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Covariant Response Tensors for Spin Zero and Spin One Boson/Anti-Boson Plasmas

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Abstract

A fully covariant and general expression for the relation between the 4-current and the 4-potential for spin 0 and spin 1 particle/anti-particle plasmas is presented. The response tensors obtained are compared with the spin $\frac{1}{2}$ case, and are shown to reproduce the classical expressions in the appropriate limit. The longitudinal and transverse parts of the linear response tensor are evaluated for an isotropic particle/anti-particle distribution, and explicit expressions are obtained for the completely degenerate limit. The dispersion equations for longitudinal and transverse waves are derived for a completely degenerate plasma. Three transverse wave modes are found as the degenerate limit is approached; one mode is identified as the familiar transverse mode of any isotropic plasma, a second mode is an analogue of the 'pair branch' for longitudinal waves, and the third mode is referred to as the 'roton-like' mode due to the form of its dispersion relation.

1. Introduction

Our purpose in this paper is to derive the linear response tensor for spin 0 and spin 1 boson/anti-boson plasmas. Our motivation for the calculation is twofold. The spin $\frac{1}{2}$ case has been treated in detail, e.g., by Tsytovich (1961), Hayes and Melrose (1984) (hereinafter referred to as (HM)) and by Kowalenko *et al.* (1985) (hereinafter referred to as (KFH)). It is thus an interesting formal problem to calculate the spin 0 and spin 1 cases and compare them with the spin $\frac{1}{2}$ case. The spin 0 case has been discussed by, e.g., Hore and Frankel (1975), Hines and Frankel (1978) and by (KFH), who reviewed earlier work. The longitudinal part of the dielectric tensor has been calculated for a gas of spin 0 bosons at zero temperature. The spin 1 case does not appear to have been discussed previously. We seek to derive a general expression for the response tensor that allows us to rederive the known results for longitudinal waves and also to treat transverse waves. The second motivation for the calculations comes from possible applications in astrophysics. Pions are spin 0 bosons and as discussed in (KFH) a pion plasma should exist in the interior of neutron stars. There has also been some recent work on boson stars (e.g., Colpi *et al.* 1986). If such stars are shown to exist, then the response tensors for the plasmas of which the stars are composed would be of relevance. However, this application is not discussed further here.

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The method used here is an averaged propagator approach discussed by Tsytovich (1961) and modified by Melrose (1974). In this method the effect of the plasma is included in the calculation of the averaged particle propagator. One then evaluates the Feynman diagrams for the vacuum polarisation with the averaged propagator replacing the usual particle propagator. The method may be generalised to higher order (i.e., nonlinear) response tensors (Melrose 1974) but here we calculate only the linear response tensor. This method gives expressions for the response tensors which are covariant, take full account of relativistic effects, and are valid for any spatially homogeneous particle/anti-particle distribution.

There are two notable differences between the boson and fermion cases. One is in the detailed form of the averaged propagator. The other is in the inclusion of additional Feynman diagrams in the boson case; these are the so-called 'seagull' diagrams.

In Section 2 the averaged propagator for a spin 0 plasma is calculated, and this is used in Section 3 to calculate the linear response tensor for a spin 0 plasma. In Section 4 we specialise to the case of an isotropic plasma and evaluate the longitudinal and transverse parts of the response tensor. In Section 5 the dispersion relations for longitudinal and transverse waves are derived for a completely degenerate plasma. In Section 6 all our results for the spin 0 case are rederived for the spin 1 case. In Section 7 we discuss the results.

We use the standard 4-tensor notation of Berestetskii *et al.* (1971), Jackson (1975) and Melrose (1982) with $\hbar = c = 1$.

2. Averaged Propagator for Spin 0

The first step in the evaluation of the response is the inclusion of the plasma effects in the particle propagator. Although the derivation here applies to any charged spin 0 particle, for clarity we refer specifically to charged pions with π^+ being the anti-particle and π^- the particle. The derivation closely follows that for the electron propagator given in (HM), and we comment only on the steps where the boson case differs from the spin $\frac{1}{2}$ case.

The pion propagator is defined by

$$G(x-x') = -i\,\mathrm{Tr}\left(\hat{\rho}_0\,\mathrm{T}\,[\hat{\psi}(x)\,\hat{\psi}^{\dagger}(x')]\right), \tag{1}$$

where T is the boson time ordering operator:

$$\mathrm{T}\left[\hat{\phi}(x)\,\hat{\phi}(x')\right] := \begin{cases} \hat{\phi}(x)\,\hat{\phi}(x') & \text{for } t' < t, \\ \hat{\phi}(x')\,\hat{\phi}(x) & \text{for } t' > t. \end{cases} \tag{2}$$

Equation (1) is valid for the vacuum state $|\,0\rangle$ with $\hat{\rho}_0 = |\,0\rangle\langle 0\,|$. In the presence of a pion plasma $\hat{\rho}_0$ is replaced by

$$\hat{\rho} = \sum_q |\,n_q^+, n_q^-\rangle\langle n_q^+, n_q^-\,|\,, \tag{3}$$

where n_q^+ and n_q^- are the occupation numbers of particles and anti-particles, respectively, in states with quantum numbers denoted collectively by q. The + sign refers to the particles (π^-) and the − sign to the anti-particles (π^+). For the spin 0 case the adjoint wavefunction $\psi^\dagger$ is the complex conjugate of ψ.

