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BD–FRW Cosmology with Bulk Viscosity

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Abstract

Exact cosmological solutions of the field equations in Brans–Dicke (BD) theory, with $k = 0$, FRW metric have been obtained in the presence of bulk viscosity. It is found that constant bulk viscosity leads to an inflationary solution for large values of the BD coupling parameter w. The case of radiative bulk viscosity during the decoupling era is investigated and the production of entropy is estimated.

1. Introduction

It has been shown (Padmanabhan and Chitre 1987) that the presence of bulk viscosity leads to inflationary-like solutions in general relativistic Friedman–Robertson–Walker (FRW) models. Here we investigate the role of bulk viscosity in cosmological evolution in the G-varying scalar–tensor theory of Brans–Dicke. It is well known (Mathiazhagan and Johri 1984) that the gravitational coupling constant G decreases very fast during the inflationary scenario in the BD theory and, as such, it is of interest to see how bulk viscosity modifies the behaviour of BD cosmological models during various stages of expansion. It is noteworthy that bulk viscosity is the only dissipation mechanism compatible with the spatial homogeneity and isotropy (Weinberg 1972) of the observable universe; another peculiar characteristic of bulk viscosity is that it acts like a negative energy field in an expanding universe (Johri and Sudharsan 1988*b*).

There are many circumstances in the evolution of the universe in which bulk viscosity could arise (Ellis 1971): (i) when neutrinos decouple from the cosmic fluid (Misner 1968); (ii) when photons decouple from matter; (iii) at the time of the formation of galaxies; (iv) when a superconducting string moves in a magnetic field (Ostriker *et al.* 1986); and (v) during particle creation in the early universe (Hu 1983) and during monopole–monopole interaction. These various processes giving rise to bulk viscosity could lead to an effective mechanism for entropy production.

Here we consider the FRW line element (with the curvature parameter zero) given by

$$ds^2 = -c^2 dt^2 + R^2(t)\{dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)\}, \tag{1}$$

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where $R(t)$ is the scale factor. This metric describes very well the observed homogeneity and isotropy of the universe over scales larger than 50 Mpc. The source energy–momentum tensor T_{ab} for the metric (1) is usually taken to be that of a perfect fluid, given by

$$T_{ab} = (\rho + p/c^2) u_a u_b + p g_{ab}, \tag{2}$$

$$u_a = (-c, 0, 0, 0); \qquad u^a u_a = -c^2, \tag{3}$$

where ρ is the energy density, p the isotropic pressure and u_a is the 4-velocity of the fluid. Investigations (Johri and Sudharsan 1988*a*) have shown that cosmological observations of the present day universe cannot preclude the existence of a tiny bulk viscosity which would be consistent with the geometry of the FRW model. Thus, a general expression for the energy–momentum tensor for a spatially homogeneous and isotropic universe is given by

$$T_{ab} = \{\rho + (p - \mu\theta)/c^2\} u_a u_b + (p - \mu\theta) g_{ab}, \tag{4}$$

with

$$\theta = 3\dot{R}/R, \tag{5}$$

where μ is the coefficient of bulk viscosity and θ the expansion scalar in the case of the metric (1). An overhead dot denotes differentiation with respect to time.

In the BD theory the field equations are given by

$$G_{ab} = -\frac{8\pi}{c^4\phi} T_{ab} - \frac{w}{\phi^2}(\phi_{;a}\phi_{;b} - \tfrac{1}{2} g_{ab}\phi_{;c}\phi_{;c})$$
$$- \frac{1}{\phi}(\phi_{;a;b} - g_{ab}\Box^2\phi), \tag{6}$$

$$\Box^2\phi = \{8\pi/(3 + 2w)c^4\} T^a{}_a, \tag{7}$$

where ϕ is the long range scalar field (which varies inversely as the gravitational constant G) and w the BD coupling parameter.

In Section 2, we solve the field equation (6), with T_{ab} given by (4) and the space-time metric by (1), when the coefficient of bulk viscosity is in general a function of time. In Section 3 we obtain solutions for two particular cases: (I) when the coefficient of bulk viscosity is a constant and (II) when the coefficient of bulk viscosity is a function of time and arises from the radiation drag during the decoupling of radiation from matter.

