

Heavy Ion Acceleration by Double-cyclotron Absorption: Some Analytic Approximations

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Abstract

It is possible to accelerate heavy ions in a magnetised plasma via a process where an ion simultaneously absorbs two gyromagnetic waves whose frequencies sum to the ion gyrofrequency. Analytic approximations to the probability of such double-cyclotron absorption are derived. A particularly simple approximation is presented for the special case where the waves all have frequencies very close to half the ion gyrofrequency. The rate of perpendicular acceleration of the ions due to double-cyclotron absorption is considered, and a simple analytic expression is derived for the case where the waves have the quasimonochromatic distribution mentioned above. One possible application of these approximations is to investigate the proposal that perpendicular acceleration of oxygen (O^+) ions in the Earth's magnetosphere, due to double-cyclotron absorption, may give rise to O^+ -conic distributions.

1. Introduction

Low frequency waves are thought to play an important role in the acceleration and heating of ions in a wide variety of space and laboratory plasma phenomena. Mechanisms which have attracted recent attention include quasilinear acceleration by lower hybrid waves (Chang and Coppi 1981; Retterer *et al.* 1986), Alfvén waves (Chang *et al.* 1986), and electrostatic ion cyclotron (EIC) waves (Ungstrup *et al.* 1979; Ashour-Abdalla and Okuda 1984); nonlinear large amplitude EIC waves (Lysak *et al.* 1980); and stochastic motion in phase space (Smith and Kaufman 1975). The aim of this paper is to develop equations describing the acceleration of ions in a magnetised plasma which results from the simultaneous absorption of two gyromagnetic waves. In particular, the equations necessary for an investigation of the role that such a double-cyclotron absorption process may play in the production of ion-conic distributions are presented.

This paper was motivated by a proposal due to Temerin (1986), that some ion conics may be produced via a two wave interaction involving waves with frequencies below the ion gyrofrequency. Temerin's mechanism involves a cyclotron resonance between an ion and two waves, the resonance condition being that the sum of the frequencies of the waves matches the ion gyrofrequency. Previous investigations of this acceleration mechanism have been performed by Roth and Temerin (1986) and by Temerin and Roth (1986). In both these papers the waves were assumed to be electrostatic,

linearly polarised and propagating perpendicular to the ambient magnetic field. The waves were also assumed to be of relatively small amplitude, to satisfy $k_{\perp}R \ll 1$ where $k_{\perp}$ is the perpendicular component of the wave vector and R is the ion gyroradius, and to have frequencies ω and ω' summing to the O^+ gyrofrequency. Temerin and Roth (1986) then deduced an expression for the average perpendicular energy gain of the ions from calculations of the orbits of ions in an ambient magnetic field and subject to the electric fields of the two waves. They presented a simple expression for the perpendicular acceleration rate for the special case where the waves all have the same perpendicular wave number $k_{\perp}$ and the wave spectrum is isotropic, with uniform power in a small bandwidth $\Delta\omega$ about $\Omega/2$ (Ω is the ion gyrofrequency).

The aim of this paper is to start from the general equations describing the processes of double-cyclotron absorption and emission (Ball and Melrose 1989, present issue p. 481), and to derive approximations which lead to a simple analytic expression for the rate of average perpendicular energy gain of heavy ions due to double-cyclotron absorption. The calculation given here is an alternative to, but is completely different from, that of Temerin and Roth (1986). The approximations presented in this paper are developed primarily with a view to applying the result to the production of O^+ -conics in the supra-auroral zones of the Earth's magnetosphere. However, the implications of this work for O^+ -conic production via double-cyclotron absorption are considered elsewhere (Ball 1989) and will not be discussed in this paper.

Since the aim of this paper is to develop analytic approximations which are applicable to the ion-conic acceleration mechanism, some of the simplifying assumptions made in this work are based firmly on experimental data relevant to ion conic production. Other assumptions are made primarily to simplify the analysis. These assumptions are consistent with those made explicitly by Temerin and Roth (1986). Since all the assumptions and approximations made in this work are made explicitly, their validity may be considered in the light of available experimental data. This makes it clear where the data are insufficient to justify or invalidate the assumptions, and where the analysis may need to be revised should new data contradict the assumptions.

In §2 the general equations derived by Ball and Melrose (1989) which form the basis for this paper are briefly presented. Approximations to the absorption probability for heavy ions in an electron-proton plasma are derived in §3. The 'small-gyroradius approximation' is assumed, and it is assumed that the wave electric field is perpendicular to the ambient magnetic field. The approximation to the nonlinear response requires that the heavy ions are test particles and do not contribute to the collective response of the plasma, and the electrons and protons which determine the collective response are assumed to be cold. The rate of perpendicular acceleration (the gain in average perpendicular energy per unit time), for singly ionised oxygen is calculated in §4. Simple analytic expressions are given in §3 and §4, for the special case of quasimonochromatic waves with $\omega \sim \Omega/2$ as considered by Temerin and Roth (1986). In §5 the assumptions made in §3 and §4 are discussed, and the conditions which must be satisfied by the wave and plasma parameters for the approximations derived in §3 and §4 to be applicable are established. A summary of the results is given in §6.