We assume plane wave solutions, so that q in (3) is interpreted as the continuous quantum number $\boldsymbol{p}$. We write the second quantised wavefunctions in the form (Berestetskii *et al.* 1971)

$$\psi_q^\xi(x) = \frac{1}{\sqrt{2\varepsilon_p}} \exp(i\xi \boldsymbol{p} \cdot \boldsymbol{x}), \tag{4}$$

with $\xi = \pm 1$ (+1 for π^- and −1 for π^+). The energy eigenvalues are $\xi\varepsilon$ with $\varepsilon = [m^2 + |\boldsymbol{p}|^2]^{1/2}$, and the occupation number is $n^\xi(\xi\boldsymbol{p})$. Note that we identify the 4-momentum as ξp in (4) so that $\boldsymbol{p}$ is the physical 3-momentum for both particles and anti-particles. It is convenient to introduce the 4-dimensional occupation numbers $N(P)$, as in (HM):

$$N(P) = \sum_\xi \frac{2\pi m}{\varepsilon}\, \delta(P^0 - \xi\varepsilon)\, n^\xi(\boldsymbol{P}). \tag{5}$$

The averaged propagator in the presence of a spin 0 plasma then reduces to

$$G(P) = \left[\frac{1}{P^2 - m^2 + i0} - \frac{iN(P)}{2m}\right]. \tag{6}$$

In the absence of the plasma ($N(P) = 0$), the propagator (6) reproduces the standard form, e.g., as given by Itzykson and Zuber (1980), apart from a difference of a factor of i in notation. The notable difference from the spin $\frac{1}{2}$ case is in the sign of the term involving $N(P)$, and this difference may be attributed to the difference between Bose and Fermi statistics.

3. Response Tensor for a Spin Zero Plasma

The linear response tensor is defined by the relation between the 4-current $J^\mu = [\rho, \boldsymbol{J}]$ and the 4-potential $A^\mu = [\phi, \boldsymbol{A}]$:

$$J^\mu(k) = \alpha^{\mu\nu}(k) A_\nu(k). \tag{7}$$

The method used to calculate the linear response tensor $\alpha^{\mu\nu}(k)$ was described by Melrose (1974). For the spin $\frac{1}{2}$ case one uses QED and there is only one type of photon–particle vertex. From the Feynman amplitude for the bubble diagram (Fig. 1) one finds (HM)

$$\alpha^{\mu\nu}_{(1/2)}(k) = -ie^2 \mathrm{Sp} \int \frac{d^4P}{(2\pi)^4}\, \gamma^\mu \bar{G}(P) \gamma^\nu \bar{G}(P-k). \tag{8}$$

Only terms arising from the product of the resonant part of one propagator and the non-resonant part of the other are to be retained in (8).

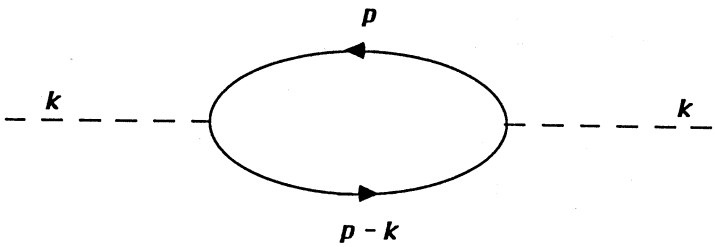

Fig. 1. The 'bubble' diagram.

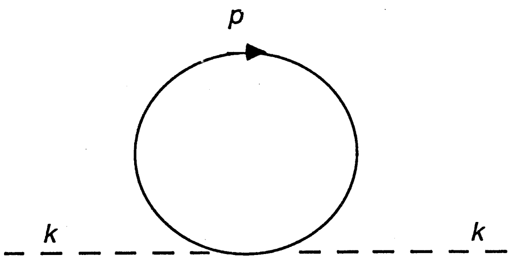

Fig. 2. The contracted 'seagull' diagram.

For the spin 0 gas there are two types of vertex, the usual one-photon vertex and a vertex corresponding to a 'seagull' diagram (e.g., Scadron 1979). The contribution from the contracted seagull diagram (Fig. 2) needs to be added to that from the bubble diagram (Fig. 1) for bosons. The bubble diagram contributes

$$\alpha^{\mu\nu}_{\rm bub}(k) = \frac{e^2}{2m}\int \frac{d^4P}{(2\pi)^4}\,(P^\mu + P'^\mu)\,(P^\nu + P'^\nu)\left[\frac{N(P)}{P'^2 - m^2} + \frac{N(P')}{P^2 - m^2}\right], \tag{9}$$

with $P' = P - k$. The seagull diagram contributes

$$\alpha^{\mu\nu}_{\rm gull}(k) = -\frac{e^2}{m}\,g^{\mu\nu}\int \frac{d^4P}{(2\pi)^4}\,N(P). \tag{10}$$

