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Riemann–Hilbert Problem with Higher Order Poles for Self-Induced Transparency with Degenerate Energy Levels

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Abstract

We have analysed the inverse problem for self-induced transparency with degenerate energy levels (DSIT) via the Riemann–Hilbert methodology with second order poles. The required formulae are deduced in detail and the corresponding soliton solutions are obtained. It is noticed that the profile of the single soliton is not of the usual sech type.

1. Introduction

At present there exist two or three different approaches to the inverse scattering problem. One of the most elegant formulations of DSIT is via the Riemann–Hilbert problem as proposed by the Russian school (Novikov *et al.* 1984), but recently it has been stressed that it is possible to extend the formalism by incorporating higher order poles in the complex eigenvalue plane (Belinsky and Zaharov 1978). Here we formulate such an extended version of the Riemann–Hilbert dressing procedure with second order poles in the case of a newly discovered nonlinear system—self-induced transparency with degenerate energy levels (Basharov and Maimistov 1984). At this point we can mention that the first attempt to study the problem of DSIT was made by McCall and Hahn (1969). Later the problem of DSIT was divided into two classes, one in an absorbing medium and the other in an amplifying medium (Courtens 1972; Drummond 1984). On the other hand, the propagation of two short and different wavelength optical pulses in a three-level absorber by the help of the three-level Maxwell–Bloch equation was studied by Konopnicki (1980). The situation is actually described by an electric field equivalent to two co-propagating plane waves each of which is in near resonance with a transition in the absorber. One then observes different wavelength optical solitons—known as simultons. In the following we describe the details of the system proposed in Basharov and Maimistov (1984). It is important to note that in the case of multiple pole dressing we also get a soliton which goes to zero as $x \to \pm\infty$, but whose profile is not of the usual sech form.

2. Physical Problem of DSIT

When an ultrashort optical pulse passes through a nonlinear medium the pulse does not change its shape. Previously this phenomenon was formulated neglecting any degeneracy of the energy levels of the medium (Lamb 1980), but as a rule almost

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all media have degeneracies and recently DSIT has been formulated taking this into account (Basharov and Maimistov 1984; Konopnicki 1980) as well as the polarisation of the electric field propagating inside the medium. The electric field is denoted as $E = e\exp\{i(k_z-\omega t)\}+\text{c.c.}$, where ω is the carrier frequency and $\omega_0 = (E_b-E_a)/\hbar$ is the frequency of the atomic transition between the levels E_a and E_b, which are degenerate with respect to m and μ of the total angular momenta j_a and j_b in the states a and b. Of all these formulations only those by Basharov and colleagues based their study on a Lax pair which is the essential point for our Riemann–Hilbert approach, and so from now on we follow their conventions. In reality media degeneracy of the energy levels is natural and this manifests itself in special features of the internal propagation of the polarised optical pulses. Equations (1) below actually describe both the effect of polarisation and degeneracy on an ultrashort optical pulse. These effects manifest themselves especially when the 2π pulses with different polarisation collide. Previously the Lax representation was used to consider the integrability of the Maxwell–Bloch equations describing such phenomena, in the case of arbitrary polarisations of the light pulses in resonance with the quantum transitions:

$$j_a = 1 \rightarrow j_b = 0, \quad j_b = 0 \rightarrow j_a = 1, \quad j_b = \tfrac{1}{2} \rightarrow j_a = \tfrac{1}{2}.$$

Incidentally it can be mentioned that it is not very logical to speak of only pure solitons in the case of a degenerate medium inside which the polarised pulse propagates. So here we try to find other types of excitations not of a purely soliton nature, but which have some similarity with soliton excitations.

In the present analysis we also make the simplification of replacing the Maxwell distribution of the z-component of the velocity of the resonance atom by a delta function. These equations are written as

$$\frac{\partial}{\partial_t} E_q = i\sum_{\mu m}\langle R_{\mu m}\rangle J^q_{\mu m},$$

$$\left(\frac{\partial}{\partial x}+i(\eta-\Delta)\right)R_{\mu m} = i\sum E_q\left(\sum_{m'} J^q_{\mu m} R_{m'm} - \sum_{\mu'} R_{\mu' m'} J^q_{\mu' m}\right),$$

$$\frac{\partial}{\partial x} R_{mm'} = i\sum_{q\mu}(E^*_q J^q_{\mu m} R_{\mu m'} - E_q R_{m\mu} J^q_{\mu m'}),$$

$$\frac{\partial}{\partial x} R_{\mu\mu'} = i\sum_{qm}(E_q J^q_{\mu m} R_{m\mu'} - E^*_q R_{\mu m} J^q_{\mu' m}). \tag{1}$$