2. The General Equations

In the semiclassical formalism the particles are described individually by their momentum $\mathbf{p}$ (velocity $\mathbf{v}$) and collectively by their distribution function $f(\mathbf{p})$, normalised so that the particle number density $n = \int d^3\mathbf{p} f(\mathbf{p})$. Waves in a wave mode M are described by their dispersion relation $\omega = \omega_M(\mathbf{k})$ and polarisation vector $\mathbf{e}_M(\mathbf{k})$, and are regarded as collections of quanta with energy $\hbar|\omega_M(\mathbf{k})|$, momentum $\hbar\mathbf{k}$ and occupation number $N_M(\mathbf{k})$. The occupation number may be formally defined in terms of the total energy $W_M(\mathbf{k})$ of waves in a mode M with wave vector $\mathbf{k}$, enclosed in a system volume V :

$$N_M(\mathbf{k}) = \frac{W_M(\mathbf{k})}{\hbar|\omega_M(\mathbf{k})|V}. \quad (1)$$

The kinetic equation for the particle distribution can be written in the form (Ball and Melrose 1989)

$$\begin{aligned} \frac{df(\mathbf{p})}{dt} = & \frac{\partial}{\partial p_{\parallel}} \left\{ A_{\parallel} f(\mathbf{p}) + \left[D_{\parallel\parallel} \frac{\partial}{\partial p_{\parallel}} + D_{\parallel\perp} \frac{\partial}{\partial p_{\perp}} \right] f(\mathbf{p}) \right\} + \\ & + \frac{1}{p_{\perp}} \frac{\partial}{\partial p_{\perp}} \left\{ p_{\perp} \left[A_{\perp} f(\mathbf{p}) + \left(D_{\perp\parallel} \frac{\partial}{\partial p_{\parallel}} + D_{\perp\perp} \frac{\partial}{\partial p_{\perp}} \right) f(\mathbf{p}) \right] \right\} \end{aligned} \quad (2)$$

with

$$\begin{aligned} \begin{pmatrix} A_{\parallel} \\ A_{\perp} \end{pmatrix} = & \sum_{s=-\infty}^{s=\infty} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{d^3\mathbf{k}'}{(2\pi)^3} [N_M(\mathbf{k}) + N_{M'}(\mathbf{k}')] \times \\ & \times \begin{pmatrix} \hbar(k_{\parallel} + k'_{\parallel}) \\ \hbar s \Omega / v_{\perp} \end{pmatrix} w_{MM'}(\mathbf{k}, \mathbf{k}', \mathbf{p}; s), \end{aligned} \quad (3)$$

and

$$\begin{aligned} \begin{pmatrix} D_{\parallel\parallel} \\ D_{\parallel\perp} = D_{\perp\parallel} \\ D_{\perp\perp} \end{pmatrix} = & \sum_{s=-\infty}^{s=\infty} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{d^3\mathbf{k}'}{(2\pi)^3} N_M(\mathbf{k}) N_{M'}(\mathbf{k}') \times \\ & \times \begin{pmatrix} [\hbar(k_{\parallel} + k'_{\parallel})]^2 \\ \hbar(k_{\parallel} + k'_{\parallel}) \hbar s \Omega / v_{\perp} \\ (\hbar s \Omega / v_{\perp})^2 \end{pmatrix} w_{MM'}(\mathbf{k}, \mathbf{k}', \mathbf{p}; s). \end{aligned} \quad (4)$$

The subscripts $\parallel$ and $\perp$ indicate components parallel and perpendicular to the magnetic field respectively. The quantity $w_{MM'}(\mathbf{k}, \mathbf{k}', \mathbf{p}; s)$ which appears in equations (3) and (4) is the probability of double-cyclotron absorption at a harmonic s of the ion gyrofrequency Ω and is of central importance in this treatment. Note that the equations of Ball and Melrose (1989) included relativistic effects. However, in this paper the strictly nonrelativistic limit is taken; the Lorentz factor is set equal to unity and the momentum $\mathbf{p}$ is set

equal to $m\mathbf{v}$. Thus $\Omega = |q|B/m$ where q is the ion charge, m is the rest mass of the ion and B is the magnitude of the ambient magnetic field $\mathbf{B}$.

The term in equation (2) of direct relevance to the postulated acceleration of oxygen ions, and hence the production of ion conics via double-cyclotron absorption, is that involving $D_{\perp\perp}$. This point is discussed in detail in §4.

