}(\lambda)\,. \tag{F9}$$

The g-boson number operator is a special case ($\lambda = \alpha = 0$) since

$$(g^\dagger_{(\nu)}\, \tilde{g}_{(\nu)})^{(0)}_0 = \Sigma_m (-)^m \tfrac{1}{3} g^\dagger_{(\nu)m}\, \tilde{g}_{(\nu)-m} = \tfrac{1}{3} g^\dagger_{(\nu)} \cdot \tilde{g}_{(\nu)}, \tag{F10}$$

so that with $\lambda = 0$ and thus $K = 2$ in (F9), we have

$$[Z_{(\nu)}, g^\dagger_{(\nu)} \cdot \tilde{g}_{(\nu)}] = -\theta_{(\nu)}\, Q^{(\pi)} \,.\, P^{(\nu)}, \tag{F11}$$

with $P^{(\nu)}$ as defined in (F7).

The s–d Coupling Operator

In the quadrupole operator, there is an s–d boson coupling term:

$$X^{(2)}_{(\nu)\alpha} = (s^{\dagger}_{(\nu)}\,\tilde{d}_{(\nu)})^{(2)}_{\alpha} + (d^{\dagger}_{(\nu)}\,\tilde{s}_{(\nu)})^{(2)}_{\alpha}\,. \tag{F12}$$

This has particularly simple commutators with $Z_{(\nu)}$ namely

$$[Z_{(\nu)}, X^{(2)}_{(\nu)\alpha}] = \theta_{(\nu)}\,\Sigma_{\mu}\,Q^{(\pi)}_{\mu}\langle 2\,4\,\mu\,\alpha-\mu|2\,\alpha\rangle$$

$$\times\,[(s^{\dagger}_{(\nu)}\,\tilde{g}_{(\nu)})^{(4)}_{\alpha-\mu} + (g^{\dagger}_{(\nu)}\,\tilde{s}_{(\nu)})^{(4)}_{\alpha-\mu}]\,. \tag{F13}$$

Operators of the Form $(d^{\dagger}_{(\nu)}\,\tilde{g}_{(\nu)})^{(\lambda)}$ *and* $(g^{\dagger}_{(\nu)}\,\tilde{d}_{(\nu)})^{(\lambda)}$

Expansion of the commutators of the first operator gives

$$[Z_{(\nu)}, (d^{\dagger}_{(\nu)}\,\tilde{g}_{(\nu)})^{(\lambda)}_{\alpha}] = \Sigma_{\epsilon}\langle 2\,4\,\epsilon\,\alpha-\epsilon|\lambda\,\alpha\rangle$$

$$\times\,\{[Z_{(\nu)}, d^{\dagger}_{(\nu)\epsilon}]\tilde{g}_{(\nu)\alpha-\epsilon} + d^{\dagger}_{(\nu)\epsilon}[Z_{(\nu)}, \tilde{g}_{(\nu)\alpha-\epsilon}]\}\,, \tag{F14}$$

and by using the basic commutator equations (F1.2) and (F1.3) we get

$$[Z_{(\nu)}, (d^{\dagger}_{(\nu)}\,\tilde{g}_{(\nu)})^{(\lambda)}_{\alpha}] = \Sigma_{\epsilon\mu}\langle 2\,4\,\epsilon\,\alpha-\epsilon|\lambda\,\alpha\rangle\theta_{(\nu)}\,Q^{(\pi)}_{\mu}$$

$$\times\,(\langle 2\,4\,\mu\,\epsilon-\mu|2\,\epsilon\rangle g^{\dagger}_{(\nu)\epsilon-\mu}\,\tilde{g}_{(\nu)\alpha-\epsilon}$$

$$-\sqrt{\tfrac{5}{9}}\langle 2\,2\,\mu\,\alpha-\epsilon-\mu|4\,\alpha-\epsilon\rangle d^{\dagger}_{(\nu)\epsilon}\,\tilde{d}_{(\nu)\alpha-\epsilon-\mu})\,,$$

and which with standard angular momentum recoupling reduces to

$$[Z_{(\nu)}, (d^{\dagger}_{(\nu)}\,\tilde{g}_{(\nu)})^{(\lambda)}_{\alpha}] = \theta_{(\nu)}\,\Sigma_{K\mu}\,Q^{(\pi)}_{\mu}\langle K\,2\,\alpha-\mu|\lambda\,\alpha\rangle$$

$$\times\,[5(2K+1)]^{\frac{1}{2}}[\,W(22K4;\,4\lambda)(g^{\dagger}_{(\nu)}\,\tilde{g}_{(\nu)})^{(K)}_{\alpha-\mu}$$

$$-(-)^{\lambda+K}\,W(24K2;\,2\lambda)(d^{\dagger}_{(\nu)}\,\tilde{d}_{(\nu)})^{(K)}_{\alpha-\mu}]\,. \tag{F15}$$

