

An Improved Scheme Using the Shooting Method to Solve Degenerate Four-wave Mixing Equations

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Abstract

A modified scheme using the shooting method to solve the nonlinear coupled-wave equations for degenerate four-wave mixing is proposed. This simple scheme significantly improves the efficiency of this numerical method and the accuracy of computed results.

The shooting method (Walsh 1977) is one of the most powerful and popular numerical methods to solve two-point boundary-value problems. It has recently been used to obtain numerical solutions of the nonlinear coupled-wave equations for degenerate two-wave mixing (DTWM), four-wave mixing (DFWM) and six-wave mixing (DSWM) (Erbschloe *et al.* 1988; Kukhtarev and Kovalenko 1980; Ja 1983*a*, 1983*b*, 1984*a*, 1986*a*). Compared with other numerical methods which have also been used for solving the same problems, such as the $O(h^7)$ method (Ja 1984*b*) (one of the general finite difference methods) and the finite element method (Ja 1986*b*, 1988), the shooting method has the following advantages:

- (a) It is straightforward and easy to implement and program, and it requires much less storage.
- (b) It can be used for constructing automatic computational procedures.
- (c) Only several starting values are needed, while in the finite difference and the finite element methods the values of every element of a matrix have to be guessed. Thus, there are more stringent requirements on the choice of the starting values in the latter methods.

In a previous scheme using the shooting method, the dependent variable in the differential equations is the beam intensity. For example, Fig. 1 shows the DFWM configuration in a reflection geometry with no externally applied electric field. With a small incident angle θ , reflection gratings such as those formed by two interfering beams $\bar{A}_2$ and $\bar{A}_3$, or $\bar{A}_1$ and $\bar{A}_4$, will be dominant. We therefore call it a reflection geometry.

The physical process taking place in the photo-refractive crystal, which is responsible for the formation of the gratings, can be described briefly as follows: when the crystal is illuminated by these coherent light beams, intensity interference fringes are formed. Photocarriers (e.g. electrons) are

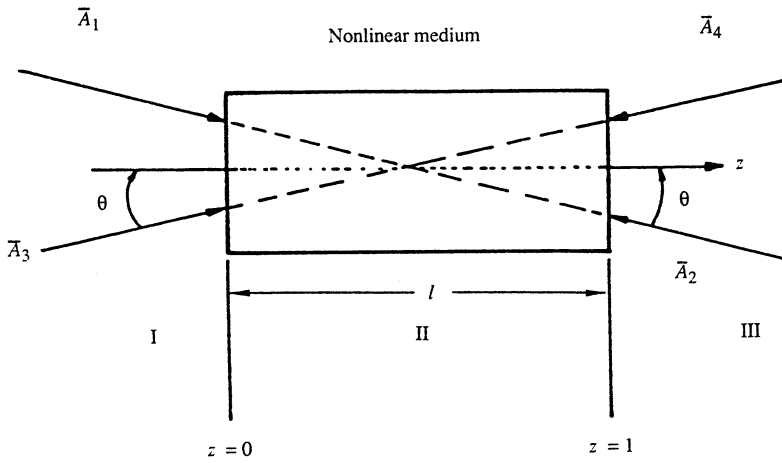


Fig. 1. Four-wave mixing arrangement in a photo-refractive crystal for the reflection geometry shown. It is assumed that the average refractive index n is the same for regions I, II and III.

released by photo-excitation from donor centres, and move in the conduction band by diffusion (macroscopically they move to darker areas). Later they are captured by the trapping centres. The redistribution of these electrons by retrapping then creates a periodic space-charge field, which in turn modulates the refractive index of the crystal through the linear electro-optic (Pockels) effect. Consequently, a phase volume grating is formed.

The reading beam $\bar{A}_1$ (one of the antipropagating pump beams) is diffracted by the reflection grating formed by the pump beams $\bar{A}_2$ and the signal beam $\bar{A}_3$. A phase-conjugate beam $\bar{A}_4$ is then generated and travels in the opposite direction to beam $\bar{A}_3$.