We set

$$e_{\pm} = -i E_{\pm 1}, \quad n_b = R_{\mu\mu'}|_{\mu=\mu'=0}, \quad n_{a\pm} = R_{mm'}|_{m=m'=\pm 1};$$

$$\nu_{\pm} = R_{\mu m}|_{\mu=0,\ m=\pm 1}, \quad \nu_a = R_{mm'}|_{m=-m'=-1};$$

$$\eta = K v t_0; \quad t_0 = (3\hbar/2\pi\omega|d|^2 N_0)^{\frac{1}{2}}, \quad \Delta = (\omega-\omega_0)/t_0.$$

The Lax equation associated with these is

$$\psi_x = L\psi, \qquad \psi_t = M\psi,$$

where

$$L = \begin{bmatrix} -\mathrm{i}\lambda & e_- & e_+ \\ -e_-^* & \mathrm{i}\lambda & 0 \\ -e_+^* & 0 & \mathrm{i}\lambda \end{bmatrix}, \tag{2}$$

$$M = \left\langle \frac{1}{2\lambda-(\eta-\Delta)} \begin{bmatrix} n_b & \nu_- & \nu_+ \\ \nu_-^* & n_{a-} & \nu_a \\ \nu_+^* & \nu_a^* & n_{a+} \end{bmatrix} \right\rangle. \tag{3}$$

In the above equations E_q represent the polarised electric field, $J^q_{\mu m}$ and $R_{\mu m}$ respectively denote the dipole moments and density matrix elements, while $R_{\mu\mu'}$, $R_{mm'}$ etc. are defined through the optical coherence matrix $\rho_{\sigma\sigma'}$ via

$$\rho_{\mu\mu'} = N_0\, f(\eta) R_{\mu\mu'},$$

where $f(\eta)$ is the Maxwell distribution of the z-component of the velocity:

$$f(\eta) = T_0\, \pi^{-\frac{1}{2}} \exp\{-(\eta\, T_0)^2\}.$$

Further, d is the induced dipole moment. In our notation μ, μ' always refer to a value of zero and m, m' to ± 1; this is why $R_{\mu\mu'}$, $R_{\mu m}$ and $R_{mm'}$ are written separately. In addition ψ is a three component *column* vector of the type (ψ_1, ψ_2, ψ_3). The λ occurring is the eigenvalue associated with the Lax equation and, in general, is a complex quantity.

3. Riemann–Hilbert Formulation

We now proceed to formulate the Riemann–Hilbert approach to the inverse scattering transform for equations (2) and (3). Suppose we have information about one set of solutions of the nonlinear equations (1), which we call the 'seed' solution. Let us denote the eigenfunctions and the fields pertaining to this case by the index 'zero', i.e. as ψ_0, e_{0-}, e_{0+} etc. Equipped with basic analyticity assumptions about ψ_1, ψ_2, the two solutions of (2) on the two sides of a given contour Γ in the complex λ-plane, we assume that ψ_1 and ψ_2 are of the form

$$\psi_1 = 1 + \frac{A_1^1}{\lambda-\lambda_1} + \frac{A_1^2}{(\lambda-\lambda_1)^2},$$

$$\psi_2 = 1 + \frac{B_1^1}{\lambda-\mu_1} + \frac{B_1^2}{(\lambda-\mu_1)^2}. \tag{4}$$

Now we impose the condition

$$\psi_1 \psi_2 = 1. \tag{5}$$

Equating to zero the coefficients of $(\lambda-\lambda_1)^{-1}$, $(\lambda-\mu_1)^{-1}$, $(\lambda-\lambda_1)^{-2}$ and $(\lambda-\mu_1)^{-2}$ we get

$$\begin{aligned}
&A_1^2+\frac{A_1^2 B_1^1}{\lambda_1-\mu_1}+\frac{A_1^2 B_1^2}{(\lambda_1-\mu_1)^2}=0,\\
&B_1^2-\frac{A_1^1 B_1^2}{\lambda_1-\mu_1}+\frac{A_1^2 B_1^2}{(\lambda_1-\mu_1)^2}=0,
\end{aligned} \tag{6}$$

$$B_1^1-\frac{A_1^1 B_1^2}{(\lambda_1-\mu_1)^2}+\frac{A_1^2 B_1^1}{(\lambda_1-\mu_1)^2}-\frac{A_1^1 B_1^1}{\lambda_1-\mu_1}-\frac{2A_1^2 B_1^2}{(\lambda_1-\mu_1)^3}=0,$$

$$A_1^1+\frac{A_1^1 B_1^2}{(\lambda_1-\mu_1)^2}-\frac{A_1^2 B_1^1}{(\lambda_1-\mu_1)^2}+\frac{A_1^1 B_1^1}{\lambda_1-\mu_1}-\frac{2A_1^2 B_1^2}{(\lambda_1-\mu_1)^3}=0.$$