An identical development gives

$$[Z_{(\nu)}, (g^{\dagger}_{(\nu)}\,\tilde{d}_{(\nu)})^{(\lambda)}_{\alpha}] = \theta_{(\nu)}\,\Sigma_{K\mu}\langle K\,2\,\alpha-\mu\,\mu|\lambda\,\alpha\rangle$$

$$\times\,(-)^{\lambda+K}[5(2K+1)]^{\frac{1}{2}}[\,W(22K4;\,4\lambda)(g^{\dagger}_{(\nu)}\,\tilde{g}_{(\nu)})^{(K)}_{\alpha-\mu}$$

$$-(-)^{\lambda+K}\,W(24K2;\,2\lambda)(d^{\dagger}_{(\nu)}\,\tilde{d}_{(\nu)})^{(K)}_{\alpha-\mu}]\,. \tag{F16}$$

Combinations Operators $P^{(\nu)}$ *and* $\tilde{P}^{(\nu)}_{K\gamma}(\lambda)$

Using the definition (F7) of $P^{(\nu)}$ and the results given in (F15) and (F16), one readily obtains for $\lambda = 2$

$$[Z_{(\nu)}, P_\alpha^{(\nu)}] = \theta_{(\nu)} \Sigma_{K\mu}\, Q_\mu^{(\pi)} \langle K\, 2\, \alpha-\mu\, \mu | 2\, \alpha\rangle [5(2K+1)]^{\frac{1}{2}}$$
$$\times [1+(-)^K][\, W(22K4;\, 42)(g_{(\nu)}^\dagger\, \tilde{g}_{(\nu)})_{\alpha-\mu}^{(K)}$$
$$-(-)^K\, W(24K2;\, 22)(d_{(\nu)}^\dagger\, \tilde{d}_{(\nu)})_{\alpha-\mu}^{(K)}]\,. \tag{F17}$$

Likewise from the definitions (F4) of $\tilde{P}^{(\nu)}$ one gets

$$[Z_{(\nu)}, \tilde{P}_{K,\gamma}^{(\nu)}(\lambda)] = \theta_{(\nu)} \Sigma_{J\mu}\, Q_\mu^{(\pi)} [5(2J+1)]^{\frac{1}{2}} \langle J\, 2\, \gamma-\mu\, \mu | K\, \gamma\rangle$$
$$\times [\, W(22J4;\, 4K)(g_{(\nu)}^\dagger\, \tilde{g}_{(\nu)})_{\gamma-\mu}^{(J)} - (-)^{K+J}\, W(24J2;\, 2K)(d_{(\nu)}^\dagger\, \tilde{d}_{(\nu)})_{\gamma-\mu}^{(J)}]$$
$$\times [(-)^{K+J} + (-)^{\lambda+K}]\,. \tag{F18}$$

The s–d–g Boson Quadrupole Operators

With components of the quadrupole operator $Q_\alpha^{(\nu)}$ as defined by (2), and using the results obtained in this Appendix, we have

$$[Z_{(\nu)}, Q_\alpha^{(\nu)}] = \theta_{(\nu)} \Sigma_{\mu K}\, Q_\mu^{(\pi)} [5(2K+1)]^{\frac{1}{2}} \langle K\, 2\, \alpha-\mu\, \mu | 2\, \alpha\rangle$$
$$\times \{\, q_1^{(\nu)}\, \delta_{K4} [(s_{(\nu)}^\dagger\, \tilde{g}_{(\nu)})_{\alpha-\mu}^{(4)} + (g_{(\nu)}^\dagger\, s_{(\nu)})_{\alpha-\mu}^{(4)}]/\sqrt{45}$$
$$+ q_2^{(\nu)}\, W(22K2;\, 42)[(g_{(\nu)}^\dagger\, \tilde{d}_{(\nu)})_{\alpha-\mu}^{(K)} + (-)^K (d_{(\nu)}^\dagger\, \tilde{g}_{(\nu)})_{\alpha-\mu}^{(K)}]$$
$$+ q_3^{(\nu)} [1+(-)^{(K)}][\, W(22K4;\, 42)(g_{(\nu)}^\dagger\, \tilde{g}_{(\nu)})_{\alpha-\mu}^{(K)} - (-)^K\, W(24K2;\, 22)(d_{(\nu)}^\dagger\, \tilde{d}_{(\nu)})_{\alpha-\mu}^{(K)}]$$
$$- q_4^{(\nu)} (-)^K\, W(24K4;\, 22)[(g_{(\nu)}^\dagger\, \tilde{d}_{(\nu)})_{\alpha-\mu}^{(K)} + (-)^K (d_{(\nu)}^\dagger\, \tilde{g}_{(\nu)})_{\alpha-\mu}^{(K)}]\}\,. \tag{F19}$$