The general expression for the absorption probability derived by Ball and Melrose (1989) may be written in the form

$$w_{MM'}(\mathbf{k}, \mathbf{k}', \mathbf{p}; s) = \frac{2\pi q^4}{m^2 \epsilon_0^2} \frac{|a_{MM'}(\mathbf{k}, \mathbf{k}', \mathbf{p}; s)|^2}{|\omega_M(\mathbf{k})\omega_{M'}(\mathbf{k}')|} R_M(\mathbf{k})R_{M'}(\mathbf{k}') \times \\ \times \delta[\omega_M(\mathbf{k}) - k_{\parallel}v_{\parallel} + \omega_{M'}(\mathbf{k}') - k'_{\parallel}v'_{\parallel} - s\Omega] \quad (5)$$

with

$$a_{MM'}(\mathbf{k}, \mathbf{k}', \mathbf{p}; s) = a_{MM'}^{TS}(\mathbf{k}, \mathbf{k}', \mathbf{p}; s) + a_{MM'}^{NL}(\mathbf{k}, \mathbf{k}', \mathbf{p}; s) \quad (6)$$

and

$$a_{MM'}^{TS}(\mathbf{k}, \mathbf{k}', \mathbf{p}; s) = e_{Mi}^*(\mathbf{k})e_{M'j}^*(\mathbf{k}')a_{ij}^{TS}(\mathbf{k}, \mathbf{k}', \mathbf{p}; s), \\ a_{MM'}^{NL}(\mathbf{k}, \mathbf{k}', \mathbf{p}; s) = e_{Mi}^*(\mathbf{k})e_{M'j}^*(\mathbf{k}')a_{ij}^{NL}(\mathbf{k}, \mathbf{k}', \mathbf{p}; s). \quad (7)$$

The quantity $R_M(\mathbf{k})$ which appears in (5) is the ratio of electric energy density to total energy density of waves in mode M . In (7), $\mathbf{e}_M(\mathbf{k})$ is the polarisation vector (e.g., Melrose 1980*a*, pp. 42–45) and the subscripts i and j refer to components in a cartesian co-ordinate system where the magnetic field is along the z -axis. The superscripts TS and NL in equations (6) and (7) indicate the Thomson-scattering-like (hereafter TS-like) contribution and 'nonlinear' contribution respectively. The TS-like term is given by

$$a_{ij}^{TS}(\mathbf{k}, \mathbf{k}', \mathbf{p}; s) = \sum_{t'=-\infty}^{\infty} \alpha_{ij}(s - t', \mathbf{k}; t', \mathbf{k}'; \mathbf{p}) \quad (8)$$

with

$$\alpha_{ij}(s - t', \mathbf{k}; t', \mathbf{k}'; \mathbf{p}) = \frac{\exp[-i\epsilon(t\psi + t'\psi')]}{\omega_t \omega_{t'}} \{ \omega_t \omega_{t'} J_t(z) J_{t'}(z') \tau_{ij}^*(\omega_{t'}) - \\ - \omega_t J_t(z) V_j(\mathbf{k}', \mathbf{p}; t') k_I \tau_{II}^*(\omega_{t'}) - \omega_{t'} J_{t'}(z') V_i(\mathbf{k}, \mathbf{p}; t) k_I' \tau_{IJ}^*(\omega_{t'}) + \\ + [k_I k_I' \tau_{In}^*(\omega_{t'}) - \omega_M(\mathbf{k}) \omega_{M'}(\mathbf{k}')/c^2] V_i(\mathbf{k}, \mathbf{p}; t) V_j(\mathbf{k}', \mathbf{p}; t') \} \quad (9)$$

where $t = s - t'$, $\epsilon = q/|q|$, $\mathbf{k} = (k_{\perp} \cos \psi, k_{\perp} \sin \psi, k_{\parallel})$, $\omega_t = \omega - t\Omega_r - k_{\parallel}v_{\parallel}$, $z = k_{\perp}v_{\perp}/\Omega_r$ and the primed quantities are defined analogously. Here $J_t(z)$ is the Bessel function of order t . The vector $\mathbf{V}(\mathbf{k}, \mathbf{p}; t)$ is the velocity function for the spiralling ion and is given by

$$\mathbf{V}(\mathbf{k}, \mathbf{p}; t) = \{v_{\perp} [e^{i\epsilon\psi} J_{t-1}(z) + e^{-i\epsilon\psi} J_{t+1}(z)]/2, \\ -i\epsilon v_{\perp} [e^{i\epsilon\psi} J_{t-1}(z) - e^{-i\epsilon\psi} J_{t+1}(z)]/2, v_{\parallel} J_t(z)\}. \quad (10)$$

The quantity $\tau_{ij}(\omega)$ may be written in the form

$$\tau_{ij}(\omega) = (\omega^2 \delta_{ij} - \Omega^2 b_i b_j + i\epsilon \omega \Omega \varepsilon_{ijl} b_l) / (\omega^2 - \Omega^2) \quad (11)$$

where $\mathbf{b} = \mathbf{B}/|\mathbf{B}|$ and ε_{ijl} is the antisymmetric rank 3 tensor. The asterisk denotes complex conjugation. The 'nonlinear' term on the right-hand side of (7) is

$$\begin{aligned} a_{ij}^{NL}(\mathbf{k}, \mathbf{k}', \mathbf{p}; s) &= \frac{2m(-1)^s}{q\epsilon_0} \frac{\exp[-i\epsilon s(\psi + \psi')]}{[\omega_M(\mathbf{k}) + \omega_M(\mathbf{k}')]^2} \alpha_{ijl}(k, -k', k+k') \\ &\times \frac{\lambda_{lm}(k+k')}{\Lambda(k+k')} V_m(\mathbf{k} + \mathbf{k}', \mathbf{p}; s) \end{aligned} \quad (12)$$

