Aust. J. Phys., 1989, **42**, 317–29

Linear Force-free Magnetic Fields and Coronal Models

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Abstract

The mathematical properties of linear force-free fields generated by the Helmholtz equation are reviewed, and the solutions in terms of spherical, cartesian and cylindrical coordinate systems are discussed. When only the normal component of the field on a single (photospheric) surface is available as a boundary condition, the solutions are not uniquely determined. If further conditions are imposed, solutions may be unique or multiple or may not exist. The limitations of various methods of modelling the coronal magnetic field of the Sun using linear force-free fields are exposed. A new upper boundary condition is proposed that guarantees a unique solution, and takes account of the solar wind effects in a manner as closely analogous as possible to that used in potential field modelling.

1. Introduction

Simple estimates reveal that the plasma in the solar corona is magnetically dominated, the magnetic pressure greatly exceeding the gas pressure. Indeed, all other forces are small compared with that which the magnetic field is capable of exerting, so that the Lorentz force must vanish to first order. Such force-free field structures have been the basis for a great deal of coronal magnetic field modelling. In particular, many attempts have been made to deduce the steady-state coronal magnetic field structure from observations of the vertical component of the magnetic flux density B_v in the photosphere by solving

$$\mathbf{j} \times \mathbf{B} = 0, \tag{1}$$

together with the Maxwell equations

$$\nabla . \mathbf{B} = 0, \tag{2}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}, \tag{3}$$

for the coronal volume with B_v specified at the lower boundary.

Non-trivial solutions of (1) and (3) require the field and current to be parallel, i.e.

$$\mu_0 \mathbf{j} = \alpha \mathbf{B}, \tag{4}$$

0004-9506/89/030317$03.00

so that

$$\nabla \times \mathbf{B} = \alpha \mathbf{B}, \tag{5}$$

with α being a scalar function.

Taking the divergence of this expression and using condition (2) shows that $\mathbf{B} \cdot \nabla\alpha = 0$. In other words, the value of α is constant along any field line. This implies that α cannot be imposed arbitrarily over a bounding surface—it may be imposed at only one end of each field line. Since the field line connections of the boundary points are not generally known in advance, the problem of constructing variable α force-free solutions is mathematically intractable.

This problem is avoided if α is assumed to be constant everywhere. In this case, the defining equation for $\mathbf{B}$ is linear. Taking the curl of (5) produces

$$(\nabla^2 + \alpha^2)\mathbf{B} = 0, \tag{6}$$

which is the Helmholtz equation for each cartesian component of $\mathbf{B}$. However, (2) implies that not all the components are independent. Indeed, the solutions of (2) may be written generally in terms of two scalar functions ϑ and ψ as

$$\mathbf{B} = \nabla \times \nabla \times \psi\mathbf{a} + \nabla \times \vartheta\mathbf{a}, \tag{7}$$

where $\mathbf{a}$ is some constant vector (cf. Chandrasekhar 1961). If α is constant, Raadu and Nakagawa (1971), Nakagawa and Raadu (1972), and Nakagawa (1973) showed that ϑ and ψ are not independent and can be chosen so that $\vartheta = \alpha\psi$ and that ψ satisfies the scalar Helmholtz equation

$$(\nabla^2 + \alpha^2)\psi = 0. \tag{8}$$

When α is not constant the defining equation for $\mathbf{B}$ is not linear and little is known about the existence and uniqueness of solutions. The special case in which $\alpha = 0$ is the potential field case; the current then vanishes and the problem of magnetic field extrapolation reduces to solving the Laplace equation, whose properties are very well known. The slightly more general case of linear force-free fields has also received attention, summarised in Priest (1982), although the astrophysical literature exhibits some confusion regarding the form of the general solutions, their existence and uniqueness.

