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Painleve Test and Discrete Boltzmann Equations

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Abstract

The Painlevé test for various discrete Boltzmann equations is performed. The connection with integrability is discussed. Furthermore the Lie symmetry vector fields are derived and group-theoretical reduction of the discrete Boltzmann equations to ordinary differentiable equations is performed. Lie Bäcklund transformations are gained by performing the Painlevé analysis for the ordinary differential equations.

1. Introduction

In the case of pairwise collisions of particles, discrete models of the Boltzmann equation are given by a system of first order partial differential equations (Godunov and Sultangazin 1971) with a quadratic non-linearity on the right-hand side, i.e.

$$\frac{\partial f_i}{\partial t} + \vec{\Omega}_i \cdot \nabla f_i = \frac{1}{\varepsilon} \sum_{jkl=1}^{N} (\sigma_{kl}^{ij} f_k f_l - \sigma_{ij}^{kl} f_i f_j), \tag{1}$$

where $f_i(t,x)$ $(i = 1, \ldots, N)$ is the distribution function of the particles flying in the direction $\vec{\Omega}_i$, the parameter ε is an analogue of the free path length and the σ_{ij}^{kl} (σ_{kl}^{ij}) describe the reaction of the pairwise collisions. Equation (1) describes an abstract gas, the molecules of which have only finitely many velocities $\vec{\Omega}_1, \vec{\Omega}_2, \ldots, \vec{\Omega}_N$ and in collision take on one velocity from this set. The most studied two-velocity models are the Carleman model

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \frac{1}{\varepsilon}(v^2 - u^2),$$

$$\frac{\partial v}{\partial t} - \frac{\partial v}{\partial x} = \frac{1}{\varepsilon}(u^2 - v^2), \tag{2}$$

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and the McKean model

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}=\frac{1}{\varepsilon}(v^2-uv),$$
$$\frac{\partial v}{\partial t}-\frac{\partial v}{\partial x}=\frac{1}{\varepsilon}(-v^2+uv), \tag{3}$$

respectively. Here we consider these two models as well as the one-dimensional one by Broadwell (1964) who considered a so-called three-velocity model

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}=\frac{1}{\varepsilon}(v^2-uw),$$
$$\frac{\partial v}{\partial t}=-\frac{1}{2\varepsilon}(v^2-uw),$$
$$\frac{\partial w}{\partial t}-\frac{\partial w}{\partial x}=\frac{1}{\varepsilon}(v^2-uw). \tag{4}$$

The H-theorem is satisfied and the conservation laws are given by

$$\frac{\partial(u-w)}{\partial t}+\frac{\partial(u+w)}{\partial x}=0,$$
$$\frac{\partial(u+4v+w)}{\partial t}+\frac{\partial(u-w)}{\partial x}=0. \tag{5}$$

Under collision there occurs the reaction $A+B\to C+C$ and $C+C\to A+B$. Note that this model contains three discrete velocities ($\Omega_x=-1,0,1$) in one space dimension. The Carleman and McKean models both contain two discrete velocities $(+1,-1)$ with the reactions

$$A+A\to B+B,\qquad B+B\to A+A,$$
$$A+B\to A+A,\qquad A+A\to A+B,$$

respectively.

The Painlevé test (Weiss *et al.* 1983; Steeb and Euler 1988) for the models given by equations (2), (3) and (4) is performed and the connection with integrability is discussed. Furthermore we give solutions.

2. Carleman and McKean Models

Before studying the one-dimensional Broadwell model let us summarise the results for the two-velocity models given above (Steeb and Euler 1987). The Carleman and McKean models can be combined to the system of partial differential equations