The differential equations for the steady state can be expressed as (Ja 1983*b*; Kukhtarev and Odulov 1979)

$$\begin{aligned}\frac{dI_1}{dz} &= -\alpha l I_1 - 2gl \frac{I_1 I_4 \pm (I_1 I_2 I_3 I_4)^{\frac{1}{2}}}{I_0}, \\ \frac{dI_2}{dz} &= \alpha l I_2 - 2gl \frac{I_2 I_3 \pm (I_1 I_2 I_3 I_4)^{\frac{1}{2}}}{I_0}, \\ \frac{dI_3}{dz} &= -\alpha l I_3 - 2gl \frac{I_2 I_3 \pm (I_1 I_2 I_3 I_4)^{\frac{1}{2}}}{I_0}, \\ \frac{dI_4}{dz} &= \alpha l I_4 - 2gl \frac{I_1 I_4 \pm (I_1 I_2 I_3 I_4)^{\frac{1}{2}}}{I_0},\end{aligned}\quad (1)$$

In equations (1) I_1 , I_2 , I_3 and I_4 are, respectively, the intensities of two antipropagating pump beams, the signal beam and the generated phase-conjugate beam, where beams 1 and 3 impinge on the crystal from the same side ($z=0$) while beams 2 and 4 are on the opposite side ($z=1$, normalised), and where $I_0 = I_1 + I_2 + I_3 + I_4$, αl is the absorption factor, gl is the gain factor and l is the crystal thickness. The plus sign applies when $g > 0$, while the minus sign applies when $g < 0$.

When using equations (1) in the shooting method scheme, drawbacks have been found:

- (1) Since the equations (1) contain square root terms, the program will not work if any value of I becomes negative in the computation procedure (physically I should be ≥ 0 because it is the intensity of an optical beam). In addition, a new set of starting values has to be guessed and the program to be tried again. Thus, convergence may be slow. The boundary condition for the intensity of the phase-conjugate beam is $I_4(1) = 0$, and I_4 usually increases gradually with z changing from unity to zero. When using the Newton-Raphson iteration to obtain a good approximation to the true value, the approximate value can be either larger or smaller than the true one. Since the true value of I_4 is very small when $z \approx 1$, its approximate value may become negative in some cases during the computation procedure. If this occurs, the program will fail.
- (2) In general, the wavefront reflectivity $W = I_4(0)/I_3(0)$ (the intensity ratio of the generated phase-conjugate beam to the signal beam at $z = 0$) is quite small, especially in the case of DFWM in a reflection geometry where W is always less than unity (Ja 1983b). Thus, $I_4(0)$ is usually much smaller than $I_1(0)$, $I_3(0)$ or $I_2(1)$. The shooting program will usually be terminated when the difference between the computed and given values of $I_1(0)$ [and $I_3(0)$] is smaller than or equal to a small prescribed value ϵ (say 10^{-4}). However, since $I_4(0) \ll I_1(0)$ or $I_3(0)$, a much larger error in $I_4(0)$ may result. Only when the values of $I_1(0)$, $I_3(0)$ and $I_4(0)$ are of the same order can we expect relative errors in them of the same order.

To remove these two drawbacks, equations (1) can be transformed into the following form by using the simple variable transform $I_i = y_i^2$ ($i = 1, 2, 3, 4$):

$$\begin{aligned}\frac{dy_1}{dz} &= -\frac{\alpha I y_1}{2} - \frac{g I y_4 (y_1 y_4 \pm y_2 y_3)}{I_0}, \\ \frac{dy_2}{dz} &= \frac{\alpha I y_2}{2} - \frac{g I y_3 (y_2 y_3 \pm y_1 y_4)}{I_0}, \\ \frac{dy_3}{dz} &= -\frac{\alpha I y_3}{2} - \frac{g I y_2 (y_2 y_3 \pm y_1 y_4)}{I_0}, \\ \frac{dy_4}{dz} &= \frac{\alpha I y_4}{2} - \frac{g I y_1 (y_1 y_4 \pm y_2 y_3)}{I_0},\end{aligned}\quad (2)$$

where now $I_0 = y_1^2 + y_2^2 + y_3^2 + y_4^2$. It can be seen that since y is not the beam intensity, it can have a negative value. Furthermore, the value of $y_4(0)$ will be closer to $y_1(0)$ and $y_3(0)$ than $I_4(0)$ to $I_1(0)$ and $I_3(0)$, because usually $I_4 \ll 1$. Therefore, the relative error in the computed wavefront reflectivity will be smaller. In other words, using equations (2), instead of (1), the program will be more likely to work and offer more accurate computed results.