In the OG approximation only $K = 0$ terms are retained and the above simplifies dramatically to

$$[Z_{(\nu)}, Q_\alpha^{(\nu)}] \sim \theta_{(\nu)}\, Q_\alpha^{(\pi)}\, q_3^{(\nu)} (\tfrac{2}{9}\, g_{(\nu)}^\dagger \cdot \tilde{g}_{(\nu)} - \tfrac{2}{5}\, d_{(\nu)}^\dagger \cdot \tilde{d}_{(\nu)})\,, \tag{F20}$$

which, averaged over a condensed neutron state, is

$$\langle [Z_{(\nu)}, Q_\alpha^{(\pi)}] \rangle = C\, Q_\alpha^{(\pi)}\,,$$

where

$$C = \theta_{(\nu)}\, q_3^{(\nu)} (\tfrac{2}{9}\, n_g - \tfrac{2}{5}\, n_d)\,. \tag{F21}$$

The s–g Hexadecapole Operator

Using

$$X_\alpha^{(4)} = (s_{(\nu)}^\dagger\, \tilde{g}_{(\nu)})_\alpha^{(4)} + (g_{(\nu)}^\dagger\, \tilde{s}_{(\nu)})_\alpha^{(4)}\,,$$

it is easy to deduce that

$$
\begin{aligned}
[Z_{(\nu)}, X_{\alpha}^{(4)}] &= s_{(\nu)}^{\dagger}[Z_{(\nu)}, \tilde{g}_{(\nu)\alpha}] + [Z_{(\nu)}, g_{(\nu)\alpha}^{\dagger}]s_{(\nu)} \\
&= -\sqrt{\tfrac{5}{9}}\,\theta_{(\nu)}\,\Sigma_{\mu}\langle 2\,2\,\mu\,\alpha-\mu|4\,\alpha\rangle Q_{\mu}^{(\pi)}(s_{(\nu)}^{\dagger}\,\tilde{d}_{(\nu)\alpha-\mu} + d_{(\nu)\alpha-\mu}^{\dagger}\,s_{(\nu)}) \\
&= -\theta_{(\nu)}\,\Sigma_{\mu K}\,Q_{\mu}^{(\pi)}\langle K\,2\,\alpha-\mu\,\mu|4\,\alpha\rangle \\
&\quad \times[5(2K+1)]^{\frac{1}{2}}\sqrt{\tfrac{1}{45}}\,\delta_{K2}[(s_{(\nu)}^{\dagger}\,\tilde{d}_{(\nu)})_{\alpha-\mu}^{(K)}(d_{(\nu)}^{\dagger}\,s_{(\nu)})_{\alpha-\mu}^{(K)}].
\end{aligned} \tag{F22}
$$

Commutator $[Z, Q^{(-\tau)}.\,P^{(\tau)}]$

We have

$$
[Z, Q^{(-\tau)}.\,P^{(\tau)}] = \Sigma_m(-)^m\{[Z_{(-\tau)}, Q_{-m}^{(-\tau)}]P_m^{(\tau)} + Q_{-m}^{(-\tau)}[Z_{(\tau)}, P_m^{(\tau)}]\}, \tag{F23}
$$

and which, by using the results of (F17) and (F19), is

$$
\begin{aligned}
[Z, Q^{(-\tau)}.\,P^{(\tau)}] &= \Sigma_m(-)^m\theta_{(-\tau)}\,\Sigma_{\mu K}\,Q_{\mu}^{(\tau)}[5(2K+1)]^{\frac{1}{2}} \\
&\quad \times\langle K\,2-m-\mu\,\mu|2-m\rangle\Omega_{K,-m-\mu}^{(-\tau)}\,P_m^{(\tau)} \\
&\quad +\Sigma_m(-)^m\theta_{(\tau)}\,\Sigma_{K\mu}\,Q_{-m}^{(-\tau)}\,Q_{\mu}^{(-\tau)}\langle K\,2\,m-\mu\,\mu|2\,m\rangle \\
&\quad \times[5(2K+1)]^{\frac{1}{2}}[1+(-)^K][W(22K4;\,42)(g_{(\tau)}^{\dagger}\,\tilde{g}_{(\tau)})_{m-\mu}^{(K)} \\
&\quad -(-)^K W(24K2;\,22)(d_{(\tau)}^{\dagger}\,\tilde{d}_{(\tau)})_{m-\mu}^{(K)}],
\end{aligned} \tag{F24}
$$

