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Constructing Large-basis Meson Wavefunctions from Perturbative Cavity Dynamics

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Abstract

Amplitudes for the mixing of the meson valence ground state with $O(g)$ states, $|q\bar{q}G\rangle$, are calculated using $S_{1/2}$ quarks and 1^+ (TE) gluon wavefunctions from perturbative cavity dynamics. Bag parameters are obtained for increasing basis size by fitting to π–ρ and K spectroscopy. The expected self-energy divergence appears to be regulated primarily by the strong coupling approaching zero. The K* and ϕ masses are also calculated.

1. Introduction

It is generally believed that large scale lattice simulations of quantum chromodynamics (QCD) will ultimately provide 'measurements' of non-perturbative quantities which are, as yet, incalculable directly from QCD. However, since lattice calculations are still in an early stage (Montvay 1987), there is considerable interest in the further development of phenomenological models of the non-perturbative regime of QCD, such as the bag (Chodes *et al.* 1974) and potential (Isgur and Karl 1979) models. There are also other non-perturbative techniques which have had some success in describing low evergy QCD phenomena such as the $1/N$ expansion (t'Hooft 1974) and, more recently, bi-local bosonisation (Praschifka *et al.* 1987).

These phenomenological models usually assume some confinement mechanism and fit model parameters to the hadron spectroscopy. The real test of any model constructed in this way is to predict the hadron dynamics from the underlying quark and gluon degrees of freedom with a minimum number of free parameters.

In this paper we concentrate on recent formulations of quark and gluon dynamics in the static cavity (bag) model for confinement—perturbative cavity dynamics (PCD) (Lee 1979; Close and Horgan 1980, 1981; Barnes *et al.* 1982; Close and Monaghan 1981; Barnes 1979; Hansson and Jaffe 1983). Using these techniques it now seems possible to calculate a range of processes which stem directly from confined quark–gluon dynamics; for example, loop effects, mixing calculations, glueball masses, etc.

Unfortunately, there is one main stumbling block for most applications of PCD, namely the question of the values of the model parameters to be used when describing the interaction of high-mode number quarks and gluons

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encountered, for example, in loop diagrams. As the energy of the constituents increases, one might expect the confinement radius and pressure to readjust to maintain equilibrium. It may also be necessary to allow for running of the effective strong coupling constant with mode energy.

It seems to us to be impossible to insert these effects into any PCD calculation of this sort without introducing an unreasonable amount of arbitrariness. We therefore take the view that, as a first step, it is more appropriate to use these techniques to calculate and improve ground-state wavefunctions by including higher modes and fitting to the hadron spectroscopy. In this way we hope to obtain a bound-state wavefunction with 'averaged' parameters that take the higher mode dynamics into account, which can then be used in further calculations of hadronic effects.

This paper is intended as a brief report on the techniques involved in, and results of, fitting the light mesons π–ρ and K to an O(g) expanded basis for $S_{1/2}$ quarks and 1^+ (TE) gluons (Barnes 1979). We hope to extend this work to include $P_{1/2}$ quarks and 1^- (TM) gluons, and ultimately to extend to O(g^2) states.

2. Perturbative Cavity Dynamics

The quark spinors for $J=\frac{1}{2}$ S and P modes confined to a spherical cavity, of radius R, are well known. To establish our notation, we write them down as

$$u_{nS}^{(\alpha)}(\boldsymbol{x})=(4\pi)^{-\frac{1}{2}}\begin{pmatrix} \mathrm{i}\, f_{nS}(r)\chi_\alpha \\ g_{nS}(r)\boldsymbol{\sigma}.\hat{\boldsymbol{r}}\chi_\alpha \end{pmatrix}, \qquad u_{nP}^{(\alpha)}(\boldsymbol{x})=(4\pi)^{-\frac{1}{2}}\begin{pmatrix} -\mathrm{i}\, g_{nP}(r)\boldsymbol{\sigma}.\hat{\boldsymbol{r}}\chi_\alpha \\ f_{nP}(r)\chi_\alpha \end{pmatrix}, \tag{1}$$

where χ_α is a two component Pauli spinor. The radial functions $f_{nL}(r)$ and $g_{nL}(r)$ are given by

$$f_{nL}(r)=R^{-\frac{3}{2}}N_{nL}\, j_0(p_{nL}\, r),$$

$$g_{nL}(r)=-R^{-\frac{3}{2}}N_{nL}\left(\frac{\omega_{nL}+\kappa m R}{\omega_{nL}-\kappa m R}\right)^{\frac{1}{2}} j_1(p_{nL}\, r), \tag{2}$$

where the shell-momentum is $p_{nL}=R^{-1}(\omega_{nL}^2-m^2R^2)^{1/2}$ and κ is the Dirac quantum number, with $\kappa=\pm1$ for $L=\mathrm{P,S}$. The mode numbers, ω_{nL}, parametrising the shell-energy, $E_{nL}=\omega_{nL}/R$, satisfy the eigen-equation imposed by the linear boundary condition

$$\tan\{(\omega_{nL}^2-m^2R^2)^{\frac{1}{2}}\}=\frac{\kappa(\omega_{nL}^2-m^2R^2)^{\frac{1}{2}}}{\omega_{nL}-\kappa m R+\kappa}, \tag{3}$$

and the normalisations are given by

$$N_{nL}=(\omega_{nL}^2-m^2R^2)[\{2\omega_{nL}(\omega_{nL}+\kappa)+mR\}\sin^{-2}\{(\omega_{nL}^2-m^2R^2)^{\frac{1}{2}}\}]^{-\frac{1}{2}}. \tag{4}$$

Anti-particle spinors are obtained according to the transformation property (up to some phase η)

$$v_{nL}^{(\alpha)}(\boldsymbol{x})=\eta C\gamma_0\, u_{nL}^{(\alpha)*}(\boldsymbol{x}), \qquad C=\mathrm{i}\gamma_2\gamma_0,$$

which gives

$$v_{nS}^{(\alpha)}(\boldsymbol{x}) = (4\pi)^{-\frac{1}{2}} \begin{pmatrix} g_{nS}(r)\boldsymbol{\sigma}.\hat{\boldsymbol{r}}\,\sigma_2\,\chi_\alpha \\ -\mathrm{i}\,f_{nS}(r)\sigma_2\,\chi_\alpha \end{pmatrix}, \quad v_{nP}^{(\alpha)}(\boldsymbol{x}) = (4\pi)^{-\frac{1}{2}} \begin{pmatrix} -f_{nP}(r)\,\sigma_2\,\chi_\alpha \\ -\mathrm{i}\,g_{nP}(r)\boldsymbol{\sigma}.\hat{\boldsymbol{r}}\,\sigma_2\,\chi_\alpha \end{pmatrix}, \tag{5}$$