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}=k_1v^2+k_2u^2+k_3uv,$$
$$\frac{\partial v}{\partial t}-\frac{\partial v}{\partial x}=-k_1v^2-k_2u^2-k_3uv, \tag{6}$$

where k_1, k_2 and k_3 are constants. Before performing the Painlevé test, let us describe certain properties of system (6). On inspection we find the conservation law

$$\frac{\partial(u+v)}{\partial t}+\frac{\partial(u-v)}{\partial x}=0. \tag{7}$$

System (6) is scale invariant under $t\rightarrow\varepsilon^{-1}t,\quad x\rightarrow\varepsilon^{-1}x,\quad u\rightarrow\varepsilon u,\quad v\rightarrow\varepsilon v$, i.e. it admits the Lie symmetry vector field

$$S=-t\partial/\partial t-x\partial/\partial x+u\partial/\partial u+v\partial/\partial v. \tag{8}$$

System (6) also admits the symmetry vector fields $X\equiv\partial/\partial x$ and $T\equiv\partial/\partial t$. The symmetry vector fields $\{S, T, X\}$ form a non-Abelian Lie algebra. Lie Bäcklund vector fields could not be found for the Carleman and McKean models, indicating that these models are not integrable. On the other hand, we find a hierarchy of Lie Bäcklund symmetry vector fields for the case $k_1=k_2=0$ (two wave interaction) (Steeb and Euler 1987). This hierarchy of Lie Bäcklund vector fields indicates that the system (6), with $k_1=k_2=0$, must be completely integrable (Steeb and Euler 1987). Since one conservation law is known, we can find a hierarchy of conservation laws with the help of the Lie point symmetries and Lie Bäcklund vector fields. This approach has been described by Steeb and Strampp (1982).

The space-independent case, namely

$$\frac{\mathrm{d}u}{\mathrm{d}t}=k_1v^2+k_2u^2+k_3uv,\qquad \frac{\mathrm{d}v}{\mathrm{d}t}=-k_1v^2-k_2u^2-k_3uv \tag{9}$$

is completely integrable where we have to impose the constraints $u\geq 0,\ v\geq 0$ with $u+v=1$. The first integral is given by $h[u, v]=u+v$. Then the integration of

$$\frac{\mathrm{d}u}{k_1v^2+k_2u_2+k_3uv}=\frac{\mathrm{d}v}{-k_1v^2-k_2u^2-k_3uv}=\frac{\mathrm{d}t}{1} \tag{10}$$

can easily be performed, where we have to distinguish between the case $k_1+k_2+k_3=0$ and $k_1+k_2+k_3\neq 0$. The first case includes the Carleman model.

Also the solution for the time-independent case can easily be found. Equation (9) passes the Painlevé test. The Kowalewski exponents (see Yoshida 1983*a*, 1983*b* for the definition) are given by $r_1=-1$ and $r_2=1$, where r_2 corresponds to the constant of motion (Steeb and Louw 1986). The resonances (see Ablowitz *et al.* 1980 for the definition) are the same.

Let us now perform the Painlevé test for system (6). Inserting the ansatz $u\propto u_0\phi^n$ and $v\propto v_0\phi^m$ yields $n=m=-1$ and

$$u_0=\frac{-(\phi_x-\phi_t)^2(\phi_t+\phi_x)}{k_1(\phi_t+\phi_x)^2+k_2(\phi_x-\phi_t)^2+k_3(\phi_x-\phi_t)(\phi_t+\phi_x)}, \tag{11a}$$

$$v_0=\frac{-(\phi_t+\phi_x)^2(\phi_t-\phi_x)}{k_1(\phi_t+\phi_x)^2+k_2(\phi_x-\phi_t)^2+k_3(\phi_x-\phi_t)(\phi_t+\phi_x)}. \tag{11b}$$

Owing to the conservation law (7), which can be written as $n_t + j_x = 0$, where $n = u + v$ (local density) and $j = u - v$ (local current density), we find, using the scaling invariance, the scaling behaviour $n[\varepsilon u, \varepsilon v] = \varepsilon n[u, v]$, $j[\varepsilon u, \varepsilon v] = \varepsilon j[u, v]$. Therefore, $r = 1$ (besides -1) is a Kowalewski exponent. The resonances are the same. Inserting the expansion

$$u = \phi^{-1} \sum_{j=0}^{\infty} u_j \phi^j, \qquad v = \phi^{-1} \sum_{j=0}^{\infty} v_j \phi^j \tag{12}$$

into system (6), we obtain at the resonance $r = 1$ the equations

$$\begin{aligned} u_{0t} + u_{0x} &= 2k_1 v_0 v_1 + 2k_2 u_0 u_1 + k_3 (u_0 v_1 + v_0 u_1), \\ v_{0t} - v_{0x} &= -2k_1 v_0 v_1 - 2k_2 u_0 u_1 - k_3 (u_0 v_1 + v_0 u_1). \end{aligned} \tag{13}$$