The total response tensor is the sum of the contributions (9) and (10):

$$\alpha^{\mu\nu}_{(0)}(k) = \frac{e^2}{2m}\int \frac{d^4P}{(2\pi)^4}\left\{F^{\mu\nu}_{(0)}(P,P')\left[\frac{N(P)}{P'^2 - m^2} + \frac{N(P')}{P^2 - m^2}\right] - 2g^{\mu\nu}N(P)\right\}, \tag{11}$$

with

$$F^{\mu\nu}_{(0)}(P,P') := (P^\mu + P'^\mu)\,(P^\nu + P'^\nu). \tag{12}$$

Making the substitution $P \to P+k$ in the term involving $N(P') \to N(P)$, one obtains the alternative form

$$\alpha^{\mu\nu}_{(0)}(k) = \frac{e^2}{2m} \int \frac{d^4P}{(2\pi)^4} N(P) \left[\frac{F^{\mu\nu}_{(0)}(P,P')}{P'^2 - m^2} + \frac{F^{\mu\nu}_{(0)}(P,P'')}{P''^2 - m^2} - 2g^{\mu\nu} \right], \tag{13}$$

with $P'' = P+k$. As noted in (HM), in (13) $2N(P)$ may be replaced by $N(P)+N(-P)$. One check on the expression (13) is that it reproduce the relevant classical limit, which in the present notation is (Melrose 1982)

$$\alpha^{\mu\nu}(k) = \frac{e^2}{2m} \int \frac{d^4P}{(2\pi)^4} N(P)\, a^{\mu\nu}(k,P/m), \tag{14}$$

with

$$\alpha^{\mu\nu}(k,P/m) = g^{\mu\nu} - \frac{(P^\mu k^\nu + P^\nu k^\mu)}{Pk} + \frac{k^2\, P^\mu P^\nu}{(Pk)^2}. \tag{15}$$

The classical limit corresponds to $k \ll P$. Taking this limit involves making the approximation

$$\frac{1}{P'^2 - m^2} \approx -\frac{1}{2(Pk)} \left[1 + \frac{k^2}{2(Pk)} \right],$$

and expanding $F^{\mu\nu}_{(0)}(P,P')$ to first order in k. A second consistency check is that the response tensor (13) satisfy the gauge-invariance and charge-continuity relations (Melrose 1974)

$$k_\mu \alpha^{\mu\nu}(k) = 0, \qquad k_\nu \alpha^{\mu\nu}(k) = 0. \tag{16}$$

4. An Isotropic Spin Zero Plasma

For an isotropic plasma there is a unique inertial frame, specifically the rest frame of the plasma. In this frame the response tensor may be separated into longitudinal and transverse parts. Melrose (1982) developed a method for separating $\alpha^{\mu\nu}(k)$ into longitudinal and transverse parts, $\alpha^l(k)$ and $\alpha^t(k)$, respectively, which are both Lorentz invariant. Before making this separation it is convenient to write the response tensor in an alternative form.

It is shown in Appendix A that for an isotropic plasma (11) may be rewritten in the form

$$\alpha^{\mu\nu}_{(0)}(k) = \frac{e^2}{2m} \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{n(\boldsymbol{p})}{\varepsilon} \times \left[\frac{F^{\mu\nu}_{(0)}(\varepsilon, \boldsymbol{p}; \varepsilon - \omega, \boldsymbol{p} - \boldsymbol{k})}{(\omega - \varepsilon - \varepsilon')\,(\omega - \varepsilon + \varepsilon')} + \frac{F^{\mu\nu}_{(0)}(-\varepsilon, \boldsymbol{p}; -\varepsilon - \omega, \boldsymbol{p} - \boldsymbol{k})}{(\omega + \varepsilon + \varepsilon')\,(\omega + \varepsilon - \varepsilon')} - 2g^{\mu\nu} \right], \tag{17}$$

with

$$\varepsilon' := [m^2 + |\boldsymbol{p} - \boldsymbol{k}|^2]^{1/2}. \tag{18}$$

In (17) we have changed notation by writing the arguments of $F^{\mu\nu}_{(0)}$ in the form

$$F^{\mu\nu}_{(0)}(P,Q) = F^{\mu\nu}_{(0)}(P^0, \boldsymbol{P}; Q^0, \boldsymbol{Q}).$$

The longitudinal and transverse parts are derived here using a form of the projection method given by Melrose (1982). The 00-component of $\alpha^{\mu\nu}(k)$ is related to the longitudinal part by

$$\alpha^l(k) = -(\omega^2/|\boldsymbol{k}|^2)\alpha^{00}(k). \tag{19}$$

The trace of the tensor then enables one to find the transverse part through

$$\alpha^t(k) = \tfrac{1}{2}\left[\alpha^\mu{}_\mu(k) + \frac{\omega^2 - |\boldsymbol{k}|^2}{|\boldsymbol{k}|^2}\,\alpha^{00}(k)\right]. \tag{20}$$