in which

$$
\begin{aligned}
\Omega_{K,\gamma}^{(\tau)} &= q_1^{(\tau)}\,\delta_{K,4}\sqrt{\tfrac{1}{45}}\,[(s_{(\tau)}^{\dagger}\,\tilde{g}_{(\tau)})_{\gamma}^{(4)} + (g_{(\tau)}^{\dagger}\,s_{(\tau)})_{\gamma}^{(4)}] \\
&\quad +[q_2^{(\tau)}\,W(22K2;\,42) - q_4^{(\tau)}(-)^K\,W(24K4;\,22)]\tilde{P}_{K,\gamma}^{(\tau)}(2) \\
&\quad +q_3^{(\tau)}[1+(-)^K][W(22K4;\,42)(g_{(\tau)}^{\dagger}\,\tilde{g}_{(\tau)})_{\gamma}^{(K)} \\
&\quad -(-)^K W(24K2;\,22)(d_{(\tau)}^{\dagger}\,\tilde{d}_{(\tau)})_{\gamma}^{(K)}].
\end{aligned} \tag{F25}
$$

Coupling the relevant operators via

$$
\begin{aligned}
&\Sigma_{m\mu}\langle K\,2\,m-\mu\,\mu|2-m\rangle(-)^m Q_{\mu}^{(\tau)}\,P_m^{(\tau)}(-)^m \\
&\qquad = \Sigma_{m\mu}(-)^{m+\mu}(-)^K\langle 2\,2\,\mu\,m|K\,m+\mu\rangle Q_{\mu}^{(\tau)}\,P_m^{(\tau)}[5/(2K+1)]^{\frac{1}{2}} \\
&\qquad = (-)^K[5/(2K+1)]^{\frac{1}{2}}\Sigma_{(m+\mu)}(-)^{m-\mu}(Q^{(\tau)}\,P^{(\tau)})_{m+\mu}^{(K)},
\end{aligned}
$$

$$\Sigma_{m\mu}\langle K\,2\;m-\mu\,\mu|1\;m\rangle(-)^m Q_{-m}^{(-\tau)} Q_{\mu}^{(-\tau)}$$

$$= \Sigma_{m\mu}(-)^K\langle 2\,2\;m-\mu\,\mu|K\;m-\mu\rangle[5/(2K+1)]^{\frac{1}{2}} Q_{-m}^{(-\tau)} Q_{\mu}^{(-\tau)}(-)^{m+\mu}$$

$$= [5/(2K+1)]^{\frac{1}{2}}\Sigma_{(-m+\mu)}(-)^{m-\mu}(Q^{(-\tau)}Q^{(-\tau)})^{(K)}_{-m+\mu},$$

gives

$$[Z, Q^{(-\tau)}\,.\,P^{(\tau)}] = 5\theta_{(\tau)}\Sigma_K[1+(-)^K](Q^{(-\tau)}Q^{(-\tau)})^{(K)}$$
$$\times[W(22K4;\,42)(g^{\dagger}_{(\tau)}\,\tilde{g}_{(\tau)})^{(K)}-(-)^K W(24K2;\,22)(d^{\dagger}_{(\tau)}\,\tilde{d}_{(\tau)})^{(K)}]$$
$$+5\theta_{(-\tau)}\Sigma_K(-)^K(Q^{(\tau)}P^{(\tau)})^{(K)}\,.\,\Omega_K^{(-\tau)}. \tag{F26}$$

Applying the $K = 0$ OG limit gives

$$[Z, Q^{(-\tau)}\,.\,P^{(\tau)}]\rightarrow\theta_{(\tau)}2(5)\surd\tfrac{1}{5}(\overline{Q^{(-\tau)2}})[\surd\tfrac{1}{45}(g^{\dagger}g)^0-\tfrac{1}{5}(d^{\dagger}d)^0]$$
$$+\theta_{(-\tau)}5\surd\tfrac{1}{5}(\overline{Q^{(\tau)}P^{(\tau)}})\Omega_0^{(-\tau)}$$
$$=\theta_{(\tau)}2(\overline{Q^{(-\tau)2}})[\tfrac{1}{9}\,g^{\dagger}_{(\tau)}\cdot\tilde{g}_{(\tau)}-\tfrac{1}{5}\,d^{\dagger}_{(\tau)}\cdot\tilde{d}_{(\tau)}]$$
$$+\theta^{\dagger}_{(-\tau)}2(\overline{Q^{+\tau)}P^{(+\tau)}})q_3^{(-\tau)}[\tfrac{1}{9}\,g^{\dagger}_{(-\tau)}\cdot\tilde{g}_{(-\tau)}-\tfrac{1}{5}\,d^{\dagger}_{(-\tau)}\cdot\tilde{d}_{(-\tau)}]$$
$$=\theta_{(\tau)}\tfrac{2}{5}(\overline{Q^{(-\tau)2}})[\tfrac{5}{9}\,g^{\dagger}_{(\tau)}\cdot\tilde{g}_{(\tau)}-d^{\dagger}_{(\tau)}\cdot\tilde{d}_{(\tau)}]$$
$$+q_3^{(-\tau)}\theta_{(-\tau)}\tfrac{2}{5}(\overline{Q^{(\tau)}P^{(\tau)}})[\tfrac{5}{9}\,g^{\dagger}_{(-\tau)}\cdot\tilde{g}_{(-\tau)}-d^{\dagger}_{(-\tau)}\cdot\tilde{d}_{(-\tau)}]. \tag{F27}$$