The expansion coefficient u_1 (or v_1) (which depends on x and t) cannot be chosen arbitrarily, since $u_{0t} + u_{0x} \neq -v_{0t} + v_{0x}$ in general. Rather, we find a constraint on ϕ from the requirement $u_{0t} + u_{0x} = -v_{0t} + v_{0x}$, namely

$$[k_1(\phi_t + \phi_x)^2 - k_2(\phi_x - \phi_t)^2] F[\phi] = 0, \tag{14}$$

where

$$F[\phi] \equiv \phi_t^2 \phi_{xx} + \phi_x^2 \phi_{tt} - 2\phi_x \phi_t \phi_{xt} = 0. \tag{15}$$

From equation (14) we draw the following conclusions. For $k_1 = k_2 = 0$, the system (6) passes the test and it is conjectured that it is integrable. This is in fact true because truncation at the constant level term (Weiss 1984) leads to the Bäcklund transformation

$$u(x, t) = (\partial/\partial t - \partial/\partial x) \ln \phi, \qquad v(x, t) = (-\partial/\partial t - \partial/\partial x) \ln \phi, \tag{16}$$

where ϕ satisfies the linear wave equation $\partial^2\phi/\partial t^2 - \partial^2\phi/\partial x^2 = 0$ with $k_3 = 1$. Since the general solution to the wave equation is well known, we find the general solution to system (6) when $k_1 = k_2 = 0$. For $k_1 \neq 0$ and (or) $k_2 \neq 0$, system (6) does not pass the Painlevé test. It is then conjectured that the system is not integrable. Notice that there are exceptions to this rule. In the present case, however, the conjecture is consistent with the fact that there are no Lie Bäcklund vector fields and no hierarchy of conservation laws for system (6) with $k_1 \neq 0$ or $k_2 \neq 0$.

Particular solutions can be found by investigating the constraint (15). Equation (15) is invariant under the Möbius group $\phi = (a\psi + b)/(c\psi + d)$ where $ad - bc = 1$ with the inverse transformation $\psi = (d\phi - b)/(-c\phi + a)$. Equation (15) is even invariant under $\phi = G(\psi)$, where G is a twice differentiable function, and admits the symmetry vector fields

$$\partial/\partial t, \quad \partial/\partial x, \quad x\partial/\partial t + t\partial/\partial x, \quad x\partial/\partial x + t\partial/\partial t + \phi\partial/\partial\phi. \tag{17}$$

Any plane wave $\phi(x, t) = g(kx - \omega t)$ is a solution to equation (15) with an arbitrary smooth function g. This is related to the fact that a group-theoretical reduction of system (6) via $u(x, t) = f_1(kx - \omega t)$, $v(x, t) = f_2(kx - \omega t)$ leads to a system of ordinary differential equations which pass the Painlevé test. Furthermore, we see that $\phi(x, t) = t/x$ [or $\phi(x, t) = x/t$] is a solution to equation (15). Consequently, the group-theoretical reduction of system (6) via

$$u(x, t) = \frac{1}{x}f_1(s), \qquad v(x, t) = \frac{1}{x}f_2(s) \tag{18}$$

($s = t/x$: similarity variable) yields a system of ordinary differential equations which pass the Painlevé test. The system is given by

$$\begin{aligned}(1-s)f_1' &= f_1 + k_1 f_2^2 + k_2 f_1^2 + k_3 f_1 f_2,\\ (1+s)f_2' &= -f_1 - k_1 f_2^2 - k_2 f_1^2 - k_3 f_1 f_2.\end{aligned} \tag{19}$$

All group-theoretical reductions of system (6) lead to systems of ordinary differential equations which pass the Painlevé test.