for the choice $\eta = i$. The spinors (1) and (5) can be used to construct the quark field operator

$$\begin{aligned}\psi^f(\boldsymbol{x}, t)_i = \sum_{nL}\sum_{\alpha} \{ & b(f, nL, \alpha, i)\, u_{nL}^{(\alpha)}(\boldsymbol{x})\, \mathrm{e}^{-\mathrm{i}E_{nL}t} \\ & + d^\dagger(f, nL, \alpha, i)\, v_{nL}^{(\alpha)}(\boldsymbol{x})\, \mathrm{e}^{\mathrm{i}E_{nL}t}\},\end{aligned} \tag{6}$$

where $b(f, nL, \alpha, i)$ are the annihilation operators for an nL quark with flavour, spin and colour labels f, α and i respectively and satisfy the usual anti-commutation relations. A similar interpretation holds for the anti-quark operators $d^\dagger(f, nL, \alpha, i)$.

Gluon wavefunctions in the static cavity are constructed as one would in the analogous problem of electrodynamics (Lee 1979; Barnes *et al.* 1982; Barnes 1979). The $j = 1$ modes in the Coulomb gauge are

(i) $J^P = 1^+$: Transverse electric (TE)

$$\boldsymbol{A}_{nj_3}^{\mathrm{TE}}(\boldsymbol{x}) = \frac{1}{R} N_{k_n}^{\mathrm{TE}}\, j_1\left(\frac{k_n}{R}r\right) \boldsymbol{Y}_{11j_3}\,; \tag{7a}$$

(ii) $J^P = 1^-$: Transverse magnetic (TM)

$$\boldsymbol{A}_{nj_3}^{\mathrm{TM}}(\boldsymbol{x}) = \frac{1}{R} N_{k_n}^{\mathrm{TM}} \left\{\sqrt{\tfrac{2}{3}}\, j_0\left(\frac{k_n}{R}r\right) \boldsymbol{Y}_{10j_3} - \sqrt{\tfrac{1}{3}}\, j_2\left(\frac{k_n}{R}r\right) \boldsymbol{Y}_{12j_3}\right\}; \tag{7b}$$

where the mode numbers satisfy

$$\begin{aligned} j_0(k_n) &= \tfrac{1}{2} j_2(k_n): \qquad \mathrm{TE} \quad k_n = \{2\cdot 7437, \ldots\}, \\ j_1(k_n) &= 0: \qquad\qquad\quad \mathrm{TM} \quad k_n = \{4\cdot 4934, \ldots\}, \end{aligned} \tag{8}$$

and the normalisations are given by

$$\begin{aligned} N_{k_n}^{\mathrm{TE}} &= [|\, j_1(k_n)\{k_n(1 - 2/k_n^2)\}^{\frac{1}{2}}\,|\,]^{-1}, \\ N_{k_n}^{\mathrm{TM}} &= [|\, j_2(k_n)\{k_n\}^{\frac{1}{2}}\,|\,]^{-1}. \end{aligned} \tag{9}$$

The vector spherical harmonics $\boldsymbol{Y}_{jlm}$ for these modes are

$$\begin{aligned} \boldsymbol{Y}_{10m} &= (4\pi)^{-\frac{1}{2}}\, \hat{\boldsymbol{e}}_m, \\ \boldsymbol{Y}_{11m} &= -3\mathrm{i}(8\pi)^{-\frac{1}{2}}\, \hat{\boldsymbol{r}} \times \hat{\boldsymbol{e}}_m, \\ \boldsymbol{Y}_{12m} &= (8\pi)^{-\frac{1}{2}}\{\hat{\boldsymbol{e}}_m - 3(\hat{\boldsymbol{r}}.\hat{\boldsymbol{e}}_m)\hat{\boldsymbol{r}}\}, \end{aligned} \tag{10}$$

where $\hat{\boldsymbol{e}}_m$ are the spin-1 unit polarisation vectors.

The field operator then for the transverse gluon field in cartesian colour basis is just

$$\boldsymbol{A}^{a}(\boldsymbol{x}, t) = \sum_{T_i=\mathrm{TE,TM}} \sum_{nj_3} \{\boldsymbol{A}^{T_i}_{nj_3}(\boldsymbol{x})\, c^{a}(n,j_3)\, \mathrm{e}^{-\mathrm{i}E_n t} + \boldsymbol{A}^{T_i *}_{nj_3}(\boldsymbol{x})\, c^{a\dagger}(n,j_3)\, \mathrm{e}^{\mathrm{i}E_n t}\}, \tag{11}$$

with the annihilation and creation operators $c^{a}(n,j_3)$ and $c^{a\dagger}(n,j_3)$ satisfying the usual commutation relations.

The components of the interaction we are interested in are

(i) Transverse interaction

$$H_{\mathrm{T}} = -\int \mathrm{d}^3x : g\bar{\psi}\gamma . \tfrac{1}{2}\lambda^{a}\boldsymbol{A}^{a}\psi :; \tag{12}$$

(ii) Instantaneous Coulomb (Lee 1979)

$$H_{\mathrm{C}}(0) = \frac{1}{8\pi}\int\int \mathrm{d}^3x_1\, \mathrm{d}^3x_2\, \rho^{a}(\boldsymbol{x}_1, 0)\, G(\boldsymbol{x}_1, \boldsymbol{x}_2)\, \rho^{a}(\boldsymbol{x}_2, 0);$$

where the cavity Green's function is

$$G(\boldsymbol{x}_1, \boldsymbol{x}_2) = \frac{1}{|\,\boldsymbol{x}_1 - \boldsymbol{x}_2\,|} + \frac{1}{R}\left\{\sum_{l=0} \frac{l+1}{l}\left(\frac{x_1 x_2}{R^2}\right)^{l} P_l(\hat{\boldsymbol{x}}_1 . \hat{\boldsymbol{x}}_2) - 1\right\},$$

$$\rho^{a} =: g\psi^{\dagger}\tfrac{1}{2}\lambda^{a}\psi : . \tag{13}$$

The gluon self-couplings will not concern us here since they lead to corrections of higher order in g.