Recently, Heyvaerts and Priest (1984) have discussed in an astrophysical context a conjecture, known as Taylor's hypothesis, that resistive effects can cause magnetic fields to evolve in a manner that approximately conserves a quantity known as the total helicity. The magnetic helicity is defined as

$$K = \int_V \mathbf{B} \cdot \mathbf{A}\,\mathrm{d}V, \tag{9}$$

where $\mathbf{A}$ is the magnetic vector potential ($\mathbf{B} = \nabla \times \mathbf{A}$). Now the minimum energy state of a system with given total helicity and with the normal component of the field B_n specified on the boundary enclosing the volume is precisely that linear force-free field satisfying those boundary conditions (Woltjer 1958;

Sakurai 1979), the value of α being related to the helicity of the system. However, it has not been shown that any portion of force-free region is able to relax to a linear state when the sources of the field in the excluded non-force-free volume are taken into account.

The use of linear force-free models to describe the slowly evolving structure of the solar coronal field has great mathematical convenience but its physical significance remains conjectural.

2. The Standard Helmholtz Problem

There are, in fact, two standard problems (cf. Koshlyakov *et al.* 1964). The *interior* problem requires the solution to (8) in the volume interior to a closed surface S, on which functions a, b and c are specified such that

$$a\psi + b\frac{\partial \psi}{\partial n} = c. \tag{10}$$

A solution to this problem exists and is unique, unless the corresponding homogeneous problem—obtained by setting $c = 0$—has a non-trivial solution. In the latter case, the inhomogeneous problem in which $c \neq 0$ is insoluble.

The *exterior* problem requires the solution to (8) in the volume exterior to a closed surface S, with the same boundary condition (10). In this problem, a solution always exists but is not unique. A unique solution to the exterior problem may be obtained only by imposing a further boundary condition, such as the 'radiation' condition

$$\lim_{r\to\infty} \psi = 0, \qquad \lim_{r\to\infty} r\left(\frac{\partial \psi}{\partial r} - \mathrm{i}\alpha\psi\right) = 0. \tag{11}$$

This condition arises from the appearance of the Helmholtz equation after a periodic time dependence has been separated out of the wave equation. Introducing an $e^{-\mathrm{i}\alpha t}$ time dependence into the solutions of (8) yields progressive waves. The radiation condition simply excludes incoming waves from infinity.

The solutions to these boundary value problems are often expressed in integral form using the Green function constructed for the problem (e.g. Barbosa 1978). The requisite Green function is unique if the solution to the problem is unique. When the solution is not unique, a generalisation of the Green function is required in order to construct an integral solution in the same form (cf. Koshlyakov *et al.* 1964).

However, most of the applications of the theory of force-free fields to the modelling of coronal fields do not match these standard problems, so these applications need to be analysed separately. This is best done explicitly in terms of different coordinate systems.

3. Spherical Coordinates

The general solution of (8) obtained by separating variables in spherical polar coordinates r, θ, ϕ can be expressed in terms of the infinite set of discrete eigenfunctions

$$\psi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} [A_{lm} j_l(\alpha r) + B_{lm} n_l(\alpha r)] Y_{lm}(\theta, \phi), \tag{12}$$

where the j and n are the spherical Bessel functions related to the half-odd integral standard Bessel functions

$$j_l(x) = \sqrt{\frac{\pi}{2x}} J_{l+\frac{1}{2}}(x), \quad n_l(x) = \sqrt{\frac{\pi}{2x}} N_{l+\frac{1}{2}}(x), \tag{13}$$

and the Y are the spherical harmonic functions. This is the form given by Chandrasekhar and Kendall (1957). Nakagawa (1973) omitted the $j_l(\alpha r)$ terms and Priest (1982) the $n_l(\alpha r)$ terms.

If we choose to set **a** equal to the unit vector in the radial direction (in view of the boundary conditions to be posed below) the magnetic flux density becomes

$$\mathbf{B} = \left[-\left(\frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial\psi}{\partial\theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2\psi}{\partial\phi^2} \right), \right.$$

$$\left. \frac{1}{r} \frac{\partial^2\psi}{\partial\theta\partial r} + \frac{\alpha}{r\sin\theta} \frac{\partial\psi}{\partial\phi}, \quad \frac{1}{r\sin\theta} \frac{\partial^2\psi}{\partial\phi\partial r} - \frac{\alpha}{r} \frac{\partial\psi}{\partial\theta} \right]. \tag{14}$$