The general solution (Cauchy problem) is not known for the Carleman and McKean models. It is commonly believed that these equations are not integrable (Ernst 1981). This agrees with our findings that the system does not pass the Painlevé test and that there are no Lie Bäcklund vector fields.

The constraint (15) also arises in other nonintegrable field equations. It seems that constraint (15) plays a central rôle in nonintegrable field equations (Steeb and Euler 1987). We also find this constraint for the three-velocity model.

Equation (15) can be linearised by a Legendre transformation which is given by

$$\begin{aligned}\varepsilon = \phi_x, \quad x = W_\varepsilon, \quad \eta = \phi_t, \quad t = W_\eta,\\ \phi(x, t) + W(\varepsilon, \eta) = x\varepsilon + t\eta.\end{aligned} \tag{20}$$

It then follows that

$$\varepsilon^2 W_{\varepsilon\varepsilon} + 2\varepsilon\eta W_{\varepsilon\eta} + \eta^2 W_{\eta\eta} = 0. \tag{21}$$

The general solution to equation (21) is given by $W(\varepsilon, \eta) = G(\varepsilon/\eta) + \eta H(\varepsilon/\eta)$, where G and H are arbitrary smooth functions.

A particular solution of the Carleman model can be found as follows (Steeb and Grauel 1985). We put $S := u + v$, $D := u - v$ and $\varepsilon = 1$. Then the conservation law takes the form

$$S_t + D_x = 0. \tag{22a}$$

Moreover, we have

$$D_t + S_x = -2DS. \tag{22b}$$

Now we assume that $S_x - D_t = 0$. From this equation and from equation (22a) we obtain $S_{tt} + S_{xx} = 0$ and $D_{tt} + D_{xx} = 0$. Thus S and D satisfy Laplace's equation and $S_t + D_x = 0$, $S_x - D_t = 0$ can be viewed as Cauchy–Riemann equations. Consequently, we can put

$$w(z) = S(x, t) + iD(x, t), \tag{23}$$

where $z = x + it$ and w is an analytic function. To satisfy equation (22b) we set

$$\frac{dw}{dz} = \frac{i}{2}(w^2 + C), \tag{24}$$

where $C \in \mathbb{R}$ and $w^2 = S^2 - D^2 + 2iSD$. Consequently, the real and imaginary parts of any solution of equation (24) satisfy equations (22a) and (22b). Equation (24) is of Ricatti type and therefore has the Painlevé property (Hille 1976). Let us now construct solutions to (24) and therefore of (2). When we insert the ansatz

$$w = \phi^{-1} w_0 + w_1 + \phi w_2 + \phi^2 w_3 + \cdots, \tag{25}$$

and put $w_j = 0$ for $j \geq 2$ [cutoff at constant level term (Weiss 1984)], we find that ($\varepsilon = 1$)

$$w_0 = 2i\phi_z, \tag{26}$$

$$\frac{dw_0}{dz} = iw_0 w_1, \tag{27}$$

$$\frac{dw_1}{dz} = \frac{i}{2}(w_1^2 + C). \tag{28}$$

This means that w_1 satisfies equation (24). The three equations are compatible. If we put $w_1 = (-C)^{1/2}$, equation (28) is satisfied and it follows that

$$w = 2i\phi^{-1}\phi_z + (-C)^{1/2}. \tag{29}$$

This equation linearises (24), namely

$$\phi_{zz} - i(-C)^{1/2}\phi_z + \tfrac{1}{4}C\phi = 0. \tag{30}$$

This linear equation can easily be solved, where we have to distinguish between the case $C = 0$ and $C \neq 0$. Inserting the solution of (30) into (29), we obtain a solution of (24). Now the real and imaginary parts of the complex function w lead to S and D and finally to u and v. By a straightforward calculation we find (Wick 1984)