3. Wavefunction Mixing

In this work we simplify the calculation by considering only S-wave quarks and TE gluons as well as ignoring $|\,q\bar{q}q\bar{q}G\rangle$ states.* The wavefunction we wish to construct has the form

$$|\,q\bar{q}\rangle^{J} = |\,1\mathrm{S}\bar{1}\bar{\mathrm{S}}\rangle^{J} + \sum_{n\bar{n}n_g} \sum_{J_{q\bar{q}}} a^{J}_{J_{q\bar{q}}}(n,\bar{n},n_g)|\,n\mathrm{S}\bar{n}\bar{\mathrm{S}}G^{\mathrm{TE}}_{n_g}\rangle^{J}_{J_{q\bar{q}}}, \tag{14}$$

where $J_{q\bar{q}}$ is the angular momentum of the $q\bar{q}$ pair which for $J=1$ can be 0 or 1.

The amplitudes are given in time-ordered perturbation theory as

$$a^{J}_{J_{q\bar{q}}}(n,\bar{n},n_g) = \frac{{}^{J}_{J_{q\bar{q}}}\langle n\mathrm{S}\bar{n}\bar{\mathrm{S}}G^{\mathrm{TE}}_{n_g}|\,H_{\mathrm{T}}\,|\,1\mathrm{S}\bar{1}\bar{\mathrm{S}}\rangle^{J}}{E(1\mathrm{S}\bar{1}\bar{\mathrm{S}}) - E(n\mathrm{S}\bar{n}\bar{\mathrm{S}}G^{\mathrm{TE}}_{n_g})}. \tag{15}$$

* In a more complete treatment these states, arising from vacuum fluctuations, should be included as they give rise to Z-graph self-energy diagrams. Note that the states $|\,q\bar{q}GGG\rangle$ due to $\mathrm{O}(g)$ gluon self-coupling do not contribute to the wavefunction as they lead to disconnected $\mathrm{O}(g^2)$ energy shifts.

To order g^2, the energy of the new ground state (14) is simply

$$E^J = {}^J\langle 1\mathrm{S}\bar{1}\bar{\mathrm{S}}|\, H\,| q\bar{q}\rangle^J = E(1\mathrm{S}\bar{1}\bar{\mathrm{S}}) + {}^J\langle 1\mathrm{S}\bar{1}\bar{\mathrm{S}}|\, H_{\mathrm{C}}\,| 1\mathrm{S}\bar{1}\bar{\mathrm{S}}\rangle^J$$
$$+ \sum_{n\bar{n}n_{\mathrm{g}}} \sum_{J_{q\bar{q}}} a^J_{J_{q\bar{q}}}(n,\bar{n},n_{\mathrm{g}})^J\langle 1\mathrm{S}\bar{1}\bar{\mathrm{S}}|\, H_{\mathrm{T}}\,| n\mathrm{S}\bar{n}\bar{\mathrm{S}}G^{\mathrm{TE}}_{n_{\mathrm{g}}}\rangle^J_{J_{q\bar{q}}}\,. \qquad (16)$$

The first step in evaluating the mixing amplitudes and the ground state energy is the calculation of the diagrams shown in Figs 1*a*–1*e*. Since we have neglected $|\, q\bar{q}q\bar{q}G\rangle$ states we are left with only 'time-forward' self-energy contributions arising from the Bremsstrahlung diagrams Figs 1*a* and 1*b*. To be consistent with this picture we also ignore the Coulomb Z-graphs so that we only evaluate the diagrams of Figs 1*c*, 1*d* and 1*e*.

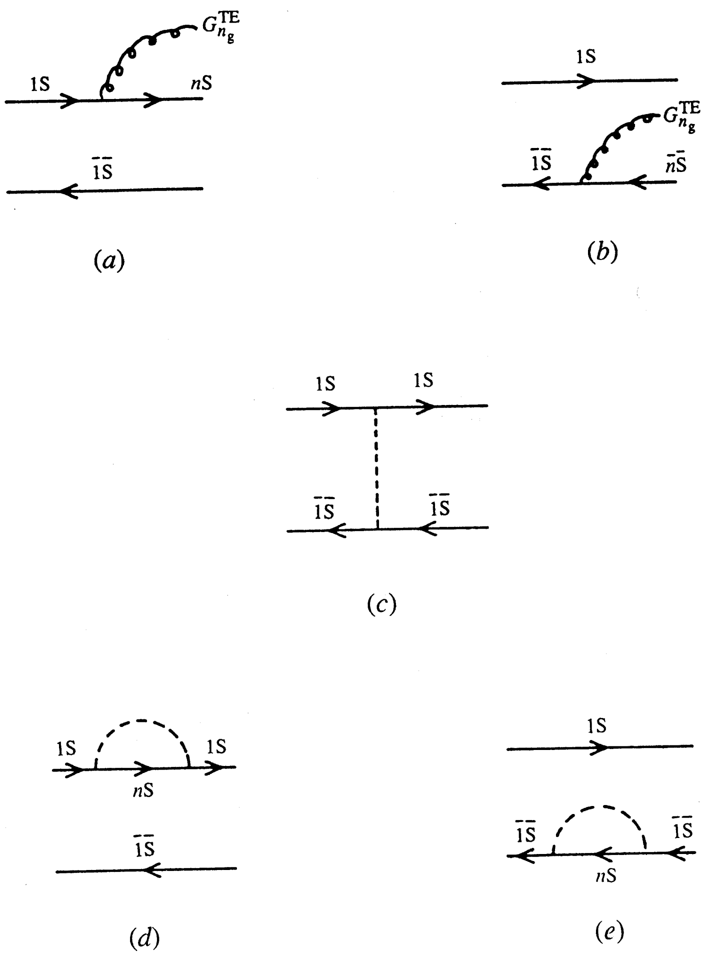


Fig. 1. (*a*) Transverse quark Bremsstrahlung. (*b*) Transverse anti-quark Bremsstrahlung. (*c*) Coulomb exchange. (*d*) Quark Coulomb self-energy. (*e*) Anti-quark Coulomb self-energy.

If one wishes to obtain realistic wavefunctions $|q\bar{q}\rangle$ to O(g), in a more complete treatment these effects, as well as the inclusion of P-wave quarks and TM gluons, must be considered. However, for the purposes of this work, it suffices to work in the approximation outlined above, since we are primarily interested in demonstrating the effects of wavefunction mixing in the simplest system possible.