For an *interior* solution, we must set $B_{lm} = 0$ in order to make ψ regular at the origin—this is the form given by Priest (1982). Then, specifying $B_r(\theta, \phi)$ on $r = R_\odot$ requires us to find the coefficients A_{lm} from

$$B_r(\theta, \phi) = \frac{1}{R_\odot^2} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} l(l+1) A_{lm} j_l(\alpha R_\odot) Y_{lm}(\theta, \phi). \tag{15}$$

If $\alpha R_\odot$ is not a zero of the Bessel function j_l, the values of A_{lm} for $l > 0$ are uniquely determined by this relation. The spherical harmonics Y_{lm} form a complete set, allowing us to express

$$B_r(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} C_{lm} Y_{lm}(\theta, \phi), \tag{16}$$

and then the coefficients can be obtained immediately using the orthogonality properties of the Y_{lm},

$$A_{lm} = R_\odot^2 C_{lm} / l(l+1) j_l(\alpha R_\odot). \tag{17}$$

For the infinite set of values of α for which $\alpha R_\odot$ is a zero of the Bessel function, no solution exists unless the corresponding set of values $C_{lm}(l = -m, \ldots, m)$ vanishes, in which case there are multiple solutions. In general though, the solution is unique up to an additive constant and hence **B** is uniquely determined throughout the volume.

For the *exterior* problem we cannot conclude that either the A_{lm} or the B_{lm} coefficients are all zero, as did Nakagawa (1973) and Priest (1982), because both the sets of Bessel functions are regular at infinity. Introducing the spherical Bessel functions of the third kind,

$$h_l^{(1)}(x) = [j_l(x) + \mathrm{i}n_l(x)], \quad h_l^{(2)}(x) = [j_l(x) - \mathrm{i}n_l(x)], \tag{18}$$

following Seehafer (1978), we may write the general solution as

$$\psi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[A'_{lm} h_l^{(1)}(\alpha r) + B'_{lm} h_l^{(2)}(\alpha r)\right] Y_{lm}(\theta, \phi). \tag{12a}$$

If the radiation condition is imposed, we must set the B'_{lm} coefficients to zero and then the previous boundary conditions are satisfied if

$$A'_{lm} = \{R_\odot^2 C_{lm}/l(l+1)\} h_l^{(1)}(\alpha R_\odot). \tag{19}$$

But now $h_l^{(1)}$ is a complex function, whose complex conjugate is $h_l^{(2)}$, so (19) may be rewritten as

$$A'_{lm} = \{R_\odot^2 C_{lm} h_l^{(2)}(\alpha R_\odot)/l(l+1)\}[j_l^2(\alpha R_\odot) + n_l^2(\alpha R_\odot)]. \tag{19a}$$

In this case, the denominator cannot vanish for $l > 0$ and hence ψ is always uniquely determined up to an arbitrary constant, i.e. **B** is always unique.

Contrary to the claim of Seehafer (1978), none of these solutions are unphysical. Whilst it is true that $\lim_{r\to\infty} |\psi| \propto e^{\mathrm{i}\alpha r}/r$, so that

$$\int_{R_\odot<r<R} |\psi|^2 \, \mathrm{d}V \tag{20}$$

becomes unbounded as $R \to \infty$, it is not true that the magnetic energy

$$\int_{R_\odot<r} \frac{B^2}{2\mu_0} \, \mathrm{d}V \tag{21}$$

is similarly unbounded. Inspection of the components of **B** reveals that the components which decay least rapidly are $B_\theta, B_\phi \propto e^{\mathrm{i}\alpha r}/r^2$. Hence B^2 behaves asymptotically like r^{-4}, which guarantees the convergence of (21).

Moreover, this model is physically plausible if the field is interpreted as a wave that slowly propagates outward due to the Sun's internal dynamo action.