$$u(x, t) = \frac{1}{2}\,\frac{\operatorname{sgn}(\mu)\sinh(t - \ln|\mu|) + \sin x}{\operatorname{sgn}(\mu)\cosh(t - \ln|\mu|) - \cos x},$$

$$v(x, t) = \frac{1}{2}\,\frac{\operatorname{sgn}(\mu)\sinh(t - \ln|\mu|) - \sin x}{\operatorname{sgn}(\mu)\cosh(t - \ln|\mu|) - \cos x}, \tag{31}$$

where $|\mu| \le 2^{1/2}-1$. This condition arises because $u \ge 0, v \ge 0$ and, therefore, $S \ge 0$ and $S^2 - D^2 \ge 0$. It can easily be seen that another solution (Wick 1984) to (24) is given by $w(z) = 2i/z$, where $C = 0$. Steeb and Grauel (1985) found a Bäcklund transformation for the ordinary differential equations of the Carleman model and thus gained a special solution, via the similarity ansatz, for this model.

3. Three-velocity Model

We consider now the three-velocity model (4). The conservation laws are given by (5). Equation (4) is scale invariant under $t \to \varepsilon^{-1} t$, $x \to \varepsilon^{-1} x$, $u \to \varepsilon u$, $v \to \varepsilon v$, $w \to \varepsilon w$, i.e. system (4) admits the Lie symmetry vector field

$$S = -t\partial/\partial t - x\partial/\partial x + u\partial/\partial u + v\partial/\partial v + w\partial/\partial w. \tag{32}$$

Moreover, system (4) admits the Lie symmetry vector fields X and T. Lie Bäcklund vector fields cannot be found for system (4).

Let us now perform the Painlevé test for the three-velocity model (4). Inserting the ansatz $u \propto u_0\phi^n$, $v \propto v_0\phi^m$ and $w \propto w_0\phi^q$ into system (4) yields $n = m = q = -1$ and

$$u_0 = -4\varepsilon\phi_t^2 \frac{\phi_x - \phi_t}{\phi_x^2 + 3\phi_t^2},$$

$$v_0 = 2\varepsilon\phi_t \frac{\phi_x^2 - \phi_t^2}{\phi_x^2 + 3\phi_t^2},$$

$$w_0 = 4\varepsilon\phi_t^2 \frac{\phi_x + \phi_t}{\phi_x^2 + 3\phi_t^2}. \tag{33}$$

Owing to the conservation laws (5) we find that $r = 1$ (twofold) must be a Kowalewaski exponent. The resonances are the same. We obtain

$$\begin{pmatrix} \varepsilon(\partial u_0/\partial t + \partial u_0/\partial x) \\ -2\varepsilon\partial v_0/\partial t \\ \varepsilon(\partial w_0/\partial t - \partial w_0/\partial x) \end{pmatrix} = \begin{pmatrix} -w_0 & 2v_0 & -u_0 \\ -w_0 & 2v_0 & -u_0 \\ -w_0 & 2v_0 & -u_0 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix} \tag{34}$$

System (4) does not pass the Painlevé test since

$$\partial u_0/\partial t + \partial u_0/\partial x \neq -2\partial v_0/\partial t, \tag{35a}$$

$$\partial u_0/\partial t + \partial u_0/\partial x \neq \partial w_0/\partial t - \partial w_0/\partial x, \tag{35b}$$

$$-2\partial v_0/\partial t \neq \partial w_0/\partial t - \partial w_0/\partial x. \tag{35c}$$

From the requirements

$$\partial u_0/\partial t + \partial u_0/\partial x = -2\partial v_0/\partial t,$$

$$\partial u_0/\partial t + \partial u_0/\partial x = \partial w_0/\partial t - \partial w_0/\partial x \tag{36}$$

on ϕ, the following constraints are found

$$(\phi_x^2 + 3\phi_t^2)^{-2}(2\phi_x\phi_t + 3\phi_t^2 - \phi_x^2)F[\phi] = 0, \tag{37}$$

and

$$(\phi^2 + 3\phi_t^2)^{-2}\phi_t\phi_x F[\phi] = 0, \tag{38}$$

where $F[\phi]$ is given by (15). The requirement $-2\partial v_0/\partial t = \partial w_0/\partial t - \partial w_0/\partial x$ follows from the requirement (35a) and (35b). A group-theoretical reduction of system (4) via

$$u(x,t) = \frac{1}{x}f_1(s),$$
$$v(x,t) = \frac{1}{x}f_2(s),$$
$$w(x,t) = \frac{1}{x}f_3(s), \tag{39}$$