From the first two equations we have

$$A_1^1 = -B_1^1. \tag{7}$$

By a simple algebraic manipulation we get from (5) and (6)

$$A_1^2 = B_1^2 \tag{8}$$

and

$$\begin{aligned}
&A_1^1+\frac{A_1^1 B_1^1}{\lambda_1-\mu_1}+\frac{A_1^2}{\lambda_1-\mu_1}+\frac{B_1^2}{\lambda_1-\mu_1}=0,\\
&A_1^2+\frac{A_1^2 B_1^1}{\lambda_1-\mu_1}+\frac{A_1^2 B_1^2}{(\lambda_1-\mu_1)^2}=0.
\end{aligned} \tag{9}$$

The immediate consequence of these two equations is

$$A_1^2 = \tfrac{1}{2}\{(A_1^1)^2-(\lambda_1-\mu_1)A_1^1\}. \tag{10}$$

So eliminating A_1^2 from (9) we get a fourth order equation for A_1^1:

$$(A_1^1)^4-4(\lambda_1-\mu_1)(A_1^1)^3+5(\lambda_1-\mu_1)^2(A_1^1)^2-2(\lambda_1-\mu_1)^3A_1^1=0. \tag{11}$$

If we get

$$A_1^1 = -(\lambda_1-\mu_1)P, \tag{12}$$

then we obtain

$$P^4-4P^3+5P^2-2P=0 \quad \text{or} \quad (P-1)(P-2)(P^2-P)=0. \tag{13}$$

The possibility $P^2 = P$ proves the projection operator character for A^i_j and so referring back to equation (5) we immediately see that

$$A^2_1 = (\lambda_1 - \mu_1)^2 P. \tag{14}$$

4. Determination of the Structure of A^i_j, B^i_j

We now use the degeneracy condition (Novikov *et al.* 1984) where the matrices A^1_1, B^1_1 etc. are given as the product of vectors and we represent them as

$$(B^1_1)_{ij} = p_i\, q_j, \qquad (B^2_1)_{ij} = r_i m_j,$$
$$(A^1_1)_{ij} = n_i\, y_j, \qquad (A^2_1)_{ij} = k_i\, z_j; \tag{15}$$

that is, we represent these matrices as the product of, for example, $\{p_1, p_2, p_3\}(q_1, q_2, q_3)$, where the first factor in braces is a *column* vector and the second factor in parentheses is a *row* vector. If we substitute these in the equations of the previous section and set

$$y_k\, p_k = \alpha, \quad z_k\, p_k = \beta, \quad z_k\, r_k = \gamma, \quad y_k\, r_k = \sigma, \tag{16}$$

then we get the following four equations:

$$n_i\, y_j + n_i \frac{\alpha}{\lambda_1 - \mu_1}\, q_j + \frac{k_i\, z_j}{\lambda_1 - \mu_1} + \frac{r_i\, m_j}{\lambda_1 - \mu_1} = 0,$$

$$p_i\, q_j - n_i \frac{\alpha}{\lambda_1 - \mu_1}\, q_j - \frac{k_i\, z_j}{\lambda_1 - \mu_1} - \frac{r_i\, m_j}{\lambda_1 - \mu_1} = 0,$$

$$z_j + \frac{\beta}{\lambda_1 - \mu_1}\, q_j + \frac{\gamma}{(\lambda_1 - \mu_1)^2}\, m_j = 0,$$

$$r_j - x_i \frac{\sigma}{\lambda_1 - \mu_1} + k_i \frac{\gamma}{(\lambda_1 - \mu_1)^2} = 0. \tag{17}$$

Equations (16) and (17) can be used to determine any four vectors of $(p_i, n_i, r_i, k_i, q_i, y_i, m_i, z_i)$ in terms of the other four, while the remaining four can be determined from the 'seed' solution via the Lax equation.