The boundary condition at infinity is avoided if a second spherical surface is introduced at $r = R_s$, and solving for ψ within the shell $R_\odot \le r \le R_s$. This is a problem of *interior* type for which both sets of Bessel functions $j_l(\alpha r)$ and $n_l(\alpha r)$ in (12) must be retained since we do not require the solution to be regular at the origin. If B_r is given on both the inner and outer boundaries a unique solution exists if $j_l(\alpha R_\odot)n_l(\alpha R_s) - j_l(\alpha R_s)n_l(\alpha R_\odot) \neq 0$ for all l. If this quantity vanishes for any l (for example, if $\alpha R_\odot$ and αR_s are both zeroes of j_l or n_l), then either no solution exists—the boundary expressions being inconsistent—or multiple solutions exist. The problem is, however, purely academic in the context of coronal modelling because observations yield no information on B_r on any surface but the photospheric at $r = R_\odot$.

As an alternative, we might consider the problem analogous to the potential problem posed by Schatten *et al.* (1969) and Altschuler and Newkirk (1969). These authors specified B_r on the lower boundary as before, but made the upper boundary a constant potential surface. This choice forces the field to be radial at the outer boundary in order to account approximately for the effect of the solar wind.

However, it is clear from (14) that setting neither $\psi = 0$ nor $\partial\psi/\partial r = 0$ on $r = R_s$ will ensure the vanishing of both B_θ and B_ϕ unless $\alpha = 0$ (the potential case). In fact, we may show, by setting both expressions to zero and eliminating $\partial\psi/\partial r$, that ψ must satisfy

$$\sin\theta\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\psi}{\partial\theta} + \frac{\partial^2\psi}{\partial\phi^2} = 0,$$

if $\alpha \neq 0$. This is the defining equation for the spherical harmonic functions with $l = 0$; thus, the only physically meaningful solution is $\psi(R_s, \theta, \phi) = Y_{00}$, i.e. ψ is a constant on $r = R_s$. To make B_θ and B_ϕ vanish, we must then require $\partial\psi/\partial r$ also to be constant on $r = R_s$. These two independent conditions on the outer boundary plus the further condition on the inner boundary overdetermine the problem so that in general it is not possible to find a solution. One could require one component, B_θ say, to vanish at the outer boundary, but the choice would appear to be arbitrary.

A somewhat less arbitrary procedure would be to minimise the horizontal field on this boundary in the least-squares sense, subject to the constraint provided by the lower boundary condition. This requires us to minimise

$$\langle B_\perp^2\rangle = \int_0^{2\pi}\int_0^{\pi}(B_\phi B_\phi^* + B_\theta B_\theta^*)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi, \tag{22}$$

subject to (16), and leads to the unique solution

$$A_{lm} = -R_\odot^2\frac{n_l'(\alpha R_s)\lambda_l' + n_l(\alpha R_s)\lambda_l}{(\lambda_l'^2 + \lambda_l^2)}\frac{C_{lm}}{l(l+1)}, \tag{23}$$

$$B_{lm} = R_\odot^2\frac{j_l'(\alpha R_s)\lambda_l' + j_l(\alpha R_s)\lambda_l}{(\lambda_l'^2 + \lambda_l^2)}\frac{C_{lm}}{l(l+1)}, \tag{24}$$

where

$$\lambda_l = j_l(\alpha R_\odot)n_l(\alpha R_s) - j_l(\alpha R_s)n_l(\alpha R_\odot), \tag{25}$$

$$\lambda_l' = j_l(\alpha R_\odot)n_l'(\alpha R_s) - j_l'(\alpha R_s)n_l(\alpha R_\odot), \tag{26}$$

the primes indicating the derivatives $j_l'(x) = \mathrm{d}j_l(x)/\mathrm{d}x$ and $n_l'(x) = \mathrm{d}n_l(x)/\mathrm{d}x$. The coefficients are undetermined when $l = 0$, but these give rise to only a constant term in the expansion for ψ and are eliminated in the expressions for $\mathbf{B}$.

This model has two desirable features. Firstly, if B_r is a given real function on the lower boundary, the complex conjugates of the coefficients C_{lm} must satisfy $C^*_{lm} = C_{l-m}$ for all l, m. From (23) and (24) it is easy to see that the coefficients A_{lm} and B_{lm} satisfy the same relation, and therefore ψ and **B** are real as well.