where the shorthand notation $\omega, \mathbf{k} \rightarrow k$ has been used. The quantity α_{ijl} is the quadratic nonlinear response tensor of the plasma (e.g., Melrose 1986, pp. 11, 81–82) and is not related to the 2-index quantity α_{ij} which appears in equations (8) and (9). The quantity $\Lambda(k)$ is the determinant of the tensor $\Lambda_{ij}(k)$ and $\lambda_{ij}(k)$ is the cofactor of the tensor element $\Lambda_{ji}(k)$. Finally, the tensor $\Lambda_{ij}(k)$ may be defined in terms of the dielectric tensor $\kappa_{ij}(k)$,

$$\Lambda_{ij}(k) = (k_i k_j - k^2 \delta_{ij}) c^2 / \omega^2 + \kappa_{ij}(k). \quad (13)$$

3. Approximations to the Absorption Probability

The equations for $w_{MM'}(\mathbf{k}, \mathbf{k}', \mathbf{p}; s)$ simplify greatly when the arguments of the Bessel functions J_t satisfy $z, z' \ll 1$. The quantities z and z' are simply related to the ion gyroradius $R = v_\perp / \Omega$ by $z = k_\perp R$ and $z' = k'_\perp R$. The approximation where only contributions to first order in R are retained is referred to as the 'small-gyroradius approximation' and is assumed hereafter. In this limit

$$J_t(z) = \begin{cases} 1 & \text{if } t = 0 \\ tz/2 & \text{if } |t| = 1 \\ 0 & \text{if } |t| > 1 \end{cases} \quad (14)$$

and substitution of (14) into (10) gives, to first order in R ,

$$\mathbf{V}(\mathbf{k}, \mathbf{p}; t) = \begin{cases} (0, 0, v_\parallel) & \text{if } t = 0 \\ R\Omega e^{it\epsilon\psi}(1, -it\epsilon, 0)/2 & \text{if } |t| = 1 \\ (0, 0, 0) & \text{if } |t| > 1 \end{cases} \quad (15)$$

In the derivation of (15) it has also been assumed that $v_\parallel$ is at most of the order of $v_\perp (= \Omega R)$.

Only the probability of double-cyclotron absorption at the fundamental of the ion gyrofrequency, $w_{MM'}(\mathbf{k}, \mathbf{k}', \mathbf{p}; 1)$, is considered in the rest of this work.

(a) The TS-like Term

It is clear from the form of (14), (15) and (9) that the only elements of $\alpha_{ij}(1-t', \mathbf{k}; t', \mathbf{k}'; \mathbf{p})$ which can contribute in the small-gyroradius approximation are $\alpha_{ij}(0, \mathbf{k}; 1, \mathbf{k}'; \mathbf{p})$ and $\alpha_{ij}(1, \mathbf{k}; 0, \mathbf{k}'; \mathbf{p})$. The argument of the δ -function in (5) implies $\omega_M(\mathbf{k}) - k_{\parallel}v_{\parallel} + \omega_{M'}(\mathbf{k}') - k'_{\parallel}v_{\parallel} - \Omega = 0$. Thus for $t = 0$ and $t' = 1$, $\omega'_{t'} = \omega'_1$ and $\omega_t = -\omega'_1$, and for $t = 1$ and $t' = 0$, $\omega'_{t'} = -\omega_1$ and $\omega_t = \omega_1$ where

$$\omega'_1 = \omega_{M'}(\mathbf{k}') - k'_{\parallel}v_{\parallel} - \Omega \quad \text{and} \quad \omega_1 = \omega_M(\mathbf{k}) - k_{\parallel}v_{\parallel} - \Omega. \quad (16)$$

Substitution of (14) and (15) into (9) gives, to lowest order in R ,

$$\alpha_{ij}(0, \mathbf{k}; 1, \mathbf{k}'; \mathbf{p}) = \frac{R}{2} [k'_{\perp} \tau_{ij}^*(\omega'_1) e^{-i\epsilon\psi'} - \frac{\Omega k_{\perp} \tau_{ij}^*(\omega'_1)}{\omega'_1} (1, -i\epsilon, 0)_j] \quad (17)$$

and

$$\alpha_{ij}(1, \mathbf{k}; 0, \mathbf{k}'; \mathbf{p}) = \frac{R}{2} [k_{\perp} \tau_{ij}(\omega_1) e^{-i\epsilon\psi} - \frac{\Omega k'_{\perp} \tau_{ij}(\omega_1)}{\omega_1} (1, -i\epsilon, 0)_i]. \quad (18)$$

If the Doppler-shift terms $k_{\parallel}v_{\parallel}$ and $k'_{\parallel}v_{\parallel}$ are sufficiently small to be neglected in (16) then the resonance condition simplifies to $\omega_M(\mathbf{k}) + \omega_{M'}(\mathbf{k}') = \Omega$ which implies

$$\omega_1 = -\omega_{M'}(\mathbf{k}') \quad \text{and} \quad \omega'_1 = -\omega_M(\mathbf{k}). \quad (19)$$