Then, with $T = \Sigma_\tau\, C_{(\tau)}\,Q^{(-\tau)}\,.\,P^{(\tau)}$, we get

$$[Z, T] = -\Sigma_\tau\,\theta_{(\tau)}\gamma_{(\tau)}[d^{\dagger}_{(\tau)}\cdot\tilde{d}_{(\tau)}-\tfrac{5}{9}\,g^{\dagger}_{(\tau)}\cdot\tilde{g}_{(\tau)}],$$

where

$$\gamma_{(\tau)} = \tfrac{2}{5}[C_{(\tau)}(\overline{Q^{(-\tau)2}})+C_{(-\tau)}(\overline{Q^{(-\tau)}P^{(-\tau)}})q_3^{(\tau)}]. \tag{F28}$$

If $(\overline{QP})$ is negligible then we get

$$[Z, Q^{(-\tau)}\,.\,P^{(\tau)}]\sim-\theta_{(\tau)}\tfrac{2}{5}(\overline{Q^{(-\tau)2}})d^{\dagger}_{(\tau)}\cdot\tilde{d}_{(\tau)}+\theta_{(\tau)}\tfrac{2}{5}(\overline{Q^{(-\tau)2}})\tfrac{5}{9}\,g^{\dagger}_{(\tau)}\cdot\tilde{g}_{(\tau)}. \tag{F29}$$

Appendix G: Commutators of the Residual Interaction

To within the scaling of $-\frac{1}{2}f$, the residual interaction has the form

$$\Delta = \Sigma_\tau\, Q^{(-\tau)} \,.\, V^{(\tau)}, \tag{G1}$$

where

$$\begin{aligned} V^{(\tau)} = {} & Q^{(\tau)} - P^{(\tau)} \\ & + q_1^{(\tau)}[(s^\dagger_{(\tau)}\, \tilde{d}_{(\tau)})^{(2)} + (d^\dagger_{(\tau)}\, \tilde{s}_{(\tau)})^{(2)}] \\ & + q_2^{(\tau)}(d^\dagger_{(\tau)}\, \tilde{d}_{(\tau)})^{(2)} + q_4^{(\tau)}(g^\dagger_{(\tau)}\, \tilde{g}_{(\tau)})^{(2)} . \end{aligned} \tag{G2}$$

We seek the commutations of Δ with Z where

$$\begin{aligned} Z &= Z_{(\pi)} + Z_{(\nu)} \\ &= \Sigma_\tau\, \theta_{(\tau)}\, Q^{(-\tau)} \,.\, [(g^\dagger_{(\tau)}\, \tilde{d}_{(\tau)})^{(2)} - (d^\dagger_{(\tau)}\, \tilde{g}_{(\tau)})^{(2)}] \\ &= \Sigma_\tau\, \theta_{(\tau)}\, Q^{(-\tau)} \,.\, E^{(\tau)}, \end{aligned} \tag{G3}$$

and these will involve four basic commutators of $Z_{(\tau)}$ with $Q^{(-\tau)}_{-\mu}$, $Q^{(\tau)}_{-\mu}$, $V^{(-\tau)}_{\mu}$ and $V^{(\tau)}_{\mu}$.

Basic Commutators

(a) $[Z_{(\tau)}, Q^{(-\tau)}_{-\mu}]$:

$$[Z_{(\tau)}, Q^{(-\tau)}_{-\mu}] = \Sigma_\epsilon\, \theta_{(\tau)}(-)^\epsilon [Q^{(-\tau)}_{-\epsilon}\, E^{(\tau)}_\epsilon, Q^{(-\tau)}_{\mu}] \equiv 0 . \tag{G4}$$

(b) $[Z_{(\tau)}, Q^{(\tau)}_{-\mu}]$:

This commutator has been developed in Appendix F to be

$$\begin{aligned} [Z_{(\tau)}, Q^{(\tau)}_{-\mu}] = {} & \theta_{(\tau)}\, \Sigma_{J\epsilon} [5(2J+1)]^{\frac{1}{2}} \\ & \times \langle J\, 2 - \mu - \epsilon\, \epsilon | 2 - \mu \rangle Q^{(-\tau)}_\epsilon\, \Omega^{(\tau)}_{J,-\mu-\epsilon}, \end{aligned} \tag{G5}$$

and which in the OG limit ($J = 0$) is

$$\begin{aligned} [Z_{(\tau)}, Q^{(\tau)}_{-\mu}] &= -\theta_{(\tau)}\, \tfrac{2}{5}\, q_3^{(\tau)}(d^\dagger_{(\tau)} \cdot \tilde{d}_{(\tau)} - \tfrac{5}{9}\, g^\dagger_{(\tau)} \cdot \tilde{g}_{(\tau)}) Q^{(-\tau)}_{-\mu} \\ &= -\theta_{(\tau)}\, X^{(\tau)} Q^{(-\tau)}_{-\mu}, \end{aligned} \tag{G6}$$

in which

$$X^{(\tau)} = \tfrac{2}{5}\, q_3^{(\tau)}(d^\dagger_{(\tau)} \cdot \tilde{d}_{(\tau)} - \tfrac{5}{9}\, g^\dagger_{(\tau)} \cdot \tilde{g}_{(\tau)}) . \tag{G7}$$

(c) $[Z_{(\tau)}, V_\mu^{(-\tau)}]$:

Using the expansion of $Z_{(\tau)}$ this commutator is

$$[Z_{(\tau)}, V_\mu^{(-\tau)}] = \theta_{(\tau)} \Sigma_\alpha (-)^\alpha [Q_{-\alpha}^{(-\tau)} E_\alpha^{(\tau)}, V_\mu^{(-\tau)}]$$
$$= \theta_{(\tau)} \Sigma_\alpha (-)^\alpha [Q_{-\alpha}^{(-\tau)}, V_\mu^{(-\tau)}] E_\alpha^{(\tau)} .$$

We recall that this commutator will be used in the evaluation of terms of the form

$$F = \Sigma_\mu (-)^\mu Q_{-\mu}^{(\tau)} [Z_{(\tau)}, V_\mu^{(-\tau)}]$$
$$= \theta_{(\tau)} \Sigma_{\alpha\mu} (-)^{\mu+\alpha} [Q_{-\alpha}^{(-\tau)}, V_\mu^{(-\tau)}] Q_{-\mu}^{(\tau)} E_\alpha^{(\tau)} , \tag{G8}$$

which, on averaging over τ-space, involves

$$\langle Q^{(\tau)} E^{(\tau)} \rangle \ll \langle Q^{(\tau)2} \rangle , \tag{G9}$$

as previously assumed. Thus we ignore terms involving the commutators $[Z_{(\tau)}, V_\mu^{(-\tau)}]$.

(d) $[Z_{(\tau)}, V_\mu^{(\tau)}]$:

Using the results derived in Appendix F for the commutators of Z with each component of the interaction operator $V_\mu^{(\tau)}$, we obtain

$$\begin{aligned}[Z_{(\tau)}, V_\mu^{(\tau)}] = \theta_{(\tau)} \Sigma_{J\epsilon} [5(2J+1)]^{\frac{1}{2}} \langle J\,2\,\mu-\epsilon\,\mu|2\,\mu\rangle Q_\epsilon^{(-\tau)} \\ \times \{ (\delta_{J4}\, q_1^{(\tau)}/\sqrt{45}) [(s_{(\tau)}^\dagger\, \tilde{g}_{(\tau)})^{(4)} + (g_{(\tau)}^\dagger\, \tilde{s}_{(\tau)})^{(4)}]_{\mu-\epsilon} \\ + [q_2^{(\tau)}\, W(22J2;\, 42) - (-)^J q_4^{(\tau)}\, W(24J4;\, 22)] \\ \times [(d_{(\tau)}^\dagger\, \tilde{g}_{(\tau)})^{(J)} + (-)^J (g_{(\tau)}^\dagger\, \tilde{d}_{(\tau)})^{(J)}]_{\mu-\epsilon} \} ,\end{aligned} \tag{G10}$$

with J restricted to values of 2 or 4 in the second term by angular momentum selection and symmetry.

This commutator is required to evaluate terms of the form

$$F = \Sigma_{\tau\mu} (-)^\mu Q_{-\mu}^{(-\tau)} [Z_{(\tau)}, V_\mu^{(\tau)}] , \tag{G11}$$

and, in the OG approximation, with

$$\langle (-)^\mu Q_{-\mu}^{(\alpha)} Q_\epsilon^{(\alpha)} \rangle \to \tfrac{1}{5} (\overline{Q^{(\alpha)2}}) \delta_{\epsilon\mu} , \tag{G12}$$