Secondly, the mean-square horizontal field over the outer boundary is

$$\langle B_\perp^2 \rangle = \alpha^2 \frac{R_\odot^4}{R_S^2} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{\left[n_l(\alpha R_s) j_l'(\alpha R_s) - n_l'(\alpha R_s) j_l(\alpha R_s)\right]^2}{(\lambda'^2 + \lambda^2)} \frac{|C_{lm}|^2}{l(l+1)}. \tag{27}$$

In the limit $\alpha \to 0$, the horizontal field over the whole surface must vanish, and we recover the condition imposed in the potential field case. The model is, in this sense, the optimal analogue of the standard potential field construction.

4. Cartesian Coordinates

If we generate the solution by separating in terms of cartesian coordinates x, y, z, we find a continuum of eigenfunctions over the horizontal plane with

$$\psi = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[A(l,m) e^{kz} + B(l,m) e^{-kz}\right] e^{ilx} e^{imy} \, dl \, dm, \tag{28}$$

where $k^2 = l^2 + m^2 - \alpha^2$.

Taking **a** to be the unit vector in the z-direction, we obtain

$$\mathbf{B} = \left[\frac{\partial}{\partial x}\frac{\partial \psi}{\partial z} + \alpha \frac{\partial \psi}{\partial y}, \ \frac{\partial}{\partial y}\frac{\partial \psi}{\partial z} - \alpha \frac{\partial \psi}{\partial x}, \ -\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right)\right]. \tag{29}$$

It is obvious that there is now no distinction between interior and exterior solutions. Moreover, the specification of $B_z(x, y)$ on the surface $z = 0$ is insufficient to determine the functions $A(l, m)$ and $B(l, m)$. Equating the expansion of B_z in (29) to the Fourier integral representation of the boundary value,

$$B_z(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} C(l, m) \, e^{ilx} e^{imy} \, dl \, dm, \tag{30}$$

produces the single relation

$$A(l, m) + B(l, m) = C(l, m)/(l^2 + m^2), \tag{31}$$

from which $A(l, m)$ and $B(l, m)$ cannot be found uniquely.

The problem posed in the infinite half-space ($z \geq 0$) is in any case physically unrealistic, as noted by Alissandrakis (1981). If we limit the eigenfunctions to those in which $lim_{z \to \infty} \psi = 0$, equation (28) becomes

$$\psi = \int \int_{l^2 + m^2 > \alpha^2} A(l, m) \exp(-\sqrt{l^2 + m^2 - \alpha^2}) \, e^{ilx} e^{imy} \, dl \, dm. \tag{32}$$

This form is compatible with the boundary value expansion (30) only if $C(l,m)$ happens to vanish for all $l^2+m^2<\alpha^2$. Unrestricted boundary conditions lead to solutions with unbounded magnetic energy.

This is not to deny that physically relevant solutions may be constructed. A half-space of infinite horizontal extent does not model a system with spherical symmetry unless we impose a periodicity on the solution over scales greater than $L \sim R_\odot$. If we consider only the finite horizontal region $-L \le x,y \le L$, the general solution becomes

$$\psi = \sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\left(A_{lm}\mathrm{e}^{kz}+B_{lm}\mathrm{e}^{-kz}\right)\exp\left(\mathrm{i}\frac{\pi lx}{L}\right)\exp\left(\mathrm{i}\frac{\pi my}{L}\right), \tag{28a}$$

where now $k^2 = \pi^2(l^2+m^2)/L^2-\alpha^2$, and the lower boundary condition (30) becomes

$$B_z(x,y) = \sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} C_{lm}\exp\left(\mathrm{i}\frac{\pi lx}{L}\right)\exp\left(\mathrm{i}\frac{\pi my}{L}\right). \tag{30a}$$

We can then see that solutions with bounded magnetic energy exist under two conditions. Firstly, the net magnetic flux across the lower boundary must vanish so that $C_{00}=0$—the corresponding coefficient in ψ, A_{00}, is then undetermined, but the constant term does not appear in the expression for **B**. Secondly, α cannot be chosen greater than π/L. Since the magnitude of α measures the departure of the force-free field configuration from the potential field case, this limit is a severe restriction on global models, for which $\pi/L \sim \pi/R_\odot \sim 10^{-8}\,\mathrm{m}^{-1}$.