The computation of the diagrams of Fig. 1 is accomplished using the quark and gluon field operators (6) and (11) in conjunction with the interactions H_{T} and H_{C}. One finds for the transverse graphs (Close and Monaghan 1981; Barnes 1979):

$$\text{(a)} \quad {}^{J}_{J_{q\bar{q}}}\langle n\mathrm{S}\bar{1}\bar{\mathrm{S}}G^{\mathrm{TE}}_{n_{\mathrm{g}}}|\,H_{\mathrm{T}}\,|\,1\mathrm{S}\bar{1}\bar{\mathrm{S}}\rangle^{J} = -\mathrm{i}\frac{1}{R}\left(\frac{8\alpha}{3}\right)^{\frac{1}{2}} S_q(J,J_{q\bar{q}})\,I_{\mathrm{T}}(1\mathrm{S},n\mathrm{S},n_{\mathrm{g}})_q; \tag{17}$$

$$\text{(b)} \quad {}^{J}_{J_{q\bar{q}}}\langle 1\mathrm{S}\bar{n}\bar{\mathrm{S}}G^{\mathrm{TE}}_{n_{\mathrm{g}}}|\,H_{\mathrm{T}}\,|\,1\mathrm{S}\bar{1}\bar{\mathrm{S}}\rangle^{J} = -\mathrm{i}\frac{1}{R}\left(\frac{8\alpha}{3}\right)^{\frac{1}{2}} S_{\bar{q}}(J,J_{q\bar{q}})\,I_{\mathrm{T}}(\bar{1}\bar{\mathrm{S}},\bar{n}\bar{\mathrm{S}},n_{\mathrm{g}})_{\bar{q}}; \tag{18}$$

where

$$S_{q,\bar{q}}(J,J_{q\bar{q}}) = \begin{cases} 1, & J=0 \\ -\sqrt{\tfrac{1}{3}}, & J=1, \quad J_{q\bar{q}}=0 \\ \mp\sqrt{\tfrac{2}{3}}, & J=1, \quad J_{q\bar{q}}=1 \end{cases} \tag{19}$$

(note that for $J=1$ and $J_{q\bar{q}}=1$ the S_q factor is negative whilst the $S_{\bar{q}}$ factor is positive) and α is the effective fine structure constant ($\alpha = g^2/4\pi$).

The transverse vertex integral is given by

$$I_{\mathrm{T}}(n\mathrm{S},n'\mathrm{S},n_{\mathrm{g}})_q = N^{\mathrm{TE}}_{k_{n_{\mathrm{g}}}} N_{n\mathrm{S}} N_{n'\mathrm{S}} \int_0^1 \xi^2\,\mathrm{d}\xi \left\{ \left(\frac{\omega_{n\mathrm{S}} - m_q R}{\omega_{n\mathrm{S}} + m_q R}\right)^{\frac{1}{2}} j_1(p_{n\mathrm{S}}R\xi)\,j_0(p_{n'\mathrm{S}}R\xi) + \left(\frac{\omega_{n'\mathrm{S}} - m_q R}{\omega_{n'\mathrm{S}} + m_q R}\right)^{\frac{1}{2}} j_1(p_{n'\mathrm{S}}R\xi)\,j_0(p_{n\mathrm{S}}R\xi) \right\} j_1(k_{n_{\mathrm{g}}}\xi)\,. \tag{20}$$

For the Coulomb diagrams one obtains:

$$\text{(c)} \quad {}^{J}\langle 1\mathrm{S}\bar{1}\bar{\mathrm{S}}|\,H_{\mathrm{C}}\,|\,1\mathrm{S}\bar{1}\bar{\mathrm{S}}\rangle^{J}|_{\mathrm{exchange}} = -\frac{4\alpha}{3}\frac{1}{R} I_{\mathrm{C}}(1\mathrm{S},1\mathrm{S}:\,\bar{1}\bar{\mathrm{S}},\,\bar{1}\bar{\mathrm{S}})_{q\bar{q}}; \tag{21}$$

$$\text{(d)} \quad {}^{J}\langle 1\mathrm{S}\bar{1}\bar{\mathrm{S}}|\,H_{\mathrm{C}}\,|\,1\mathrm{S}\bar{1}\bar{\mathrm{S}}\rangle^{J}|_{\mathrm{self}-q} = \tfrac{1}{2}\,\frac{4\alpha}{3}\frac{1}{R}\sum_n I_{\mathrm{C}}(1\mathrm{S},n\mathrm{S}:\,n\mathrm{S},1\mathrm{S})_{qq}; \tag{22}$$

$$\text{(e)} \quad {}^{J}\langle 1\mathrm{S}\bar{1}\bar{\mathrm{S}}|\,H_{\mathrm{C}}\,|\,1\mathrm{S}\bar{1}\bar{\mathrm{S}}\rangle^{J}|_{\mathrm{self}-\bar{q}} = \tfrac{1}{2}\,\frac{4\alpha}{3}\frac{1}{R}\sum_n I_{\mathrm{C}}(\bar{1}\bar{\mathrm{S}},\bar{n}\bar{\mathrm{S}}:\,\bar{n}\bar{\mathrm{S}},\bar{1}\bar{\mathrm{S}})_{\bar{q}\bar{q}}; \tag{23}$$

where the Coulomb integral is defined by

$$I_{\mathrm{C}}(n_1\,\mathrm{S},\,n_2\,\mathrm{S}:n_3\,\mathrm{S},n_4\,\mathrm{S})_{q_{\mathrm{a}}q_{\mathrm{b}}} \equiv \int_0^1 \xi_1^2\,\mathrm{d}\xi_1 \int_0^1 \xi_2^2\,\mathrm{d}\xi_2 \left(\frac{1}{\xi_>} - 1\right) A^{q_{\mathrm{a}}}_{n_1 n_2}(\xi_1 R)\,A^{q_{\mathrm{b}}}_{n_3 n_4}(\xi_2 R),$$

with

$$A^q_{mn}(x) \equiv N_{m\mathrm{S}}\, N_{n\mathrm{S}} \left\{ j_0(p_{m\mathrm{S}}\, x)\, j_0(p_{n\mathrm{S}}\, x) + \left(\frac{\omega_{m\mathrm{S}} - m_q R}{\omega_{m\mathrm{S}} + m_q R}\right)^{\frac{1}{2}} \left(\frac{\omega_{n\mathrm{S}} - m_q R}{\omega_{n\mathrm{S}} + m_q R}\right)^{\frac{1}{2}} j_1(p_{m\mathrm{S}}\, x)\, j_1(p_{n\mathrm{S}}\, x) \right\}. \tag{24}$$

The mode summations appearing in the self-energy terms are divergent and require some regulating procedure; this has been considered by several groups (Hansson and Jaffe 1983; Baacke *et al.* 1983*a*, 1983*b*) where a multiple reflection expansion is used to isolate the divergence.

We choose, instead, to regulate these divergent mode summations by introducing a cut-off in the form of truncating the basis size to $\{n, \bar{n}, n_{\mathrm{g}} \le N_{\mathrm{B}}\}$ and demanding that no loop-quark mode exceeds this limit. In this way the wavefunction for each N_{B} is consistent with the energy of the maximum mode number allowed. Divergences are isolated in the large N_{B} behaviour of the parameters. Since these parameters are adjusted, at a fixed N_{B}, to fit the data we believe that our procedure follows the spirit of the usual renormalisation procedure in (unconfined) quantum field theory.