For DFWM in a reflection geometry with the phase shift angle $\psi = \pi/2$ between the intensity interference pattern and the generated phase grating and

with no absorption ($\alpha = 0$), an analytical (exact) expression for the wavefront reflectivity has been given by Cronin-Golomb *et al.* (1984) as

$$W = \frac{I_1(0) \sinh^2 U}{I_3(0)(1 + \sinh^2 U)},$$

(3)

where

$$\sinh U = \frac{\{I_2(1)I_3(0)\}^{\frac{1}{2}}(1 - e^{-gl})}{I_2(1) + \{I_1(0) + I_3(0)\}e^{-gl}}.$$

(4)

The computed results, using equations (1) and (2) respectively, in the shooting method scheme can then be compared with those obtained by using the exact formula (3). In all computations, only single precision is used.

Table 1. Computed values of the wavefront reflectivity W in various schemes

Scheme	N	$gl = 0.0005$		$gl = 0.05$		$gl = 5$		Time ^E
		$W (\times 10^{-8})$	Error ^D	$W (\times 10^{-4})$	Error ^D	W	Error ^D	
Exact ^A	—	4.0004312	—	4.0377957	—	0.65180300	—	—
Scheme 1 ^B	40	3.9975037	7.5×10^{-4}	4.0352589	5.0×10^{-4}	0.65160830	3.0×10^{-4}	7.7
Scheme 2 ^C	40	4.0003997	7.9×10^{-6}	4.0377951	1.5×10^{-7}	0.65180279	3.2×10^{-7}	8.2
	20	4.0004001	7.8×10^{-6}	4.0377953	1.0×10^{-7}	0.65179871	6.6×10^{-6}	5.4

^A See equation (3). ^B Equations (1). ^C Equations (2). ^D Relative error in W . ^E Total computer time (relative units).

Table 1 shows the computed wavefront reflectivity using these three sets of equations. For generality, a large range of values of gl (0.0005, 0.05 and 5) is chosen. The pump ratio is $I_2(1)/I_1(0) = 1$, the signal to pump beam ratio $I_3(0)/I_1(0)$ is 0.5 and the absorption $\alpha = 0$ with $I_1(0) \equiv 1$ (normalised).

An inspection of Table 1 shows the following.

- (1) When dividing the range of z (from 0 to 1) into 40 small subsections ($N = 40$) in the integration of equations (1) and (2) for all values of gl , scheme 2 using equations (2) produces much more accurate results than scheme 1, while the computer time is virtually the same.
- (2) In scheme 2, the same set of starting values of $I_1(1)$ and $I_3(1)$ is used for all values of gl , while this is not the case for scheme 1. For instance, when $gl = 5$ where the saturation range (W is nearly independent of gl) is approached, a different set of starting values of $I_1(1)$ and $I_3(1)$ have to be used. This means that scheme 2 is much easier to work with.
- (3) With N reduced from 40 to 20, the accuracy of the computed results using scheme 2 is approximately the same, but less computer time is used. If high accuracy is not required (e.g. a relative error of 0.01 or only three digits is required), use of scheme 2 can save considerable computer time.
- (4) In scheme 1 when W is small its computed value depends on the approximate value of $I_4(1)$, though the computed values of $I_1(1)$, $I_2(0)$ and $I_3(1)$ do not. However, in scheme 2, W is virtually independent of $I_4(1)$.

In conclusion, scheme 2 using equations (2) is more advantageous than scheme 1 using equations (1) with respect to both accuracy and computer time. Although we have only dealt with DFWM in a reflection geometry, it is expected that similar results would be obtained with DFWM in a transmission geometry. With regard to the finite difference method (Ja 1984*b*), an improved computer program based on the same principle has been written and a similar conclusion can be drawn as well. For two-wave mixing, because there are no square root terms in the differential equation, a similar variable transform may not be necessary.

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