($s = t/x$: similarity variable) yields a system of ordinary differential equations which pass the Painlevé test. The system is given by

$$(1-s)f_1' = f_1 + \frac{1}{\varepsilon}(f_2^2 - f_1 f_3),$$
$$f_2' = -\frac{1}{2\varepsilon}(f_2^2 - f_1 f_3),$$
$$(1+s)f_3' = -f_3 + \frac{1}{\varepsilon}(f_2^2 - f_1 f_3). \tag{40}$$

System (40) passes the Painlevé test. We can find a Bäcklund transformation for the system. This is given by

$$f_1(s) = \varepsilon\phi'\frac{4(1+s)}{3+s^2}\phi^{-1} + f_{11},$$
$$f_2(s) = -\varepsilon\phi'\frac{2(1-s^2)}{3+s^2}\phi^{-1} + f_{21},$$
$$f_3(s) = \varepsilon\phi'\frac{4(1-s)}{3+s^2}\phi^{-1} + f_{31}, \tag{41}$$

where ϕ, f_{11}, f_{21} and f_{31} satisfy the equations

$$\varepsilon(1-s^2)(3+s^2)\phi'' - 8\varepsilon s\phi' = -(3+s^2)\phi'[(1-s)f_{11} + (1-s^2)f_{21} + (1+s)f_{31}],$$
$$(1-s)f_{11}' = f_{11} + \frac{1}{\varepsilon}(f_{21}^2 - f_{11}f_{31}),$$
$$f_{21}' = -\frac{1}{2\varepsilon}(f_{21} - f_{11}f_{31}),$$
$$(1+s)f_{31}' = -f_{31} + \frac{1}{\varepsilon}(f_{21}^2 - f_{11}f_{31}). \tag{42}$$

A special solution of system (40) can be given for $f_{11} = f_{21} = f_{31} = 0$. Then we have

$$\phi(s) = c_1 \left(2 \ln \frac{1+s}{1-s} - s + c_2 \right), \tag{43}$$

so that

$$\begin{aligned} f_1(s) &= \frac{4\varepsilon}{1-s} \left[2 \ln \frac{1+s}{1-s} - s + c_2 \right]^{-1}, \\ f_2(s) &= -2\varepsilon \left[2 \ln \frac{1+s}{1-s} - s + c_2 \right]^{-1}, \\ f_3(s) &= \frac{4\varepsilon}{1+s} \left[2 \ln \frac{1+s}{1-s} - s + c_2 \right]^{-1}, \end{aligned} \tag{44}$$

where c_1 and c_2 are constants of integration. Consequently, when taking into account (42), we have found a special solution for the one-space dimensional Broadwell model. The space-independent three-velocity model can be solved by quadratures as done for the two-velocity models.

Cornille (1987) studied the one-space dimensional Broadwell model. He found three classes of positive exact solutions by determining 'solitons' (one-dimensional shock wave solutions) and 'bisolitons' (two-dimensional, space plus time solutions) for the system.

4. Conclusions

We have shown that the Carleman model, McKean model and Broadwell model do not pass the Painlevé test. The 'nonpassing' of the Painlevé test agrees with the fact that the general solution (Cauchy problem) cannot be given. The constraints (see equation 15) which one finds at the resonance are the same for all three models. This constraint also appears in various nonintegrable relativistic field equations (Steeb and Euler 1988). Furthermore, the models studied do not admit Lie Bäcklund vector fields. We also gave the Lie symmetry vector fields and performed group-theoretical reductions. The resulting ordinary differential equations pass the Painlevé test and can be integrated.

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