The resulting explicit expression for $\alpha^l_{(0)}(k)$ involves $\varepsilon' = [m^2 + |\boldsymbol{p}-\boldsymbol{k}|^2]^{1/2}$ and $\varepsilon'' = [m^2 + |\boldsymbol{p}+\boldsymbol{k}|^2]^{1/2}$. By applying the transformation $\boldsymbol{p} \rightarrow -\boldsymbol{p}$ one interchanges these two quantities, allowing one to express the result in a form involving only one of them. Such a form is

$$\alpha^l_{(0)}(k) = -\frac{e^2\omega^2}{2|\boldsymbol{k}|^2}\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\,\frac{n(\boldsymbol{p})}{\varepsilon}\left[\frac{(2\varepsilon-\omega)^2}{(\omega-\varepsilon)^2-\varepsilon'^2} + \frac{(2\varepsilon+\omega)^2}{(\omega+\varepsilon)^2-\varepsilon'^2} - 2\right]. \tag{21}$$

Three alternative forms for $\alpha^l_{(0)}(k)$ are derived and discussed in Appendix B.

The resulting explicit expression for $\alpha^t_{(0)}(k)$ is

$$\alpha^t_{(0)}(k) = -\frac{e^2}{|\boldsymbol{k}|^2}\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\,\frac{n(\boldsymbol{p})}{\varepsilon}$$
$$\times\left[|\boldsymbol{k}|^2 + |\boldsymbol{k}\times\boldsymbol{p}|^2\left(\frac{1}{(\omega-\varepsilon)^2-\varepsilon'^2} + \frac{1}{(\omega+\varepsilon)^2-\varepsilon'^2}\right)\right]. \tag{22}$$

Several of the steps involved in the derivation of (22) are outlined in Appendix B. As with the longitudinal part, there is considerable freedom in the way the result may be expressed algebraically, and different choices can lead to superficially different explicit forms.

5. A Completely Degenerate Spin Zero Plasma

At absolute zero a Bose distribution involves only particles or anti-particles, but not a mixture of the two, and all are in the ground state, which is $\boldsymbol{p} = 0$ here. The appropriate distribution function is

$$n(\boldsymbol{p}) = n^+(\boldsymbol{p}) + n^-(\boldsymbol{p}) = \frac{n}{4\pi|\boldsymbol{p}|^2}(2\pi)^3\delta(|\boldsymbol{p}|), \tag{23}$$

where n is the total number density of bosons or anti-bosons.

On substituting (23) into (21) and (22) the integrals are trivial, and in the integrand one has $\varepsilon = m$, $\varepsilon' = \varepsilon'' = \varepsilon_k = [m^2 + |\boldsymbol{k}|^2]^{1/2}$. It is convenient to write the results in terms of dielectric response functions:

$$\varepsilon^{l,t}(\omega, \boldsymbol{k}) = 1 + \frac{\mu_0}{\omega^2}\,\alpha^{l,t}(k). \tag{24}$$

We find

$$\varepsilon^{l}(\omega, \boldsymbol{k}) = 1 + \frac{\omega_p^2}{4|\boldsymbol{k}|^2\varepsilon_k}\left\{(m+\varepsilon_k)^2\left[\frac{1}{\omega - m + \varepsilon_k} - \frac{1}{\omega + m - \varepsilon_k}\right]\right.$$
$$\left. -(m-\varepsilon_k)^2\left[\frac{1}{\omega - m - \varepsilon_k} - \frac{1}{\omega + m + \varepsilon_k}\right]\right\}, \tag{25}$$

and

$$\varepsilon^{t}(\omega, \boldsymbol{k}) = 1 - \frac{\omega_p^2}{\omega^2}, \tag{26}$$

where we introduce the plasma frequency:

$$\omega_p = \left[\frac{e^2 n}{\varepsilon_0 m}\right]^{1/2}. \tag{27}$$

The longitudinal dielectric response for a completely degenerate Bose gas was obtained in (KFH), and the form (25) reproduces their expression (4.21) when differences in notation are taken into account. The dispersion relation for longitudinal waves is $\varepsilon^{l}(\omega, \boldsymbol{k}) = 0$, and the solutions of this equation were discussed in (KFH).

The dispersion equation for transverse waves is given by

$$\varepsilon^{t}(\omega, \boldsymbol{k}) = \frac{|\boldsymbol{k}|^2}{\omega^2}. \tag{28}$$

In the completely degenerate limit substituting (26) in (28) leads to the familiar form

$$\omega^2 = \omega_p^2 + |\boldsymbol{k}|^2. \tag{29}$$

However, in this case the dispersion equation in the limit of complete degeneracy is not the same as the limiting form of the dispersion equation as complete degeneracy is approached. The reason for this may be seen from (22). The terms involving $|\boldsymbol{k}\times\boldsymbol{p}|^2$ are non-zero as the limit is approached. Inclusion of these terms changes the dispersion equation from one linear in ω^2 to one cubic in ω^2, and thus introduces two additional solutions of the dispersion equation for transverse waves. For $|\boldsymbol{k}\times\boldsymbol{p}|^2$ arbitrarily small these additional solutions are arbitrarily close to the poles of the terms inside the parentheses in (22), that is, arbitrarily close to the solutions of

$$[(\omega - m)^2 - {\varepsilon_k}^2][(\omega + m)^2 - {\varepsilon_k}^2] = 0. \tag{30}$$