The limit can be raised by confining attention to more restricted areas of the solar surface (Nakagawa and Raadu 1972), though then the imposition of strict periodicity is not generally valid. The boundary conditions on the vertical surfaces —i.e. the selection of lm modes to represent the solution—cannot be determined uniquely by physical considerations and the choice varies from author to author. Care must also be taken not to the extrapolate the solution beyond its domain of validity in the vertical direction. For any given α, the artificial truncation of the horizontal extent of the field distribution will not remove the presence of larger scale fields in the observed coronal structure. These will be significant at heights greater than the scale height of the largest scale Fourier component considered, i.e. for $z > H_{\max} \sim k_{\min}^{-1} = L/\sqrt{\pi^2-(\alpha^2/L^2)}$. If α is small compared with π/L the model will be valid over a vertical extent comparable with its horizontal dimensions. As $|\alpha|$ approaches its limiting value of π/L the domain of validity increases indefinitely in the vertical direction.

The effect of adding an upper boundary at $z=Z_s$ can be studied with greater algebraic simplicity in the cartesian system than in the spherical system. Suppose that $B_z(x,y,Z_s)$ is given as

$$B_z(x,y,Z_s) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} D(l,m)\,\mathrm{e}^{\mathrm{i}lx}\mathrm{e}^{\mathrm{i}my}\,\mathrm{d}l\,\mathrm{d}m, \tag{33}$$

requiring

$$A(l,m)\,\mathrm{e}^{kZ_s}+B(l,m)\,\mathrm{e}^{-kZ_s} = D(l,m)/(l^2+m^2). \tag{34}$$

This, together with (31), has a unique solution for $A(l,m)$ and $B(l,m)$ if k is real, i.e. if $l^2+m^2-\alpha^2>0$. If $l^2+m^2-\alpha^2<0$, k is imaginary and the solution is unique if $\sin\kappa Z_s \neq 0$, where $\mathrm{i}\kappa = k$. If $\kappa Z_s = j\pi$ for any integer j, the solution either does not exist or is not unique.

As in the spherical system, it is not possible to make both horizontal components of the field vanish on the upper boundary unless $\alpha = 0$. If we choose instead to minimise the horizontal field in the least-squares sense, i.e. minimise

$$\langle B_\perp^2\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(B_xB_x^* + B_yB_y^*)\,\mathrm{d}x\,\mathrm{d}y, \tag{35}$$

subject to (32), we obtain the unique solution

$$A(l,m) = \frac{(k^2+\alpha^2)\mathrm{e}^{-2kZ_s}+(k^2-\alpha^2)}{(k^2+\alpha^2)(\mathrm{e}^{2kZ_s}+\mathrm{e}^{-2kZ_s})+2(k^2-\alpha^2)}\,\frac{C(l,m)}{(l^2+m^2)}, \tag{36}$$

$$B(l,m) = \frac{(k^2+\alpha^2)\mathrm{e}^{2kZ_s}+(k^2-\alpha^2)}{(k^2+\alpha^2)(\mathrm{e}^{2kZ_s}+\mathrm{e}^{-2kZ_s})+2(k^2-\alpha^2)}\,\frac{C(l,m)}{(l^2+m^2)}. \tag{37}$$

Inspection of the common denominator of these expressions reveals that it vanishes for no real or imaginary value of k, except $k=0$. In the latter case, the two terms in the expansion (28) are not independent and their combined coefficient is determined uniquely by (31). Thus our assertion that this solution is unique is proven. Of course, the relation (31) cannot be satisfied for $l=0, m=0$ unless $C(0,0)=0$, so that the existence of a solution requires the net magnetic flux across the lower boundary to vanish. The indeterminacy of the constant component in the expansion (28) again does not affect the solution for **B**.

This model has the same features in the cartesian system as in the spherical system. Firstly, if B_z is specified as a real function on the lower boundary, the complex conjugates of the coefficients $C(l,m)$ must satisfy $C^*(l,m) = C(-l,-m)$ for all l,m. Then the coefficients given by (36) and (37) satisfy

$$A^*(l,m) = A(-l,-m), \qquad B^*(l,m) = B(-l,-m), \tag{38}$$

if k is real, and

$$A^*(l,m) = B(-l,-m), \qquad B^*(l,m) = A(-l,-m), \tag{38a}$$

if k is imaginary. These relations then guarantee that ψ, and hence **B**, is real throughout the volume.