4. Fitting Procedure

In terms of bag parameters the $|\, q\bar{q}\rangle$ ground-state energy (16), for spin state, J, takes the form

$$E^J_{N_{\mathrm{B}}} = \frac{1}{R}\left(\omega^q_{1\mathrm{S}} + \omega^{\bar{q}}_{1\mathrm{S}} + \frac{4\pi}{3} B R^4 - Z_0 + \alpha S^{N_{\mathrm{B}}}_{\mathrm{C}} + \alpha S^{N_{\mathrm{B}}}_{\mathrm{T},J}\right), \tag{25}$$

where B is the bag pressure and Z_0 is the parametrisation of the zero-point energy (DeGrand *et al.* 1975). The transverse and Coulomb energy corrections are respectively

$$\frac{1}{R}\,\alpha S^{N_{\mathrm{B}}}_{\mathrm{T},J} \equiv \sum_{n\bar{n}n_{\mathrm{g}}}^{N_{\mathrm{B}}} \sum_{J_{q\bar{q}}} \frac{\left|{}^J_{J_{q\bar{q}}}\langle n\mathrm{S}\bar{n}\bar{\mathrm{S}}G^{\mathrm{TE}}_{n_{\mathrm{g}}}|\, H_{\mathrm{T}}\,|\, 1\mathrm{S}\bar{1}\bar{\mathrm{S}}\rangle^J\right|^2}{E(1\mathrm{S}\bar{1}\bar{\mathrm{S}}) - E(n\mathrm{S}\bar{n}\bar{\mathrm{S}}G^{\mathrm{TE}}_{n_{\mathrm{g}}})},$$

$$\frac{1}{R}\,\alpha S^{N_{\mathrm{B}}}_{\mathrm{C}} \equiv \Big({}^J\langle 1\mathrm{S}\bar{1}\bar{\mathrm{S}}|\, H_{\mathrm{C}}\,|\, 1\mathrm{S}\bar{1}\bar{\mathrm{S}}\rangle^J|_{\mathrm{exchange}} + {}^J\langle 1\mathrm{S}\bar{1}\bar{\mathrm{S}}|\, H_{\mathrm{C}}\,|\, 1\mathrm{S}\bar{1}\bar{\mathrm{S}}\rangle^J|_{\mathrm{self}-q} + {}^J\langle 1\mathrm{S}\bar{1}\bar{\mathrm{S}}|\, H_{\mathrm{C}}\,|\, 1\mathrm{S}\bar{1}\bar{\mathrm{S}}\rangle^J|_{\mathrm{self}-\bar{q}}\Big)_{\text{Max loop mode}=N_{\mathrm{B}}} \tag{26}$$

The transverse energy-shift can be rewritten in the following manner. We first separate the first terms of the quark and anti-quark mode summations, i.e.

$$\frac{1}{R}\,\alpha S^{N_{\mathrm{B}}}_{\mathrm{T},J} = \sum_{J_{q\bar{q}}}\left(\sum_{n_{\mathrm{g}}}^{N_{\mathrm{B}}} \frac{\left|{}^J_{J_{q\bar{q}}}\langle 1\mathrm{S}\bar{1}\bar{\mathrm{S}}G^{\mathrm{TE}}_{n_{\mathrm{g}}}|\, H_{\mathrm{T}}\,|\, 1\mathrm{S}\bar{1}\bar{\mathrm{S}}\rangle^J\right|^2}{-E(G^{\mathrm{TE}}_{n_{\mathrm{g}}})} + \sum_{(n,\bar{n})>1}^{N_{\mathrm{B}}} \sum_{n_{\mathrm{g}}}^{N_{\mathrm{B}}} \frac{\left|{}^J_{J_{q\bar{q}}}\langle n\mathrm{S}\bar{n}\bar{\mathrm{S}}G^{\mathrm{TE}}_{n_{\mathrm{g}}}|\, H_{\mathrm{T}}\,|\, 1\mathrm{S}\bar{1}\bar{\mathrm{S}}\rangle^J\right|^2}{E(1\mathrm{S}\bar{1}\bar{\mathrm{S}}) - E(n\mathrm{S}\bar{n}\bar{\mathrm{S}}G^{\mathrm{TE}}_{n_{\mathrm{g}}})}\right). \tag{27}$$

The first term contains the gluon exchange piece and the $(n,\bar{n})=1$ term of the quark and anti-quark self-energies. We perform the sum over $J_{q\bar{q}}$, remembering that the $|\,1\mathrm{S}\bar{1}\bar{\mathrm{S}}G^{\mathrm{TE}}_{n_g}\rangle$ state receives contributions from Figs 1a and 1b and using the spin factors $S_{q,\bar{q}}(J,J_{q\bar{q}})$ accordingly, to obtain

$$\frac{1}{R}\,\alpha S^{N_{\mathrm{B}}}_{\mathrm{T},J}=\frac{8\alpha}{3}\,\frac{1}{R}\langle\sigma_1\,.\,\sigma_2\rangle^J\sum_{n_g}^{N_{\mathrm{B}}}\left(\tfrac{2}{3}\,\frac{I_{\mathrm{T}}(1\mathrm{S},1\mathrm{S},n_g)_q\,I_{\mathrm{T}}(\bar{1}\bar{\mathrm{S}},\bar{1}\bar{\mathrm{S}},n_g)_{\bar{q}}}{k_{n_g}}\right)$$
$$+\,\frac{8\alpha}{3}\,\frac{1}{R}\sum_{n,n_g}^{N_{\mathrm{B}}}\left(\frac{I^2_{\mathrm{T}}(1\mathrm{S},n\mathrm{S},n_g)_q}{\omega_{1\mathrm{S}}-\omega_{n\mathrm{S}}-k_{n_g}}+\frac{I^2_{\mathrm{T}}(\bar{1}\bar{\mathrm{S}},\bar{n}\bar{\mathrm{S}},n_g)_{\bar{q}}}{\omega_{\bar{1}\bar{\mathrm{S}}}-\omega_{\bar{n}\bar{\mathrm{S}}}-k_{n_g}}\right). \tag{28}$$

The above form clearly demonstrates the gluon exchange spin splitting and the spin-independent self-energy contributions.