For waves with frequencies much lower than the electron plasma frequency the component of wave electric field parallel to the ambient magnetic field $\mathbf{B}$ is zero, i.e. $\mathbf{e}_M(\mathbf{k}) \cdot \mathbf{B} = \mathbf{e}_{M'}(\mathbf{k}') \cdot \mathbf{B} = 0$, and the polarisation vector can be written in the form

$$\mathbf{e}_M(\mathbf{k}) = (\cos \psi - it_M \sin \psi, \sin \psi + it_M \cos \psi, 0) / (1 + t_M^2)^{1/2} \quad (20)$$

where t_M is a real parameter describing the degree of polarisation; $\mathbf{e}_{M'}(\mathbf{k}')$ is given by (20) with M, ψ replaced by M', ψ' . Substitution of these expressions, together with (11), (17), (18) and (19), into (8) and then (7) gives

$$\begin{aligned} a_{MM'}^{TS}(\mathbf{k}, \mathbf{k}', \mathbf{p}; 1) = & \frac{R}{2[(1 + t_M^2)(1 + t_{M'}^2)]^{1/2}} \times \\ & \times \left\{ \left[(1 - t_M t_{M'}) \left(\frac{\omega^2 k'_{\perp} e^{-i\psi'}}{\omega^2 - \Omega^2} + \frac{\omega'^2 k_{\perp} e^{-i\psi}}{\omega'^2 - \Omega^2} \right) - \right. \right. \\ & - (t_M - t_{M'}) \left(\frac{\omega \Omega k'_{\perp} e^{-i\psi'}}{\omega^2 - \Omega^2} - \frac{\omega' \Omega k_{\perp} e^{-i\psi}}{\omega'^2 - \Omega^2} \right) \Big] \cos(\psi - \psi') + \\ & + i \left[(t_M - t_{M'}) \left(\frac{\omega^2 k'_{\perp} e^{-i\psi'}}{\omega^2 - \Omega^2} + \frac{\omega'^2 k_{\perp} e^{-i\psi}}{\omega'^2 - \Omega^2} \right) - \right. \\ & - (1 - t_M t_{M'}) \left(\frac{\omega \Omega k'_{\perp} e^{-i\psi'}}{\omega^2 - \Omega^2} - \frac{\omega' \Omega k_{\perp} e^{-i\psi}}{\omega'^2 - \Omega^2} \right) \Big] \sin(\psi - \psi') + \\ & + \frac{(1 - t_{M'}) (\omega \Omega - t_M \Omega^2)}{\omega^2 - \Omega^2} k_{\perp} e^{-i\psi} + \\ & \left. + \frac{(1 - t_M) (\omega' \Omega - t_{M'} \Omega^2)}{\omega'^2 - \Omega^2} k'_{\perp} e^{-i\psi'} \right\}. \quad (21) \end{aligned}$$

In (21) the shorthand notation $\omega_M(\mathbf{k}) \rightarrow \omega$ and $\omega_{M'}(\mathbf{k}') \rightarrow \omega'$ has been used (and it is used hereafter) and $\epsilon = +1$ (positive ions) has been assumed.

It is clear from (21) that the interaction vanishes if the waves are identically right circularly polarised $t_M = t_{M'} = 1$. In this case the electric field vectors of the waves rotate about $\mathbf{B}$ in the opposite sense to the gyration of the ions.

For the sake of analytic simplicity, it is useful to consider the special case of linearly polarised waves, i.e. $t_M = t_{M'} = 0$, with equal frequencies, $\omega = \omega' = \Omega/2$ (as considered by Temerin and Roth 1986). In this case (21) simplifies considerably and the resulting expression may be written in the form

$$a_{MM'}^{TS}(\mathbf{k}, \mathbf{k}', \mathbf{p}; 1)_{\Omega/2} = R[k_{\perp} f(\psi, \psi') + k'_{\perp} f(\psi', \psi)] \quad (22)$$

where

$$f(\psi, \psi') = -\frac{1}{6} \{e^{-i\psi} [\cos(\psi - \psi') + 2i \sin(\psi - \psi')] + 2e^{-i\psi'}\} \quad (23)$$

and $f(\psi', \psi)$ denotes the function obtained by interchanging ψ and ψ' in (23).

(b) The 'Nonlinear' Term

The terms $\alpha_{ijl}(k, -k', k + k')$ and $\lambda_{lm}(k + k')/\Lambda(k + k')$, which appear in (12), describe the collective response of the plasma to the waves labelled k and k' . Analytic calculations involving the nonlinear plasma response are generally very cumbersome and are often only manageable when the cold plasma approximation is used. In a self-consistent formulation the collective response of the plasma contains contributions from all the charged species present, including the absorbing ions. However, considerable difficulties result if it is assumed that the contribution of the absorbing ions to the collective plasma response is determined by the cold plasma approximation. These difficulties arise because the cold plasma approximations to α_{ijl}^{σ} and $\lambda_{ij}^{\sigma}/\Lambda^{\sigma}$, where σ denotes the contribution from a particular species, both exhibit singularities at Ω_{σ} , and both must be evaluated at $\omega_M(\mathbf{k}) + \omega_{M'}(\mathbf{k}')$ —very close to the gyrofrequency of the absorbing ions. This complication is avoided in the following work by making a 'test-particle approximation'. It is assumed that the absorbing ions are unable to behave collectively due to either extremely low density or thermal disruption, and all contributions of the absorbing ions to the collective plasma response are neglected.