Secondly, the mean-square horizontal field over the outer boundary is

$$\langle B_\perp^2\rangle = \alpha^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{4k^2}{(k^2+\alpha^2)(\mathrm{e}^{2kZ_s}+\mathrm{e}^{-2kZ_s})+2(k^2-\alpha^2)}\,\frac{|C(l,m)|^2}{(l^2+m^2)}\,\mathrm{d}l\,\mathrm{d}m, \tag{39}$$

so that we again recover the potential field case in the limit $\alpha\to 0$.

5. Cylindrical Coordinates

The case of cylindrical coordinates ρ, ϕ, z exactly parallels the cartesian case; it has already been discussed by several authors (e.g. Raadu and Nakagawa 1971; Chiu and Hilton 1977) and is included here for completeness. The general solution which is regular on the axis is

$$\psi = \sum_{m=-\infty}^{\infty} \int_0^{\infty} \left[A_m(l) e^{kz} + B_m(l) e^{-kz}\right] J_m(l\rho)\, e^{im\phi}\, dl, \tag{40}$$

where $k^2 = l^2 - \alpha^2$.

Taking **a** again as the unit vector in the z-direction, we get

$$\mathbf{B} = \left[\frac{\partial}{\partial\rho}\frac{\partial\psi}{\partial z} + \frac{\alpha}{\rho}\frac{\partial\psi}{\partial\phi},\ \ \frac{1}{\rho}\frac{\partial}{\partial\phi}\frac{\partial\psi}{\partial z} - \alpha\frac{\partial\psi}{\partial\rho},\ \ -\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial\psi}{\partial\rho} + \frac{1}{\rho}\frac{\partial^2\psi}{\partial\phi^2}\right)\right]. \tag{41}$$

The lower boundary condition can now be written in terms of the complete set of functions $J_m(l\rho)e^{im\phi}$ as

$$B_z(\rho, \phi) = \sum_{m=-\infty}^{\infty} \int_0^{\infty} C_m(l) J_m(l\rho)\, e^{im\phi}\, dl. \tag{42}$$

Equating this to the expression in (41) using the expansion (40) yields the single relation

$$A_m(l) + B_m(l) = C_m(l)/l^2, \tag{43}$$

from which $A_m(l)$ and $B_m(l)$ cannot be found uniquely.

Limiting the eigenfunctions to those for which $\lim_{z\to\infty} \psi = 0$, equation (40) becomes

$$\psi = \sum_{m=-\infty}^{\infty} \int_{|\alpha|}^{\infty} B_m(l) e^{-kz} J_m(l\rho)\, e^{im\phi}\, dl, \tag{44}$$

where the domain of integration excludes the origin, and is compatible with the boundary condition (42) only if $C_m(l) = 0$ for all $l < |\alpha|$. If, on the contrary, the expression (42) contains any components with non-vanishing $C_m(l)$ for $l < |\alpha|$, then ψ contains terms with a purely sinusoidal z-dependence. When used to model the coronal field, such solutions possess the physically unacceptable property of unbounded total magnetic energy (cf. Nakagawa *et al.* 1971).

As in the cartesian case, any cylindrical representation of the corona can possess only components whose scales are less than $R_\odot$, i.e. components whose first zero $j_{m,1}$ lie within $l\rho \sim lR_\odot$; hence we must set $C_m(l) = 0$ for $l < j_{m,1}/R_\odot$. In order to produce only exponentially decaying terms in (41) we must also restrict the value of α to $|\alpha| < l_{\min} \sim j_{0,1}/R_\odot \sim 2.4/R_\odot$. This is essentially the same restriction as before.

Once again, any attempt to model a finite horizontal region, $\rho < R$ say, that would enable larger values of α to be employed (up to $\alpha \sim 2.4/R$), introduces a degree of arbitrariness by way of choice of boundary conditions that cannot be resolved by observations. All such solutions are also valid for only a finite extent in the vertical direction, unless $|\alpha|$ is equal to its maximum value. The largest significant scale is $H_{\max} \sim L/\sqrt{5.76-(\alpha/L)^2}$, closely analogous to the cartesian case.