The confinement radius R_0 is obtained by minimising the energy (25) with respect to R:

$$\left.\frac{\partial E^J_{N_{\mathrm{B}}}(R)}{\partial R}\right|_{R=R_0}=0\,. \tag{29}$$

Finally, the mass of the state is given by

$$M_J(R_0)=[\{E^J_{N_{\mathrm{B}}}(R_0)\}^2-\langle P^2\rangle]^{\frac{1}{2}}\,. \tag{30}$$

(Note that the physical observable, the mass, is independent of the cut-off N_{B} by construction.)

In the original MIT bag model fit (DeGrand *et al.* 1975) there were no centre-of-mass corrections. Consequently, the pion was difficult to fit since these effects are very important for this state. Ideally one would use Peierls–Yoccoz projection methods to calculate $\langle P^2\rangle$, but this procedure is lengthy. Several authors (Bartelski *et al.* 1984; Carlson *et al.* 1983) have simply put

$$\langle P^2\rangle_{1\mathrm{S}\bar{1}\bar{\mathrm{S}}}\equiv p^2_{1\mathrm{S}}+p^2_{\bar{1}\bar{\mathrm{S}}}\,, \tag{31}$$

which seems to accommodate the pion very well. We generalise this choice, so as to include the mixing of $|\,q\bar{q}G\rangle$ states, and put

$$\langle P^2\rangle^J_{N_{\mathrm{B}}}=N^2_J\left(\langle P^2\rangle_{1\mathrm{S}\bar{1}\bar{\mathrm{S}}}+\sum_{n\bar{n}n_g}\sum_{J_{q\bar{q}}}|\,a^J_{J_{q\bar{q}}}(n,\bar{n},n_g)|^2\langle P^2\rangle_{n\bar{n}n_g}\right), \tag{32}$$

where the generalisation of (31) is taken to be

$$\langle P^2\rangle_{n\bar{n}n_g}\equiv p^2_{n\mathrm{S}}+p^2_{\bar{n}\bar{\mathrm{S}}}+k^2_{n_g}/R^2 \tag{33}$$

(i.e. we sum $\langle P^2\rangle$ over occupied states weighting according to their probability) and the state normalisation is

$$N_J=\left(1+\sum_{n\bar{n}n_g}\sum_{J_{q\bar{q}}}|\,a^J_{J_{q\bar{q}}}(n,\bar{n},n_g)|^2\right)^{-\frac{1}{2}} \tag{34}$$

The strategy we employ in fitting the light mesons π–ρ and K is to first assume that the non-strange quark masses are zero and fit the π–ρ system. The $m_q = 0$ case is particularly simple because the confinement radius dependence in the mode numbers (3), and overlap integrals, (20) and (24), enters as mR [these results should not change appreciably for m_q up to ~10 MeV since for $R \sim 5\ \mathrm{GeV}^{-1}$ we still have $(mR)^2 \ll 1$]. Hence, from (25), the minimisation condition immediately yields

$$R_0^J = \left(\frac{2\omega_{1S} - Z_0 + \alpha S_C^{N_B} + \alpha S_{T,J}^{N_B}}{4\pi B}\right)^{\frac{1}{4}} . \tag{35}$$

Once the π–ρ system is fitted for some parameter set $(\alpha, B, Z_0)_{N_B}$, these values are then used to determine the strange quark, m_s, required to fit the kaon.

5. Results

We begin by considering the $N_B = 1$ case which is easily written down and is pedagogically useful. For $m_q = 0$ (and $q = \bar{q}$) the relevant overlap integrals are

$$I_T(1S, 1S, 1) = 0\cdot6030, \qquad I_C(1S, 1S; 1S, 1S) = 0\cdot2784. \tag{36}$$

Note that for $q = \bar{q}$, the first term of the Coulomb self-energy mode sum cancels the exchange contribution.

The matrix elements for $J = 0, 1$ states are

$$\begin{aligned} {}^{J}_{J_{q\bar{q}}}\langle 1S\bar{1}\bar{S}G_1^{TE}|\,H_T\,|\,1S\bar{1}\bar{S}\rangle^J &= -i(0\cdot6030)\,\frac{1}{R}\left(\frac{8\alpha}{3}\right)^{\frac{1}{2}} \{S_q(J, J_{q\bar{q}}) + S_{\bar{q}}(J, J_{qq})\}, \\ {}^{J}\langle 1S\bar{1}\bar{S}|\,H_C\,|\,1S\bar{1}\bar{S}\rangle^J\big|_{N_B=1} &= 0, \end{aligned} \tag{37}$$

where the spin factors in the transverse matrix element add due to the two possible Bremsstrahlung diagrams, Figs 1a and 1b, which reach the same state for $N_B = 1$. For the mixing amplitudes we then obtain

$$a^J_{J_{q\bar{q}}}(1, \bar{1}, 1) = i\left(\frac{8\alpha}{3}\right)^{\frac{1}{2}} \begin{cases} 0\cdot4396, & J = 0, \\ -0\cdot2538, & J = 1, \quad J_{q\bar{q}} = 0 \\ 0, & J = 1, \quad J_{q\bar{q}} = 1 \end{cases} \tag{38}$$