The solutions are

$$\omega^2 = 2m^2 + |\boldsymbol{k}|^2 + 2\sqrt{m^4 + m^2|\boldsymbol{k}|^2} \approx 4m^2 + 2|\boldsymbol{k}|^2 + \cdots, \tag{31}$$

$$\omega^2 = 2m^2 + |\boldsymbol{k}|^2 - 2\sqrt{m^4 + m^2|\boldsymbol{k}|^2} \approx \frac{|\boldsymbol{k}|^4}{4m^2} + \cdots, \tag{32}$$

where the approximations apply for $|\boldsymbol{k}|^2 \ll m^2$. The solution (31) has a counterpart for longitudinal waves and was referred to as the 'pair branch' in (KFH). The dispersion relation (32) is similar in form to the dispersion relation for rotons in a Bose gas at very low temperatures (see e.g. Lifshitz and Pitaevskii 1980). However, rotons are longitudinal and the mode found here is transverse. We refer to it as a 'roton-like' mode.

The fact that there are three transverse modes as the completely degenerate limit is approached can be obscured by considering only the completely degenerate limit itself. The two additional modes are present as the limit is approached, and so they are allowed in any actual boson plasma no matter how close it is to complete degeneracy. It is interesting to note that in the case of spin 1, unlike the case of spin 0, all three modes are present in the completely degenerate limit itself, cf. Section 6.

6. A Spin One Plasma

The generalisation to a plasma of bosons and anti-bosons of spin 1 is as follows. The averaged propagator for the spin 1 case may be evaluated using the same method as for the spin 0 case in Section 2. In place of (19) we find

$$\bar{G}^{\mu\nu}(P) = -\left(g^{\mu\nu} - \frac{P^\mu P^\nu}{m^2}\right)\left[\frac{1}{P^2 - m^2 + i0} - \frac{iN(P)}{2m}\right]. \tag{33}$$

The linear response tensor is obtained from the amplitudes of the two Feynman diagrams, Figs 1 and 2, as in the spin 0 case. These give

$$\alpha^{\mu\nu}_{(1)}(k) = \frac{e^2}{2m}\int \frac{d^4P}{(2\pi)^4}\left\{\left[\frac{N(P')}{P^2 - m^2} + \frac{N(P)}{P'^2 - m^2}\right]F^{\mu\nu}_{(1)}(P,P') - N(P)\left[4g^{\mu\nu} + 2\frac{P^\mu P^\nu}{m^2}\right]\right\}, \tag{34}$$

with $P' = P - k$, and with

$$F^{\mu\nu}_{(1)}(P,Q) = \frac{1}{m^2}\left[2PQ\,(P^\mu Q^\nu + P^\nu Q^\mu) - Q^2P^\mu P^\nu - P^2Q^\mu Q^\nu - 2(PQ)^2 g^{\mu\nu}\right] + g^{\mu\nu}(P^2 + Q^2) + 2(P^\mu + Q^\mu)(P^\nu + Q^\nu) + P^\mu Q^\nu + P^\nu Q^\mu. \tag{35}$$

An alternative expression in place of (34) is, cf. (13),

$$\alpha^{\mu\nu}_{(1)}(k) = \frac{e^2}{2m}\int \frac{d^4P}{(2\pi)^4}N(P)\left[\frac{F^{\mu\nu}_{(1)}(P,P')}{P'^2 - m^2} + \frac{F^{\mu\nu}_{(1)}(P,P'')}{P''^2 - m^2} - 4g^{\mu\nu} - 2\frac{P^\mu P^\nu}{m^2}\right], \tag{36}$$

with $P'' = P + k$. A difficulty occurred in the derivation of (34). The two terms in square brackets in (34) arise from the 'bubble' and 'seagull' diagrams respectively. Using the vertex factors given by Bjorken and Drell (1964) and Scadron (1979), the latter term has the opposite sign to the one given in (36). This is clearly incorrect because the response tensor would then neither reproduce the classical limit correctly nor satisfy the charge-continuity and gauge-independence relations (16). As a confirmation of the correctness of the choice of sign of the 'seagull' contribution in (34) we have recalculated it using the vertex factors given by Lee and Yang (1962) for spin 1 bosons, and this calculation does give the correct relative sign of the bubble and seagull terms.

Note that in taking the classical limit one identifies the number density as

$$n = 3\int \frac{d^4P}{(2\pi)^4}\, N(P),$$

where the factor 3 arises from the three spin states.