The collective plasma response is hereafter assumed to be that due to a cold, charge-neutral plasma consisting solely of electrons and protons. For the application of interest here ω , ω' , $\omega + \omega'$ are all of order Ω . In general, $\Omega \ll \Omega_e$ and because of the test-particle approximation $\Omega \ll \Omega_p$ is also satisfied. (When used without a subscript, Ω denotes the gyrofrequency of the absorbing ions; Ω_e and Ω_p denote the electron and proton gyrofrequencies respectively.) From these observations alone it can be shown (Ball 1988) that the electron contribution to the nonlinear response tensor may be neglected and only the proton contribution need be retained.

If one writes

$$\begin{aligned} \mathbf{k}^+ &= \mathbf{k} + \mathbf{k}' = (k_{\perp} \cos \psi + k'_{\perp} \cos \psi', k_{\perp} \sin \psi + k'_{\perp} \sin \psi', k_{\parallel} + k'_{\parallel}) \\ &= (k_{\perp}^+ \cos \psi^+, k_{\perp}^+ \sin \psi^+, k_{\parallel}^+), \end{aligned} \quad (24)$$

then in general, $\mathbf{V}(\mathbf{k}^+, \mathbf{p}; t)$ is given by (10) with ψ replaced by ψ^+ and z replaced by $z^+ = k_\perp^+ v_\perp / \Omega$, and in the "small-gyroradius approximation" $\mathbf{V}(\mathbf{k}^+, \mathbf{p}; t)$ may be obtained from (15) by making the same replacements. Setting $t = 1$, $\epsilon = +1$ then gives

$$\mathbf{V}(\mathbf{k} + \mathbf{k}', \mathbf{p}; 1) \approx R\Omega e^{i\psi^+} \mathbf{v}/2 \quad (25)$$

where

$$\mathbf{v} = (1, -i, 0). \quad (26)$$

The choice of cartesian co-ordinates has so far involved only the specification that the magnetic field lies along the z -axis. The x and y -axes may still be chosen such that any one of ψ , ψ' , ψ^+ is zero. It is most convenient to choose axes such that $\psi^+ = 0$, in which case (24) implies

$$k_\perp \sin \psi + k'_\perp \sin \psi' = 0. \quad (27)$$

When $\psi^+ = 0$, substitution of (25) into (12) and then (7) implies

$$\begin{aligned} a_{MM'}^{NL}(\mathbf{k}, \mathbf{k}', \mathbf{p}; 1) = & -e^{-i(\psi+\psi')} \frac{mR\Omega}{q\epsilon_0} e_{Mi}^*(\mathbf{k}) e_{M'j}^*(\mathbf{k}') \frac{\alpha_{ijl}^p(k, -k', k+k')}{(\omega + \omega')^2} \times \\ & \times \frac{\lambda_{lm}(k+k')}{\Lambda(k+k')} v_m. \end{aligned} \quad (28)$$

The quantity α_{ijl}^p is the proton contribution to the nonlinear response tensor which may be written in the form (Ball 1988)

$$\alpha_{ijl}^p(k, -k', k+k') = \frac{e^3 n_p}{2m_p^2} [k_r T_{ijlr}(\omega, \omega') + k'_r T_{jilr}(\omega', \omega)] \quad (29)$$

where

$$T_{ijlr}(\omega, \omega') = \frac{\bar{\tau}_{jr}^p(\omega') [\bar{\tau}_{il}^p(\omega) - \bar{\tau}_{il}^p(\omega^+)]}{\omega'} + \frac{\bar{\tau}_{rl}^p(\omega^+) \bar{\tau}_{jl}^p(\omega')}{\omega^+} + \frac{\bar{\tau}_{ir}^p(\omega) \bar{\tau}_{jl}^p(\omega')}{\omega}, \quad (30)$$

with $\omega^+ = \omega + \omega'$, and

$$\bar{\tau}_{ij}^p(\omega) = (\omega^2 \delta_{ij} + i\omega \Omega_p \epsilon_{ijl} b_l) / (\omega^2 - \Omega_p^2). \quad (31)$$