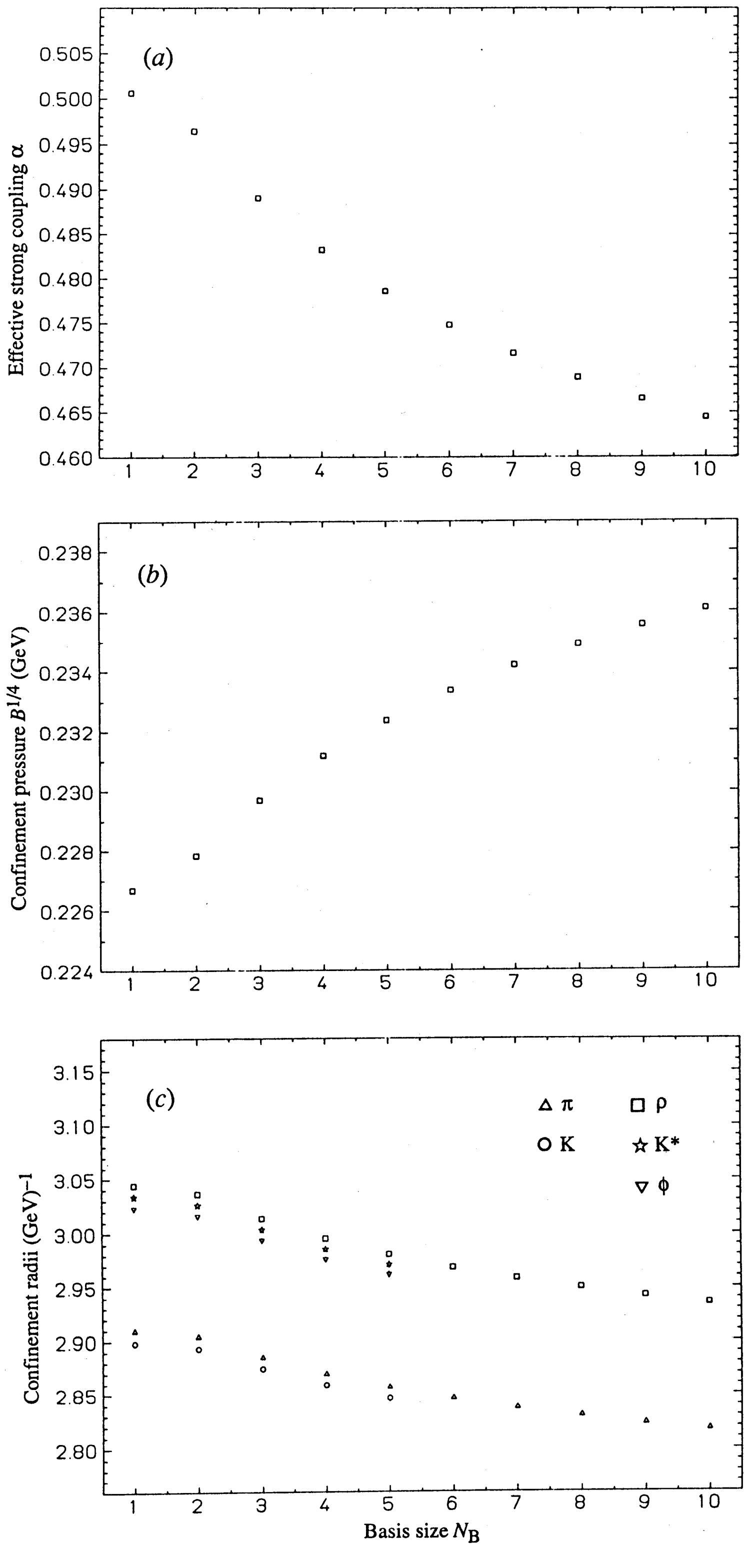
and the transverse energy contributions are

$$\frac{1}{R}\,\alpha S_{T,J}^{N_B} = -\frac{1}{R}\,\frac{8\alpha}{3} \begin{cases} 0\cdot5301, & J = 0 \\ 0\cdot1767, & J = 1 \end{cases} \tag{39}$$

which demonstrates the splitting of scalar and vector states.

The transverse integrals are easily checked by considering the gluon exchange contribution to the transverse energy shift. For example, taking $J = 0$ and 1, this gives

$$E_{\text{exchange}} = \frac{4\alpha}{3}\,\frac{1}{R}\langle\sigma_1 . \sigma_2\rangle^J(0\cdot1767), \tag{40}$$


(a)
Effective strong coupling α
0.505
0.500
0.495
0.490
0.485
0.480
0.475
0.470
0.465
0.460
1 2 3 4 5 6 7 8 9 10
(b)
Confinement pressure $B^{1/4}$ (GeV)
0.238
0.236
0.234
0.232
0.230
0.228
0.226
0.224
1 2 3 4 5 6 7 8 9 10
(c)
Confinement radii $(\mathrm{GeV})^{-1}$
△ π
□ ρ
○ K
☆ K*
▽ φ
3.15
3.10
3.05
3.00
2.95
2.90
2.85
2.80
1 2 3 4 5 6 7 8 9 10
Basis size N_B

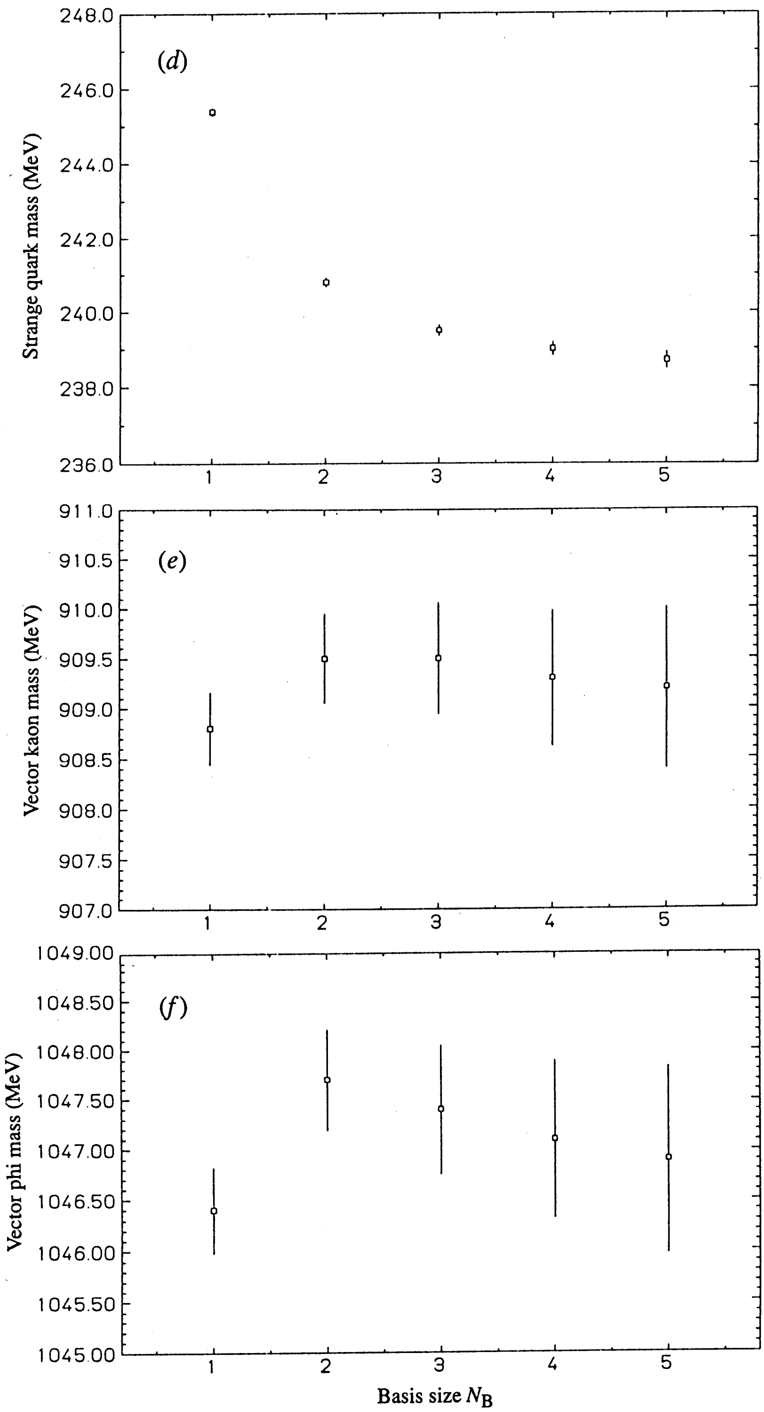


Fig. 2. (*a*)–(*d*) Behaviour of the fitting parameters with the cut-off parameter N_B. (*e*) Vector kaon mass K*(892). (*f*) Vector phi mass $\phi(1020)$.