The counterparts of the expressions (17), (21) and (24) for an isotropic spin 1 plasma are, respectively,

$$\alpha^{\mu\nu}_{(1)}(k) = \frac{e^2}{2m}\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\,\frac{n(\boldsymbol{p})}{\varepsilon}\left\{\frac{F^{\mu\nu}_{(1)}(\varepsilon,\boldsymbol{p};\varepsilon-\omega,\boldsymbol{p}-\boldsymbol{k})}{(\omega-\varepsilon-\varepsilon')\,(\omega-\varepsilon+\varepsilon')}\right.$$
$$\left. + \frac{F^{\mu\nu}_{(1)}(-\varepsilon,\boldsymbol{p};-\varepsilon-\omega,\boldsymbol{p}-\boldsymbol{k})}{(\omega+\varepsilon+\varepsilon')\,(\omega+\varepsilon-\varepsilon')} - 4g^{\mu\nu} - 2\frac{P^\mu P^\nu}{m^2}\right\}, \qquad (37)$$

with $P^\mu = [\varepsilon, \boldsymbol{p}]$,

$$\alpha^{l}_{(1)}(k) = -\frac{e^2\omega^2}{2|\boldsymbol{k}|^2}\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\,\frac{n(\boldsymbol{p})}{\varepsilon}\left\{-4-2\frac{\varepsilon^2}{m^2}\right.$$
$$\left. +\left[\frac{a^l_+(\varepsilon,\boldsymbol{p};\omega,\boldsymbol{k})}{(\omega-\varepsilon-\varepsilon')(\omega-\varepsilon+\varepsilon')} + \frac{a^l_-(\varepsilon,\boldsymbol{p};\omega,\boldsymbol{k})}{(\omega+\varepsilon+\varepsilon')(\omega+\varepsilon-\varepsilon')}\right]\right\}, \qquad (38)$$

with

$$a^l_\pm(\varepsilon,\boldsymbol{p};\omega,\boldsymbol{k}) = 12\varepsilon^2 + 3\omega^2 \mp 12\varepsilon\omega$$
$$+\frac{1}{m^2}\left[|\boldsymbol{p}|^2\left(|\boldsymbol{k}|^2 \pm 2\boldsymbol{k}\cdot\boldsymbol{p} + \omega^2 \mp 2\varepsilon\omega\right) - 2(\boldsymbol{k}\cdot\boldsymbol{p})^2\right], \qquad (39)$$

and

$$\alpha^{t}_{(1)}(k) = -\frac{3e^2}{4|\boldsymbol{k}|^2}\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\,\frac{n(\boldsymbol{p})}{\varepsilon}$$
$$\times\left\{|\boldsymbol{k}|^2 + \left[|\boldsymbol{p}\times\boldsymbol{k}|^2 - \tfrac{1}{3}(\omega^2-|\boldsymbol{k}|^2)\left(|\boldsymbol{k}|^2 + \frac{|\boldsymbol{p}\times\boldsymbol{k}|^2}{2m^2}\right)\right]\right.$$
$$\left.\times\left(\frac{1}{(\omega-\varepsilon)^2-\varepsilon'^2} + \frac{1}{(\omega+\varepsilon)^2-\varepsilon'^2}\right)\right\}. \qquad (40)$$

In the completely degenerate limit one has

$$\alpha^{l}_{(1)}(k) = -\frac{\omega^2\omega_p^2}{6\mu_0\,|\boldsymbol{k}|^2}\left[\frac{12m^2-12m\omega+3\omega^2}{(\omega-m)^2-\varepsilon_k^2}+\frac{12m^2+12m\omega+3\omega^2}{(\omega+m)^2-\varepsilon_k^2}-6\right], \tag{41}$$

and

$$\alpha^{t}_{(1)}(k) = -\frac{\omega_p^2}{\mu_0}\left[1-\tfrac{1}{3}(\omega^2-|\boldsymbol{k}|^2)\left(\frac{1}{(\omega-m)^2-\varepsilon_k^2}+\frac{1}{(\omega+m)^2-\varepsilon_k^2}\right)\right]. \tag{42}$$

The longitudinal and transverse dispersion equations in the completely degenerate limit are

$$\omega^4-\omega^2(2\,|\boldsymbol{k}|^2+\omega_p^2+4m^2)+|\boldsymbol{k}|^4+|\boldsymbol{k}|^2\omega_p^2+4m^2\omega_p^2=0, \tag{43}$$

and

$$3\omega^6-\omega^4(9\,|\boldsymbol{k}|^2+\omega_p^2+12m^2)$$
$$+\,\omega^2(9\,|\boldsymbol{k}|^4+2\omega_p^2\,|\boldsymbol{k}|^2+12\,|\boldsymbol{k}|^2m^2+12\omega_p^2m^2)-|\boldsymbol{k}|^4\omega_p^2-3\,|\boldsymbol{k}|^6=0, \tag{44}$$

respectively. The longitudinal dispersion equation is the same as for the spin 0 case. The dispersion equation for transverse waves is cubic in ω^2, and there are three solutions in the completely degenerate limit. In place of (29) to (32) one can obtain approximate solutions by first solving in the limit of zero $|\boldsymbol{k}|$ and then finding corrections for small $|\boldsymbol{k}|$. For simplicity we also assume $\omega_p^2 \ll m^2$. The resulting approximate solutions for transverse waves are

$$\omega^2=\omega_p^2+|\boldsymbol{k}|^2+\frac{\omega_p^4}{6m^2}+\cdots, \tag{45}$$