5. Dressing up the Seed Solution

Suppose we now construct another solution of the Lax equation as

$$\psi' = \psi_2\, \psi_0,$$

whence for the seed solution we assume that

$$e_+ = e_- = 0; \qquad \nu_a = 0, \quad \nu_+ = \nu_- = 0;$$

$$n_b = \alpha_1, \quad n_{a-} = \alpha_2, \quad n_{a+} = \alpha_3.$$

Then the Lax equation leads to

$$\tilde{L} = L+[B_1^1, E], \qquad E = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{18}$$

where we have assumed that ψ' satisfies

$$\psi'_x = \tilde{L}\psi'. \tag{19}$$

Also it is to be noted that one can construct the vectors $(n)_i$, $(k)_i$, $(q)_i$ and $(m)_i$ from the seed solution matrices $\psi_0(\lambda_1)$ and $\bar{\psi}_0(\mu_1)$ as

$$\begin{aligned} (n)_i &= \psi_0(\lambda_1)a_i, \qquad & (k)_i &= \psi_0(\lambda_1)b_i, \\ (q)_i &= \bar{a}_i\bar{\psi}_0(\mu_1), \qquad & (m)_i &= \bar{b}_i\bar{\psi}_0(\mu_1). \end{aligned} \tag{20}$$

So when B_1^1 is given via the projection operator constructed with the aid of the vectors $(n)_i, (k)_i, (m)_i$ etc., we at once obtain the new solution $\tilde{L}$ or $\tilde{e}_-$, $\tilde{e}_+$ etc.

From equations (20) we express the components of the vectors (q_j, m_j) etc. in terms of (y, p) etc., and hence the elements of the B_1^1, A_1^1 matrices, so that we get

$$\begin{aligned} (S_1^1)_{ij} = &-\frac{\lambda_1-\mu_1}{2}\left\{\frac{1}{\alpha}\left(p_i\,y_j+\frac{\lambda_1-\mu_1}{\gamma}\,p_i\,z_j+\frac{\beta}{\gamma}\,y_j\,r_i\right)\right\} \\ &-\frac{\lambda_1-\mu_1}{2}\left\{\frac{1}{\alpha^2}\left(p_i\,y_j+\frac{\lambda_1-\mu_1}{\gamma}\,p_i\,z_j+\frac{\beta}{\gamma}\,y_j\,r_i\right)^2-\frac{8}{\alpha\gamma}(p_i r_i)(y_j\,z_j)\right\}^{\frac{1}{2}}. \end{aligned} \tag{21}$$

Now, as in the case of the Riemann–Hilbert problem with first order poles, we explicitly construct the vectors p_i, y_j etc.:

$$p_i = \begin{bmatrix} a_1\,\mathrm{e}^{-\mathrm{i}\mu_1 x+\mathrm{i}t\eta(a_1,\mu_1)} \\ a_2\,\mathrm{e}^{\mathrm{i}\mu_1 x+\mathrm{i}t\eta(a_2,\mu_1)} \\ a_3\,\mathrm{e}^{\mathrm{i}\mu_1 x+\mathrm{i}t\eta(a_3,\mu_1)} \end{bmatrix}, \tag{22}$$

$$y_j = [\bar{a}_1\,\mathrm{e}^{\mathrm{i}\lambda_1 x-\mathrm{i}t\eta(a_1,\lambda_1)},\ \bar{a}_2\,\mathrm{e}^{-\mathrm{i}\lambda_1 x-\mathrm{i}t\eta(a_2,\lambda_1)},\ \bar{a}_3\,\mathrm{e}^{-\mathrm{i}\lambda_1 x-\mathrm{i}t\eta(a_3,\lambda_1)}], \tag{23}$$

where (a_1, a_2, a_3) is any arbitrary vector and

$$\eta(a_i, \mu_i) = \frac{a_i}{2\mu_i-(\eta-\Delta)}. \tag{24}$$

We now use these in (21) to obtain

$$\tfrac{1}{2}\tilde{e}_+ = (B_1^1)_{13} = -\frac{\mathrm{i}\,a}{\alpha\gamma}\exp\left(-\frac{2a(\alpha_1+\alpha_3)+\mathrm{i}\,k(\alpha_1-\alpha_3)}{k^2+4a^2}\,t\right)$$

$$\times[a_1\,\bar{a}_3\,\gamma+2\mathrm{i}\,aa_1\,\bar{b}_3+b_1\,a_3\,\beta+\{(a_1\,\bar{a}_3\,\gamma+2\mathrm{i}aa_1\,\bar{b}_3+b_1\,\bar{a}_3\,\beta)^2$$

$$-8a_1\,\bar{a}_3\,b_1\,\bar{b}_3\,\alpha\gamma\}^{\frac{1}{2}}],$$

where

$$\alpha = a_1\,\bar{a}_1\exp\left(-2ax-\frac{4a\alpha_1}{k^2+4a^2}\,t\right)+a_2\,\bar{a}_2\exp\left(2ax-\frac{4a\alpha_2}{k^2+4a^2}\,t\right)$$