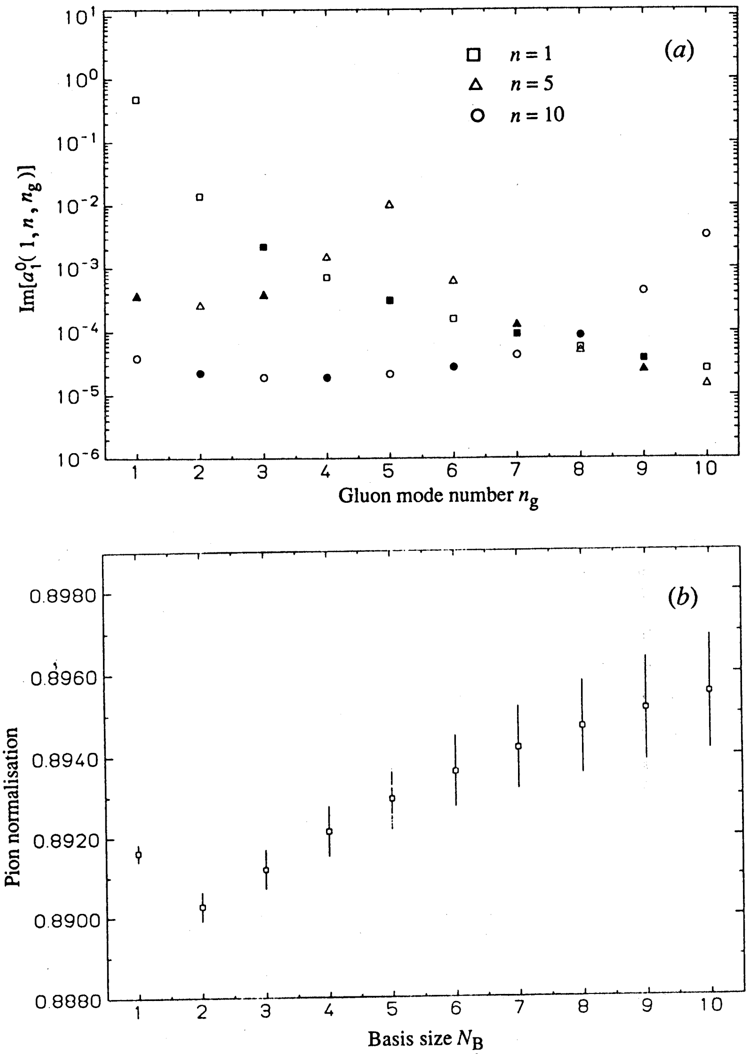


Fig. 3. (*a*) Pion wavefunction amplitudes ($N_B = 10$) for $n = 1$, 5 and 10 as a function of n_g. Negative values are plotted as shaded points. (*b*) Pion wavefunction normalisation as a function of basis size N_B.

which is the first term in the usual mode sum (Close and Monaghan 1981; Barnes 1984).

A check of the Coulomb integrals can be done by performing the integration without the $-R^{-1}$ term in the Coulomb Green's function. The result for the exchange term then agrees with other authors (Barnes 1984) who omitted the constant in $G(\boldsymbol{x}_1, \boldsymbol{x}_2)$; since we are considering self-energy diagrams we keep the full cavity Green's function .

It remains to find parameter sets $(\alpha, B, Z_0)_{N_B}$ which fit π–ρ and then fit K with m_s. Since there is actually a range of Z_0 values here $(-1 \cdot 5 \sim 1 \cdot 5)$ which will fit the light mesons we have chosen to set the zero point parameter,

Z_0, to some reasonable value,* $Z_0 = 1$, and investigate the evolution of the remaining parameters with N_B for this simple system.

The results† for the $Z_0 \equiv 1$ system parameters are plotted in Figs $2a$–$2d$ whilst the predictions for K* and ϕ are shown in Figs $2e$ and $2f$. The wavefunction mixing amplitudes and normalisation, for the pion only, are shown in Figs $3a$ and $3b$.

6. Discussion and Conclusions

From the large N_B behaviour of the bag parameters shown in Figs $2a$–$2d$ for our simple $Z_0 = 1$ system, we see that the self-energy divergences are being regulated primarily by α approaching zero. The bag pressure and confinement radius appear to be locked together in maintaining a regulated volume energy with opposing N_B dependence.

The minimisation of the bag energy for the strange system, K–K* and ϕ, is less straightforward than the π–ρ ($m_q = 0$) case since the overlap integrals for $m_q = 0$ depend on R. We have fitted m_s to K(496) up to $N_B = 5$.

An interesting feature is the behaviour of m_{K^*} and m_ϕ with N_B. Since we have set $Z_0 = 1$ and fitted m_s with m_K, the K* and ϕ masses are not fixed at each N_B; rather, these quantites are subject to the way the system responds to the regularisation procedure. It appears that the predictions for m_{K^*} and m_ϕ are independent of N_B to within the errors quoted, although taking $Z_0(N_B)$ not constant may change this. The fact that these masses are about 2% above the data may indicate a lower value of α is required for the higher momentum stange quark.

In Fig. $3a$ we see that the wavefunction Bremsstrahlung amplitudes are a maximum when $n = n_g$ and appear to oscillate uniformly either side of this maximum. The Fock state normalisation still appears to have some residual N_B behaviour, though very slight.

To construct a more realistic wavefunction to O(g) we would have to include P-wave quark and TM gluon modes as well as the Z-graphs we have so far ignored. The calculations presented here give us the confidence to proceed to more complex situations and use the resulting wavefunctions to compute various hadronic effects involving the light mesons.

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* The original MIT fit used $Z_0 = 1 \cdot 84$ whilst the fit of Bartelski *et al.* (1984) with centre-of-mass corrections had $Z_0 = 0 \cdot 76$.

† The errors shown are our estimate of the numerical errors involved in the integrations and the minimisation of the bag energy for the strange quark systems.

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