Consider again the special case of linearly polarised waves with $t_M = t_{M'} = 0$ and $\omega = \omega' = \Omega/2$. When $t_M = t_{M'} = 0$, (20) implies

$$\mathbf{e}_M(\mathbf{k}) = (\cos \psi, \sin \psi, 0), \quad \mathbf{e}_{M'}(\mathbf{k}') = (\cos \psi', \sin \psi', 0), \quad (32)$$

and when $\omega = \omega' = \Omega/2$, (31) implies

$$\bar{\tau}_{ij}^p(\Omega) = A_1 \delta_{ij} + i \mathcal{A} A_1 \epsilon_{ijl} b_l, \quad \bar{\tau}_{ij}^p(\Omega/2) = A_2 \delta_{ij} + 2i \mathcal{A} A_2 \epsilon_{ijl} b_l \quad (33)$$

where $A_1 = 1/(1 - \mathcal{A}^2)$, $A_2 = 1/(1 - 4\mathcal{A}^2)$, and $\mathcal{A} = \Omega_p/\Omega = me/m_p q$. After some manipulation of equations (28-30) with (32) and (33), the details of which are

given in Ball (1988), it is possible to obtain an expression for $a_{MM'}^{NL}(\mathbf{k}, \mathbf{k}', \mathbf{p}; 1)_{\Omega/2}$ which can be written in the form

$$a_{MM'}^{NL}(\mathbf{k}, \mathbf{k}', \mathbf{p}; 1)_{\Omega/2} = R(k_{\perp}\chi_l + k'_{\perp}\chi'_l)\mu_l \quad (34)$$

which exploits its symmetry with respect to $k_{\perp}$ and $k'_{\perp}$, and is similar to the form of the TS-like contribution given by (22). The vectors $\boldsymbol{\mu}$ and $\boldsymbol{\chi}$ are defined as follows:

$$\mu_l = -e^{-i(\psi+\psi')} \frac{\lambda_{lm}(k+k')}{\Lambda(k+k')} v_m, \quad (35)$$

and

$$\begin{aligned} \chi_l = & \frac{\omega_{pp}^2}{\Omega_p^2} \frac{\mathcal{A}^3 A_2}{2} [e_l 2(A_2 - A_1)U(\psi, \psi') + \frac{k_l}{k_{\perp}} A_1 U(\psi, \psi') + e'_l 2A_2 + \\ & + ia_l \mathcal{A}(4A_2 - A_1)U(\psi, \psi') + ia'_l 4 \mathcal{A} A_2], \end{aligned} \quad (36)$$

where

$$U(\psi, \psi') = \cos(\psi - \psi') + 2i \mathcal{A} \sin(\psi - \psi'), \quad (37)$$

$$\mathbf{a} = (-\sin \psi, \cos \psi, 0), \quad \mathbf{a}' = (-\sin \psi', \cos \psi', 0), \quad (38)$$

and $\mathbf{e}$ and $\mathbf{e}'$ are short for $\mathbf{e}_M(\mathbf{k})$ and $\mathbf{e}_M(\mathbf{k}')$. The symbol $\boldsymbol{\chi}'$ denotes the vector obtained from (36) by interchanging the roles of k and k' , i.e. by making the following changes: pairwise interchange of e_l and e'_l ; pairwise interchange of a_l and a'_l ; replacing $k_l/k_{\perp}$ by $k'_l/k'_{\perp}$; replacing $U(\psi, \psi')$ by $U(\psi', \psi) = U^*(\psi, \psi')$.

All the quantities appearing in equations (34–36), with the exception of $\lambda_{lm}(k+k')/\Lambda(k+k')$, are simple physical parameters. The quantities λ and Λ have been formally defined elsewhere (equation (13) and above) and may be calculated from the cold plasma dielectric tensor. The relevant equations, for the case where $\psi^+ = 0$, may be obtained directly from Melrose (1986, equations 10.25–30) by making the replacements $\omega \rightarrow \omega^+ = \omega + \omega'$, $N \rightarrow N^+ = |\mathbf{k}^+|c/\omega$ and $\theta \rightarrow \theta^+$ where $\mathbf{k}^+ \cdot \mathbf{b} = |\mathbf{k}^+| \cos \theta^+ = k_{\parallel}^+$.

(c) Form of the Probability

Substitution of the expression for the TS-like contribution (22), and the 'nonlinear' contribution (34) into (6) yields

$$a_{MM'}(\mathbf{k}, \mathbf{k}', \mathbf{p}; 1)_{\Omega/2} = R(k_{\perp}g + k'_{\perp}g') \quad (39)$$

where

$$g = f(\psi, \psi') + \mu_l \chi_l, \quad g' = f(\psi', \psi) + \mu_l \chi'_l. \quad (40)$$

The functions g and g' are essentially geometrical factors.

If the wave population is axially symmetric about $\mathbf{B}$ it is useful to average the absorption probability (5) over azimuthal angles ψ and ψ' . If this average

is denoted by angle brackets, then from (39):

$$\langle |a_{MM'}(\mathbf{k}, \mathbf{k}', \mathbf{p}; 1)_{\Omega/2}|^2 \rangle = R^2 \langle |k_{\perp} g + k'_{\perp} g'|^2 \rangle. \quad (41)$$

Recall that since $\psi^+ = 0$ the angles ψ and ψ' are related by (27). Therefore, the first step in calculating the average is to express ψ' as a function of ψ . This expression is then substituted into (40) and the average is found by calculating the integral $(1/2\pi) \int_0^{2\pi} d\psi |k_{\perp} g + k'_{\perp} g'|^2$. Whilst this may be difficult to do analytically, it should be straightforward to calculate the azimuthal average numerically for any given situation.

Hereafter only the special case where the waves are distributed with axial symmetry about $\mathbf{B}$, and where $k'_{\perp} = k_{\perp}$, is considered. This special case was discussed by Temerin and Roth (1986).