then

$$\begin{aligned}\langle F \rangle \sim \Sigma_{\tau\mu}\, \theta_{(\tau)} (\overline{Q^{(-\tau)2}}) \Sigma_J [5(2J+1)]^{\frac{1}{2}} \\ \times \langle J\,2\,0\,\mu|2\,\mu\rangle \{ \delta_{J4} (q_1^{(\tau)}/\sqrt{45}) [(s_{(\tau)}^\dagger\, \tilde{g}_{(\tau)})^{(4)} + (g_{(\tau)}^\dagger\, \tilde{s}_{(\tau)})^{(4)}] \\ + \text{`}q_5^{(\tau)}\text{'} [(d_{(\tau)}^\dagger\, \tilde{g}_{(\tau)})_0^{(J)} + (-)^{(J)} (g_{(\tau)}^\dagger\, \tilde{d}_{(\tau)})_0^{(J)}] \} = 0 ,\end{aligned} \tag{G13}$$

since, for J equal to 2 or 4,

$$\Sigma_{\mu}\langle J\,2\,0\,\mu|2\,\mu\rangle = 0. \tag{G14}$$

Single Commutator of Z

With Δ as defined by (G1), the single commutator with Z is

$$\begin{aligned}
[Z, \Delta] &= [Z_{(\pi)}, Q^{(\nu)}\,.\,V^{(\pi)} + Q^{(\pi)}\,.\,V^{(\nu)}] \\
&\quad + [Z_{(\nu)}, Q^{(\nu)}\,.\,V^{(\pi)} + Q^{(\pi)}\,.\,V^{(\nu)}] \\
&= \Sigma_{\mu}(-)^{\mu}\{[Z_{(\pi)}, Q^{(\nu)}_{-\mu}]\,V^{(\pi)}_{\mu} + Q^{(\nu)}_{-\mu}[Z_{(\pi)}, V^{(\pi)}_{\mu}] \\
&\quad + [Z_{(\pi)}, Q^{(\pi)}_{-\mu}]\,V^{(\nu)}_{\mu} + Q^{(\pi)}_{-\mu}[Z_{(\pi)}, V^{(\nu)}_{\mu}] \\
&\quad + [Z_{(\nu)}, Q^{(\nu)}_{-\mu}]\,V^{(\pi)}_{\mu} + Q^{(\nu)}_{-\mu}[Z_{(\nu)}, V^{(\pi)}_{\mu}] \\
&\quad + [Z_{(\nu)}, Q^{(\pi)}_{-\mu}]\,V^{(\nu)}_{\mu} + Q^{(\pi)}_{-\mu}[Z_{(\nu)}, V^{(\nu)}_{\mu}]\},
\end{aligned} \tag{G15}$$

and using (G4), mindful that averages will be used, we get

$$\begin{aligned}
[Z, \Delta] &\sim \Sigma_{\mu}(-)^{\mu}\{[Z_{(\pi)}, Q^{(\pi)}_{-\mu}]\,V^{(\pi)}_{\mu} + [Z_{(\nu)}, Q^{(\nu)}_{-\mu}]\,V^{(\pi)}_{\mu}\} \\
&= \Sigma_{\tau\mu}(-)^{\mu}[Z_{(\tau)}, Q^{(\tau)}_{-\mu}]\,V^{(-\tau)}_{\mu}.
\end{aligned} \tag{G16}$$

The result given in (G6) thus makes

$$[Z, \Delta] = \Sigma_{\tau}(-\theta_{(\tau)}\,X^{(\tau)})\,Q^{(-\tau)}\,.\,V^{(-\tau)}, \tag{G17}$$

where $X^{(a)}$ is as defined in (G7). This result may now be used to obtain a convenient approximate form for the double commutator.

Structure of the Double Commutator

$$\begin{aligned}
[Z,[Z,\Delta]] &= [Z_{(\pi)} + Z_{(\nu)}, \Sigma_{\tau}(-\theta_{(\tau)}\,X^{(\tau)})\,Q^{(-\tau)}\,.\,V^{(-\tau)}] \\
&= -\theta_{(\pi)}\{[Z_{(\pi)}, X^{(\pi)}Q^{(\nu)}\,.\,V^{(\nu)}] + [Z_{(\nu)}, X^{(\pi)}Q^{(\nu)}\,.\,V^{(\nu)}]\} \\
&\quad -\theta_{(\nu)}\{[Z_{(\pi)}, X^{(\nu)}Q^{(\pi)}\,.\,V^{(\pi)}] + [Z_{(\nu)}, X^{(\nu)}Q^{(\pi)}\,.\,V^{(\pi)}]\} \\
&= \Sigma_{\tau}(-\theta_{(\tau)})\{[Z_{(\tau)}, X^{(\tau)}Q^{(-\tau)}\,.\,V^{(-\tau)}] + [Z_{(-\tau)}, X^{(\tau)}Q^{(-\tau)}\,.\,V^{(-\tau)}]\} \\
&= \Sigma_{\tau}(-\theta_{(\tau)})\{[Z_{(\tau)}, X^{(\tau)}]\,Q^{(-\tau)}\,.\,V^{(-\tau)} + X^{(\tau)}[Z_{(\tau)}, Q^{(-\tau)}\,.\,V^{(-\tau)}] \\
&\quad + [Z_{(-\tau)}, X^{(\tau)}]\,Q^{(-\tau)}\,.\,V^{(-\tau)} + X^{(\tau)}[Z_{(-\tau)}, Q^{(-\tau)}\,.\,V^{(-\tau)}]\}.
\end{aligned} \tag{G18}$$