$$+a_3\,\bar{a}_3\exp\left(2ax-\frac{4a\alpha_3}{k^2+4a^2}\,t\right),$$

$$\beta = a_1\,\bar{b}_1\exp\left(-2ax-\frac{4a\alpha_1}{k^2+4a^2}\,t\right)+a_2\,\bar{b}_2\exp\left(2ax-\frac{4a\alpha_2}{k^2+4a^2}\,t\right)$$

$$+a_3\,\bar{b}_3\exp\left(2ax-\frac{4a\alpha_3}{k^2+4a^2}\,t\right),$$

$$\gamma = b_1\,\bar{b}_1\exp\left(-2ax-\frac{4a\alpha_1}{k^2+4a^2}\,t\right)+b_2\,\bar{b}_2\exp\left(2ax-\frac{4a\alpha_2}{k^2+4a^2}\,t\right)$$

$$+b_3\,\bar{b}_3\exp\left(2ax-\frac{4a\alpha_3}{k^2+4a^2}\,t\right).$$

We have written the form of the solution after some simplification as

$$\tfrac{1}{2}\tilde{e}_+ = \nu_3\Big/\left\{\nu_1\cosh\left(4ax+\frac{4a\alpha_1}{k^2+4a^2}\,t+\eta_1\right)+\exp\left(2ax-\frac{4a\alpha_1}{k^2+4a^2}\,t\right)\right.$$

$$\left.\times\cosh(2ax+\eta_2)+\nu_2\right\}, \tag{25}$$

where we have assumed $\alpha_1 = \alpha_3, \alpha_2 = 0, \bar{a}_3 = 0$. Similarly, we get

$$\tfrac{1}{2}\tilde{e}_- = (B_1^1)_{12} = -\frac{\mathrm{i}\,a}{\alpha\gamma}\exp\left(-\frac{2a(\alpha_1+\alpha_2)+\mathrm{i}\,k(\alpha_1-\alpha_2)}{k^2+4a^2}\,t\right)$$

$$\times[a_1\,\bar{a}_2\,\gamma+2\mathrm{i}\,aa_1\,\bar{b}_2+b_1\,\bar{a}_2\,\beta+\{(a_1\,\bar{a}_2\,\gamma+2\mathrm{i}\,aa_1\,\bar{b}_2$$

$$+\beta b_1\,\bar{a}_2)^2-8a_1\,\bar{a}_2\,b_1\,\bar{b}_2\,\alpha\gamma\}^{\frac{1}{2}}],$$

with the choice of $\alpha_1 = \alpha_3, \alpha_2 = 0, \bar{\alpha}_2 = 0$. This equation reduces to

$$\tfrac{1}{2}\tilde{e}_- = \nu_3' \exp\left(-\frac{\mathrm{i}\,k\alpha_1}{k^2+4a^2}\right)$$
$$\times \Big/ \left\{\nu_1' \cosh\left(4ax + \frac{6a\alpha_1}{k^2+4a^2}\,t + \eta_1'\right) + \nu_2' \exp\left(-\frac{6a\alpha_1}{k^2+4a^2}\,t\right)\right.$$
$$\left. + \exp\left(2ax - \frac{2a\alpha_1}{k^2+4a^2}\,t\right)\cosh(2ax+\eta_2')\right\}. \tag{26}$$

It may be noted that $\tilde{e}_-$ in the first case and $\tilde{e}_+$ in the second case can also be similarly calculated.

6. Conclusions

In our discussion we have obtained the profiles of the polarisation vectors in self-induced transparency with degenerate medium via a new formulation of the dressing operator approach of the Riemann–Hilbert problem with second order poles. It is important to note that the shape of the fields are no longer of the simple sech type, but as $x \to \infty$ they tend to zero. Also they do not have the 'one wavefront' structure pertaining to the one soliton solution. At this point it should be noted that the multitude of solutions with various peculiar features can be easily harnessed with the powerful Riemann–Hilbert technique. Our present result is also in conformity with the observation of Belinsky *et al.* (see Belinsky and Fraviglia 1982) that a higher order dressing through the Riemann–Hilbert approach leads not to a single simple soliton but a combination of solitons. This in fact is reflected in the form of the solutions written in equations (25) and (26). Lastly, we can mention that it has been shown by Novikov *et al.* (1984) that every solution of Riemann–Hilbert dressing is a solution of the nonlinear problem, so by that theorem it is quite evident that our solution (25) is also a solution of the DSIT problem.

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