2. Field Equations and the General Solutions

The BD field equations (6) with an equation of state

$$P = \gamma\rho c^2; \qquad 0 \le \gamma \le 1 \tag{8}$$

reduce to

$$\frac{3\dot{R}^2}{R^2} + \frac{3\dot{R}\dot{\phi}}{R\phi} - \frac{w(\dot{\phi})^2}{2(\phi)^2} = \frac{8\pi}{\phi}\rho, \tag{9}$$

$$\frac{2\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{\ddot{\phi}}{\phi} + \frac{w(\dot{\phi})^2}{2(\phi)^2} + \frac{2\dot{R}\dot{\phi}}{R\phi} = -\frac{8\pi}{\phi}\gamma\rho + \frac{24\pi}{c^2\phi}\mu\frac{\dot{R}}{R}, \tag{10}$$

$$\ddot{\phi} + \frac{3\dot{R}\dot{\phi}}{R} = \frac{8\pi}{(3+2w)c^2}\left((1-3\gamma)\rho c^2 + 9\mu\frac{\dot{R}^2}{R^2}\right), \tag{11}$$

$$\dot{\rho} + (1+\gamma)\rho\theta = (\mu/c^2)\theta^2, \tag{12}$$

where (9) is the time–time component and (10) the space–space component of the field equation (6); equation (7) leads to (11) whereas the conservation equation $T^{ab}{}_{;b} = 0$ yields (12).

If we assume that the coefficient of bulk viscosity is a function of time only, as expected in a homogeneous universe, we have four unknowns $R(t)$, $\phi(t)$, $\rho(t)$ and $\mu(t)$ to be determined from equations (9)–(12), out of which only three are independent; as such the problem remains unsolved unless we can find out the functional form of $\mu(t)$.

In homogeneous space–time, the scalar field ϕ is a function of time only as such it can be expressed as a function of the scale factor $R(t)$ as well. For the sake of simplicity we assume the power law relation

$$\phi(t) = KR^{\alpha}, \tag{13}$$

where K is a constant and α the power index.

With (13), the field equations (9)–(11) reduce to

$$(8\pi/\phi)\rho = (3 + 3\alpha - w\alpha^2/2)\dot{R}^2/R^2, \tag{14}$$

$$-(8\pi/\phi)\gamma\rho + (24\pi/c^2\phi)\mu\dot{R}/R = (2+\alpha)\ddot{R}/R + (1 + \alpha + \alpha^2 + w\alpha^2/2)\dot{R}^2/R^2, \tag{15}$$

$$(8\pi/c^2\phi)\{(1-3\gamma)\rho c^2 + 9\mu\dot{R}/R\} = (3+2w)\{(\alpha\ddot{R}/R) + \alpha(\alpha+2)\dot{R}^2/R^2\}. \tag{16}$$

Eliminating $\rho(t)$ and $\mu(t)$ from (16) with the help of (14) and (15), we obtain the simple differential equation in $R(t)$

$$\ddot{R}/R + \beta\dot{R}^2/R^2 = 0, \tag{17}$$

which readily yields on integration

$$R(t) = (t/t_c)^{1/(1+\beta)}; \qquad t_c = \text{constant} \tag{18}$$

with the initial condition $R(0) = 0$, where

$$\beta = (w\alpha^2 + 4w\alpha - 6)/2(w\alpha - 3); \qquad (w\alpha - 3) \neq 0. \tag{19}$$

Equations (14) and (18) lead to

$$\rho(t) = \rho_0(t/t_c)^{\alpha/(\beta+1)-2}, \tag{20}$$

where

$$\rho_0 = (K/8\pi)(3 + 3\alpha - w\alpha^2/2)/(\beta + 1)^2 t_c^2, \tag{21}$$

whereas (18) and (20) in conjunction with (15) give

$$\mu(t) = \mu_0(t/t_c)^{\alpha/(\beta+1)-1}, \tag{22}$$

where

$$\mu_0 = \{Kc^2/24\pi(\beta + 1)t_c\}\{(1 + \alpha + \alpha^2 + w\alpha^2/2) + \gamma(3 + 3\alpha - w\alpha^2/2) - (\alpha + 2)\beta\}. \tag{23}$$

In general, once the functional form of $\mu(t)$ is known, which depends upon the specific dissipative process under consideration during the epoch, the solutions are completely specified in terms of the coupling parameter w.

3. Particular Solutions

In this section we consider the following two cases: the coefficient of bulk viscosity is a constant μ_0 and, secondly, that it is a function of time arising from radiative viscosity during the decoupling era.