$$\omega^2=4m^2+2\,|\boldsymbol{k}|^2+\cdots, \tag{46}$$

$$\omega^2=\frac{|\boldsymbol{k}|^4}{12m^2}+\cdots. \tag{47}$$

7. Discussion and Conclusions

It is of interest here to compare our results for spin 0 and spin 1 particle/anti-particle plasmas with those for a spin $\frac{1}{2}$ plasma. Firstly, for the averaged propagator all the results may be summarised as follows. In each case the propagator in vacuo is of the form

$$G^{\mu\nu}(P)=\frac{\Gamma^{\mu\nu}(P)}{P^2-m^2+i0}, \tag{48}$$

with

$$\Gamma^{\mu\nu}(P)=\begin{cases}1, & \text{for spin 0,}\\ \gamma^\mu P_\mu+m, & \text{for spin } \tfrac{1}{2},\\ -[g^{\mu\nu}-P^\mu P^\nu/m^2], & \text{for spin 1.}\end{cases} \tag{49}$$

The averaged propagator in the presence of a plasma is then

$$\bar{G}^{\mu\nu}(P) = \Gamma^{\mu\nu}(P)\left[\frac{1}{P^2 - m^2 + i0} \pm \frac{iN(P)}{2m}\right], \tag{50}$$

where the upper sign is for fermions and the lower sign is for bosons. This difference in sign may be interpreted in terms of the presence of particles suppressing or enhancing the resonant part of the propagator for Fermi and Bose statistics, respectively. Secondly, for bosons one needs to include the contribution from the contracted seagull diagram (Fig. 2), in addition to that for the bubble diagram (Fig. 1).

Although we have presented general expressions for isotropic plasmas, these have been evaluated only for the degenerate limit. For this case the longitudinal dispersion equation is the same for both spin 0 and spin 1 plasmas; there are two wave modes corresponding to the familiar Langmuir mode and to a 'pair branch'. There are three solutions for transverse polarisation. One solution corresponds to the familiar transverse mode for a classical plasma, and a second is a 'pair branch'. The third solution is a 'roton-like' mode in that the dispersion relation is analogous to that for rotons, but the mode is transverse whereas rotons are longitudinal. For spin 0 the latter two modes are present in the completely degenerate limit only in a limiting sense: they are present as the limit is approached but disappear in the limit itself. For spin 1 these modes are present even in the completely degenerate limit. The 'roton-like' mode does not appear to have been identified hitherto. Its possible physical significance has yet to be explored.

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Appendix A: Derivation of Equation (17)

Here we express the linear response tensor for an isotropic spin 0 plasma in a form where the poles are in the same positions as (14) of (HM). Starting from the expression (13) and using the definition (18) of $N(P)$, we integrate over P^0 to find, omitting the final term involving $2g^{\mu\nu}$ for the present,

$$\frac{e^2}{2}\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\sum_{\xi}\Bigg[F^{\mu\nu}_{(0)}(\xi\varepsilon,\boldsymbol{p};\xi\varepsilon-\omega,\boldsymbol{p}-\boldsymbol{k})\,n^{\xi}(\boldsymbol{p})\,\frac{1}{\varepsilon(P'^2-m^2)}\Bigg|_{P^0=\xi\varepsilon}$$

$$+F^{\mu\nu}_{(0)}(\omega+\xi\varepsilon',\boldsymbol{p};\xi\varepsilon',\boldsymbol{p}-\boldsymbol{k})\,n^{\xi}(\boldsymbol{p}-\boldsymbol{k})\,\frac{1}{\varepsilon'(P^2-m^2)}\Bigg|_{P^0=\omega+\xi\varepsilon'}\Bigg].$$

The denominators are factorised and reproduce those in (17). The second term in square brackets is rewritten by making the substitution $\boldsymbol{p}\to-\boldsymbol{p}+\boldsymbol{k}$ in the integral. The remaining steps involve using two symmetry properties. One is

$$F^{\mu\nu}_{(0)}(P^0,\boldsymbol{P};Q^0,\boldsymbol{Q})=F^{\mu\nu}_{(0)}(-P^0,-\boldsymbol{P};-Q^0,-\boldsymbol{Q})=F^{\mu\nu}_{(0)}(Q^0,\boldsymbol{Q};P^0,\boldsymbol{P}).$$

The other applies for an isotropic plasma:

$$n^{\pm}(-\boldsymbol{p})=n^{\pm}(\boldsymbol{p}).$$

In the text we require only the weaker condition

$$n^{+}(\boldsymbol{p})+n^{-}(-\boldsymbol{p})=n^{+}(-\boldsymbol{p})+n^{-}(\boldsymbol{p}).$$

The first two terms in (17) are then reproduced. The final term involving $g^{\mu\nu}$ follows by integrating over P^0 and making the substitution $\boldsymbol{p}\to-\boldsymbol{p}$ in all the terms involving n^-.