The dilemmas may be resolved in the same manner as before.

6. Discussion

Whereas potential field modelling of the global coronal field is a matter of routine (Altschuler *et al.* 1977), with the exception of Levine and Altschuler (1974) little attention has been given to use of linear force-free fields to model the global corona. In global models, the effect of the solar wind at $r \sim 2R_\odot$ cannot be ignored but the standard method of obtaining a unique solution in spherical geometry—the imposition of the 'radiation' condition—takes no account of it. The difficulty cannot be circumvented by employing cartesian or cylindrical geometry. Although solutions can then be found with all components decaying exponentially with height, we are then restricted to such small values of $|\alpha|$ that the linear force-free and potential models will barely differ. Moreover, the effect of the solar wind is still not incorporated. This dilemma may be resolved by creating an outer boundary over which the horizontal components of the field are minimised, a procedure which leads to a unique solution for a given normal component of $\mathbf{B}$ on the lower boundary, as well as mimicking as far as possible the effect of the solar wind source surface introduced in potential models. The position of the outer boundary which leads to the best representation of the solar wind effects must be found by trial and error. In the potential field case, it is found to be at $R_S \sim 2.5R_\odot$.

Recent considerations of the global electrodynamics of coronal heating and activity have suggested that the solar corona may support relatively large permanent current systems, and that the current systems might relax conserving their magnetic helicity into a linear force-free form (Heyvaerts and Priest 1984). These ideas might be tested by a systematic investigation of global models with $\alpha \neq 0$.

Although few comparisons of global models with coronal morphology have been made, many authors have compared linear force-free field extrapolations of the photospheric field over limited regions of the solar disk with the observed structure of features such as active regions (Raadu and Nakagawa 1971; Nakagawa *et al.* 1973; Levine 1976). Our discussion of the cartesian and cylindrical systems has exposed the limitations of such investigations. No account is taken of an outer boundary introducing the gross effect of the solar wind. All demand that the field components vanish as $r \to \infty$, so that the range of α values that can be accommodated has an upper limit of the order of L^{-1}, where L is the horizontal dimension of the area under consideration. Several authors (e.g. Levine 1974; Seehafer 1978, 1982) have pointed out the uncertainties introduced by the indeterminacy of the boundary conditions at the vertical bounding surfaces. The first two limitations can be lifted by introducing an outer horizontal boundary as before, but the last difficulty is insurmountable and the extent to which it may influence any conclusions drawn from comparisons can be judged only by numerical experiments.

A final point that calls for comment is the means of determining the value of α. The analyses summarised above assume a value for α, and allow us to generate solutions for any choice within the range of the model. Chiu and Hilton (1977) suggested that the value of α may be extracted from observations of the photospheric field, at least in principle. If, for instance, the horizontal field components were known there, a best-fit value of α could be obtained by a least-squares comparison of the computed and measured components. (Indeed, one could contemplate foregoing an outer boundary and determining ψ completely by some extension of this method.) However, it should be realised that the field is not force-free throughout the photosphere, where the magnetic measurements are made. The photosphere and chromosphere together form a very thin transition layer of relatively low ionisation between a plasma dominated regime in the subsurface regions and a magnetically dominated regime in the corona. If we allow for a surface current distribution at such a 'discontinuity', the Maxwell equations require the normal or radial component of **B** to be continuous, but not the horizontal components. Thus the coronal field is properly determined by measurements of B_r or B_z, but is not properly determined by measurements of the horizontal components in the photosphere. For this reason, the use of line-of-sight components in place of the true normal components should be avoided in the calculation of linear force-free and potential fields. We need to fix α by reference to the coronal conditions, and the only legitimate procedure appears to be that adopted by most authors, i.e. it is found *a posteriori* by comparing the calculated field structures with observations of the morphology of the actual coronal features.

Acknowledgments

The author wishes to thank an anonymous referee for critical comments on the physical significance of linear force-free fields.

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Manuscript received 14 September 1988, accepted 2 February 1989