Case I: Here we assume that the universe might contain a primordial component of bulk viscosity μ_0 as suggested by Padmanabhan and Chitre (1987); such a primordial bulk viscosity might be inherent in the cosmic fluid analogous to the cosmical constant. By virtue of (22), the assumption $\mu = \mu_0$ implies that

$$\alpha = \beta + 1, \tag{24}$$

which leads to a solution for α in terms of w,

$$\alpha = \{3(w + 1) \pm (9w^2 + 6w + 9)^{\frac{1}{2}}\}/w. \tag{25}$$

In order that $R(t)$ is an increasing function of time and that $\rho(t)$, $p(t)$ etc. are positive definite, we can consider only the minus sign in (25). Thus, the complete solution in the case of constant bulk viscosity is

$$R(t) = (t/t_c)^{1/\alpha}, \qquad \phi(t) = K(t/t_c), \tag{26}$$

$$\rho(t) = (t/t_c)^{-1}, \qquad p(t) = \gamma c^2 \rho(t). \tag{27}$$

It is seen from (25) that, for large values of w, α is very small and is of the order of $1/w$. Hence we find that, for sufficiently large values of w, $R(t)$ exhibits an inflationary nature, although of the power-law type, since $R \sim (t/t_c)^{O(w)}$ would give rise to very rapid expansion.

Now we proceed to estimate the amount of entropy produced due to the presence of a constant bulk viscosity, which is given by (Weinberg 1971)

$$\dot{\sigma} = \mu\theta^2/nkT, \tag{28}$$

where σ is the entropy per baryon, k the Boltzmann constant, T the temperature and n the baryon number density; when conserved it is given by

$$n = n_0 R^{-3}; \qquad n_0 = \text{constant}. \tag{29}$$

Here we consider the amount of entropy produced during the decoupling era, which is characterised by $\gamma \approx \frac{1}{3}$ [see the equation of state (8)]; the energy density is given by Stefan's law

$$\rho = aT^4/c^2, \tag{30}$$

where a is Stefan's constant.

Substituting from (26), (27), (29) and (30) in (28), we have

$$\dot{\sigma} = \sigma_0 (t/t_c)^{q-1}; \qquad \sigma_0 = 9\mu_0\, a^{\frac{1}{4}}/kn_0\, c^{\frac{1}{2}} \rho_0^{\frac{1}{4}}\, \alpha^2 t_c^2. \tag{31}$$

Integrating (31) between the limits t_1 and t_2 we have

$$\sigma_2 - \sigma_1 = \Delta\sigma = (t_c\,\sigma_0/q)\{(t_2/t_1)^q - 1\}(t_1/t_c)^q; \qquad q = 3(4-\alpha)/4\alpha. \tag{32}$$

For positive entropy production we must have $q > 0$. This constrains the coupling parameter w to lie within the limits $(-1\cdot 5, \infty)$ with $w \neq 0$. It is seen that for large values of w the exponent is $q \sim 1/\alpha \sim O(w)$ and, as such, a very large amount of entropy is produced provided $t_c < t_1 < t_2$, but if the condition $0 < t_1 < t_2 < t_c$ holds we may not have sufficient entropy produced. It is also seen from (26) that inflation takes place only if $t/t_c > 1$.

Case II: Here we consider the effect of bulk viscosity during the decoupling era, where the bulk viscosity arises due to a longer mean free time for photons than for matter. Since around the decoupling era matter is radiation dominated, we assume an equation of state characterised by $\gamma \approx \frac{1}{3}$, such that $\delta \equiv |\frac{1}{3} - \gamma| \neq 0$, but $\delta \ll 1$.

The bulk viscosity arising due to radiation drag is given by (Weinberg 1971)

$$\mu(t) = 4aT^4\tau\delta^2, \tag{33}$$

where τ is the mean free time for photons given by

$$\tau = 1/\sigma_T\, nc, \tag{34}$$

where σ_T is the Thomson cross section for the scattering of photons by nonrelativistic electrons. With the help of equations (34), (30), (29), (20) and (18), equation (33) becomes

$$\mu(t) = (4c\delta^2\rho_0/\sigma_T\, n_0)(t/t_c)^{\{(\alpha+3)/(1+\beta)\}-2}. \tag{35}$$

For equations (35) and (22) to be compatible we must have

$$\beta = 2. \tag{36}$$

Substituting (36) in (19) we obtain

$$\alpha = \pm(-6/w)^{\frac{1}{2}}. \tag{37}$$

Inserting the value of α, equations (18)–(22) lead to

$$R(t) = (t/t_c)^{\frac{1}{3}}, \qquad \phi(t) = K(t/t_c)^b, \tag{38}$$

$$\rho(t) = \rho_0(t/t_c)^{b-2}, \qquad \mu(t) = \mu_0(t/t_c)^{b-1}, \tag{39}$$

where $b = \alpha/3$ is real only if $w < 0$. These solutions are physically viable for an expanding universe only if the following conditions hold:

(i) $\phi(t)$ is an increasing function of time (i.e. G decreases with time);

(ii) $\mu(t)$ and $\rho(t)$ are decreasing functions of time; and

(iii) $\mu(t) > 0$ for positive entropy production.