Appendix B: Derivation of Alternative Forms for $\alpha^{l}_{(0)}(k)$

Alternative forms for $\alpha^{l}_{(0)}(k)$ may be derived from (21) as follows. First rewrite the integrand in the form

$$\frac{A}{\omega-\varepsilon+\varepsilon'}-\frac{A}{\omega-\varepsilon'+\varepsilon}+\frac{B}{\omega+\varepsilon'+\varepsilon}-\frac{B}{\omega-\varepsilon'-\varepsilon},$$

where A and B do not depend on ω. This is achieved by equating the integrand in (21) to this form and solving for A and B. It is convenient to make the replacement $\boldsymbol{p}\to-\boldsymbol{p}$, implying $\varepsilon'=[m^2+|\boldsymbol{p}-\boldsymbol{k}|^2]^{1/2}\to\varepsilon''=[m^2+|\boldsymbol{p}+\boldsymbol{k}|^2]^{1/2}$. Thus one obtains the form

$$\alpha^{l}_{(0)}(k)=-\frac{e^2\omega^2}{2|\boldsymbol{k}|^2}\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{n(\boldsymbol{p})}{\varepsilon\varepsilon''}\Bigg\{-(\varepsilon+\varepsilon'')^2\left[\frac{1}{\omega-\varepsilon+\varepsilon''}-\frac{1}{\omega+\varepsilon-\varepsilon''}\right]$$

$$+(\varepsilon-\varepsilon'')^2\left[\frac{1}{\omega-\varepsilon-\varepsilon''}-\frac{1}{\omega+\varepsilon+\varepsilon''}\right]\Bigg\}. \tag{B1}$$

Another form that may be derived from (B1) by elementary manipulations is

$$\alpha^{l}_{(0)}(k)=\frac{e^2\omega^2}{2|\boldsymbol{k}|^2}\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\,n(\boldsymbol{p})\left\{\frac{\varepsilon-\varepsilon''}{\omega^2-(\varepsilon-\varepsilon'')^2}\left[1+\frac{\varepsilon^2+\boldsymbol{k}\cdot\boldsymbol{p}+\frac{1}{2}|\boldsymbol{k}|^2}{2\varepsilon\varepsilon''}\right]\right.$$
$$\left.+\frac{\varepsilon+\varepsilon''}{\omega^2-(\varepsilon+\varepsilon'')^2}\left[1-\frac{\varepsilon^2+\boldsymbol{k}\cdot\boldsymbol{p}+\frac{1}{2}|\boldsymbol{k}|^2}{2\varepsilon\varepsilon''}\right]\right\}.\tag{B2}$$

Note that these two forms have factors ω^2 outside the integrals on the right hand sides. A further form may be derived by using the identity

$$\frac{\lambda^3}{\omega^2-\lambda^2}+\lambda=\frac{\lambda\omega^2}{\omega^2-\lambda^2},$$

and noting that for an isotropic distribution, the integral over $\boldsymbol{k}\cdot\boldsymbol{p}$ is zero. Thus one finds

$$\alpha^{l}_{(0)}(k)=-\frac{e^2}{2|\boldsymbol{k}|^2}\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\,\frac{n(\boldsymbol{p})}{\varepsilon}\left\{2|\boldsymbol{k}|^2\right.$$
$$\left.+\frac{(|\boldsymbol{k}|^2+2\boldsymbol{k}\cdot\boldsymbol{p})^2}{\varepsilon''}\left[\frac{\varepsilon''-\varepsilon}{\omega^2-(\varepsilon-\varepsilon'')^2}+\frac{\varepsilon''+\varepsilon}{\omega^2-(\varepsilon+\varepsilon'')^2}\right]\right\}.\tag{B3}$$

Expression (B1) is of the form found in (4.17) of (KFH), and expression (B2) is similar to (2.27) of (KFH) for a spin $\frac{1}{2}$ plasma. By applying the transformation $\boldsymbol{p}\to-\boldsymbol{p}$ to expression (B3), implying $\varepsilon''\to\varepsilon'$, it reduces to a form similar to (14) of (HM).

In the derivation of the transverse part (22) for spin 0, and also of the transverse part (40) for spin 1, one seeks to rewrite the terms so that the numerators are the same and the $\pm$ signs appear only in the denominators. This involves rewriting two types of term. One arises from the identity (in 4-tensor notation with $P^2=m^2$)

$$\sum_{\pm}\frac{\pm 2Pk}{(P\pm k)^2-m^2}=2-k^2\sum_{\pm}\frac{1}{(P\pm k)^2-m^2},\tag{B4}$$

and the other is (in 3-vector notation)

$$\sum_{\pm}\frac{\pm 2\varepsilon\omega}{\pm 2\varepsilon\omega\mp\boldsymbol{p}\cdot\boldsymbol{k}+\omega^2-|\boldsymbol{k}|^2}=$$
$$-1+\left(\frac{2\boldsymbol{p}\cdot\boldsymbol{k}-2\varepsilon^2\omega^2}{\omega^2-|\boldsymbol{k}|^2}-\tfrac{1}{2}(\omega^2-|\boldsymbol{k}|^2)\right)\sum_{\pm}\frac{1}{\pm 2\varepsilon\omega\mp\boldsymbol{p}\cdot\boldsymbol{k}+\omega^2-|\boldsymbol{k}|^2}.\tag{B5}$$

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