Thus there are four commutators to consider:

(a) $[Z_{(\tau)}, Q^{(-\tau)} \cdot V^{(-\tau)}]$:

As $Z_{(\tau)}$ involves $E^{(\tau)}$, this commutator will always be a function of $E^{(\tau)}$ the average of which we assume is negligible. The term in the expression of the double commutator (G18) involving this commutator may then be neglected.

(b) $[Z_{(\tau)}, Q^{(\tau)} \cdot V^{(\tau)}]$:

Expanding the scalar product gives

$$[Z_{(\tau)}, Q^{(\tau)} \cdot V^{(\tau)}] = \Sigma_\mu (-)^\mu \{[Z_{(\tau)}, Q^{(\tau)}_{-\mu}]\, V^{(\tau)}_\mu + Q^{(\tau)}_{-\mu}[Z_{(\tau)}, V^{(\tau)}_\mu]\}, \tag{G19}$$

and, by using the results obtained previously of (G6) and (G10)–(G14), we may use

$$\begin{aligned}[Z_{(\tau)}, Q^{(\tau)} \cdot V^{(\tau)}] &\to \Sigma_\mu (-)^\mu (-\theta_{(\tau)}\, X^{(\tau)}) Q^{(-\tau)}_{-\mu}\, V^{(\tau)}_\mu \\ &= -\theta_{(\tau)}\, X^{(\tau)} Q^{(-\tau)} \cdot V^{(\tau)}\end{aligned} \tag{G20}$$

or, as is required in (G18),

$$[Z_{(-\tau)}, Q^{(-\tau)} \cdot V^{(-\tau)}] \equiv -\theta_{(-\tau)}\, X^{(-\tau)} Q^{(\tau)} \cdot V^{(-\tau)}. \tag{G21}$$

(c) $[Z_{(-\tau)}, X^{(\tau)}]$:

The definition of $Z_{(-\tau)}$ shows that this commutator will involve the antiquadrupole operator $E^{(\tau)}$ and so given the assumptions used heretofore, this component in the expression (G18) for the double commutator can be eliminated.

(d) $[Z_{(\tau)}, X^{(\tau)}]$:

Using the definition of $X^{(\tau)}$ this commutator is

$$\begin{aligned}[Z_{(\tau)}, X^{(\tau)}] = \tfrac{2}{5}\, q_3^{(\tau)}\, \theta_{(\tau)} \{ & [Z_{(\tau)}, d^\dagger_{(\tau)} \cdot \tilde{d}_{(\tau)}] \\ & -\tfrac{5}{9}[Z_{(\tau)}, g^\dagger_{(\tau)} \cdot \tilde{g}_{(\tau)}]\}.\end{aligned} \tag{G22}$$

Using the results (F6) and (F11) this gives

$$[Z_{(\tau)}, X^{(\tau)}] = \tfrac{2}{5}\, q_3^{(\tau)}\, \tfrac{14}{9}\, \theta^2_{(\tau)}\, Q^{(-\tau)} \cdot P^{(\tau)}. \tag{G23}$$

Thus the double commutator as given by (G18) becomes

$$\begin{aligned}[Z,[Z,\Delta]] \sim \Sigma_\tau (-\theta_{(\tau)}) \{ & \tfrac{2}{5}\, q_3^{(\tau)}\, \tfrac{14}{9}\, \theta^2_{(\tau)}\, Q^{(-\tau)} \cdot P^{(\tau)}\, Q^{(-\tau)} \cdot V^{(-\tau)} \\ & + X^{(\tau)}(-\theta_{(\tau)}) X^{(-\tau)} Q^{(\tau)} \cdot V^{(\tau)}\}.\end{aligned} \tag{G24}$$

With the requirement that

$$(-)^\alpha\, Q^{(-\tau)}_\alpha\, Q^{(-\tau)}_\beta = \tfrac{1}{5}(\overline{Q^{(-\tau)2}})\delta_{\alpha,-\beta},$$

the leading term above involves an operator of the form $P^{(\tau)} \, . \, V^{(-\tau)}$ whose expectation in the condensate vanishes. Thus, we have our approximate form for the double commutator as

$$[Z,[Z,\Delta]] \sim \Sigma_{\tau} (-\theta_{(\tau)} X^{(\tau)})(-\theta_{(-\tau)} X^{(-\tau)}) Q^{(\tau)} \, . \, V^{(-\tau)} \, . \qquad \text{(G25)}$$

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