The constraint (i) implies that $b > 0$ and (ii) and (iii) imply that

$$-\tfrac{3}{2} < w < -\tfrac{2}{3}. \tag{40}$$

Further, by virtue of (39), (38), (30), (29) and (28), the variation in entropy is given by

$$\dot{\sigma} = \sigma'_0(t/t_c)^{m-1}; \qquad \sigma'_0 = \mu_0\, a^{\frac{1}{4}}/kn_0\, c^{\frac{1}{2}} t_c^2 \rho_0^{\frac{1}{4}}. \tag{41}$$

Integrating between t_1 and t_2, the increase in entropy is

$$\Delta\sigma \approx (\sigma'_0\, t_c/m)\{(t_2/t_1)^m - 1\}(t_1/t_c)^m; \qquad m = (3b-2)/4. \tag{42}$$

Thus, in the case of radiative bulk viscosity, the entropy produced turns out to be negligibly small for the narrow range of w prescribed in (40), simply because it leads to $m < 1$. This is similar to the result obtained in the relativistic case (Johri and Sudharsan 1988*a*). Moreover, the negative range of values imposed by the above physical considerations is too restrictive and unrealistic in view of the observational constraints set on w (Reasenberg *et al.* 1979; Will 1981; Alley 1983).

4. Discussion

The cosmological solutions in the presence of bulk viscosity are completely determined only if the functional dependence of $\mu(t)$ on time is known, which in turn depends on the physics of the dissipation mechanism giving rise to $\mu(t)$. Two particular cases are discussed in Section 3. The solution with constant bulk viscosity is a power-law type and exhibits inflationary growth for sufficiently large values of w. The power-law dependence of $\phi(t)$ and $\rho(t)$ is independent of w, and as expected these solutions do not reduce to the relativistic case, unlike the perfect fluid BD–FRW solutions which reduce to the relativistic case in the limit $w \to \infty$.

It is found that the behaviour of the solution obtained in the case of constant bulk viscosity is similar to the one obtained (Mathiazhagan and Johri 1984) in the BD theory in the presence of the cosmological constant Λ, which acts as the vacuum energy density in inflationary cosmologies. The rate of entropy production due to the presence of constant bulk viscosity can be sufficiently high to account for the presently observed entropy density, provided the value of w lies within the prescribed range $(-1 \cdot 5 < w < \infty)$.

The solutions obtained in the case of a radiative bulk viscosity satisfy the condition that $\rho(t)$ and $\mu(t)$ are decreasing functions of time in an expanding universe; also, the positive entropy production condition $\mu(t) > 0$ is satisfied provided the BD coupling parameter lies in the range given by (40). This range for w is too restrictive and unrealistic and not at all compatible with the present observational limit set on w as cited above and, also, does not lead to sufficient entropy production.

5. Conclusions

We have discussed in general the compatibility of the BD theory with the entropy production in FRW viscous cosmologies. During the course of our investigations we analysed the BD coupling parameter w under two cases:

(i) $\mu = \mu_0$: these solutions are valid for a wide range of w $(-1 \cdot 5 < w < \infty)$. We find that the presence of a constant bulk viscosity coefficient acts like a cosmological constant and leads to an 'inflationary' like power-law solution. It is also seen that the entropy produced due to the presence of constant bulk viscosity could be sufficiently high to account for the observed entropy per baryon. Thus, the inflation induced by a constant bulk viscosity leads to sufficient entropy production, as in the case of relativistic models (Johri and Sudharsan 1988*a*).

(ii) $\mu = \mu(t)$: the investigations showed that a radiative bulk viscosity alone cannot lead to inflation. It is seen that sufficient entropy cannot be produced, and moreover for $\rho(t)$ and $\mu(t)$ to be decreasing functions of time the coupling parameter w must be restricted to a very narrow negative range (40); hence, these solutions are too restrictive and observationally unacceptable.

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