Equation (27) implies $\sin \psi' = -\sin \psi$ when $k'_{\perp} = k_{\perp}$. Thus one suitable choice of axes gives

$$\psi' = -\psi, \quad (42)$$

in which case (41) can be written in the form

$$\langle |a_{MM'}(\mathbf{k}, \mathbf{k}', \mathbf{p}; 1)_{\Omega/2, k_{\perp}=k'_{\perp}}|^2 \rangle = k_{\perp}^2 R^2 G \quad (43)$$

where $G = \langle |(g + g')_{\psi'=-\psi}|^2 \rangle$. Substitution of (43) into (5) leads to the following simple expression for the absorption probability:

$$\begin{aligned} \langle w_{MM'}(\mathbf{k}, \mathbf{k}', \mathbf{p}; 1)_{\Omega/2, k_{\perp}=k'_{\perp}} \rangle &= \frac{2\pi q^4}{m^2 \epsilon_0^2} \frac{k_{\perp}^2 v_{\perp}^2}{\omega^2} G \frac{R_M(\mathbf{k}) R_{M'}(\mathbf{k}')}{\omega \omega'} \times \\ &\times \delta(\omega + \omega' - \Omega) \end{aligned} \quad (44)$$

where $v_{\perp}/\Omega$ is the gyroradius R , and by construction only $\omega = \omega' = \Omega/2$ contribute when (44) is integrated over ω and ω' .

If the functions $F(\psi)$ and $\eta(\psi)$ are defined as follows:

$$\dot{F}(\psi) = f(\psi, -\psi) + f(-\psi, \psi), \quad (45)$$

$$\eta(\psi) = (\mu_I \chi_I + \mu_I \chi_I')_{\psi'=-\psi}, \quad (46)$$

then one may write G in the form

$$G = \langle |F(\psi)|^2 \rangle + \langle |\eta(\psi)|^2 \rangle + 2\langle \text{Re}[F^*(\psi)\eta(\psi)] \rangle. \quad (47)$$

The first term of (47) is purely TS-like and its evaluation is straightforward. Substitution of (23) into (45) gives

$$F(\psi) = -\frac{1}{3} [\cos \psi \cos(2\psi) + 2 \sin \psi \sin(2\psi) + 2 \cos \psi] \quad (48)$$

and it follows that

$$\langle |F(\psi)|^2 \rangle = \frac{25}{36}. \quad (49)$$

Note that from (48) $F(\psi)$ is real and so the third term of (47) becomes $2\langle F(\psi)\text{Re}[\eta(\psi)] \rangle$. Evaluation of the second and third terms in (47) is more complicated because of the dependence of $\eta(\psi)$ on the quantity λ_{lm}/Λ . The calculations could be carried out numerically. However, in order to obtain a simple analytic approximation to G an approximation to $\lambda_{lm}(k+k')/\Lambda(k+k')$ is required. There are two standard approximations to this term which may be called the 'low-frequency limit' (e.g., Melrose 1986, pp. 175-179) and the 'electrostatic approximation' (e.g., Melrose and Sy 1972). These two approximations, and the simplifications which result from them, are the subject of the rest of this section.

(d) Low-frequency Limit

In a cold electron-proton plasma (Melrose 1986, equation 10.26)

$$P = 1 - (\omega_{pe}^2 + \omega_{pp}^2)/\omega^2 \quad (50)$$

so at sufficiently low frequencies P is large and negative, and tends to $-\infty$ in the limit of zero frequency. In this limit the other quantities which appear in the equations for λ_{ij} and Λ generally remain finite, so the equations may be approximated by retaining only terms of highest order in P . Since λ_{lm} and Λ are both of order P the expansion is trivial and the low-frequency limit of λ_{lm}/Λ is independent of P :

$$\frac{\lambda_{lm}(k+k')}{\Lambda(k+k')} = \frac{1}{(S - N^{+2})(S - N^{+2} \cos^2 \theta^+) - D^2} \begin{pmatrix} S - N^{+2} & iD & 0 \\ -iD & S - N^{+2} \cos^2 \theta^+ & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (51)$$

Substitution of (51) into (35), with (26), gives

$$\mu^{LF} = \frac{-e^{-i(\psi+\psi')}}{S'(S' - k_{\perp}^{+2} c^2/\omega^2) - D^2} [S' + D - k_{\perp}^{+2} c^2/\omega^2, -i(S' + D), 0] \quad (52)$$

where the superscript LF denotes the low-frequency limit. Note that in (52) ψ and ψ' are related by (27), $k_{\perp}^+ = k_{\perp} \cos \psi + k'_{\perp} \cos \psi'$, and S' is defined by

$$S' = S - k_{\parallel}^{+2} c^2/\omega^2. \quad (53)$$

Since the collective plasma response is assumed to be that of a charge-neutral electron-proton plasma, and since $\omega^+ \sim \Omega \ll \Omega_p \ll \Omega_e$, the quantity S , given by (Melrose 1986, equation 10.26) may be approximated by

$$S \approx 1 + \omega_{pp}^2/\Omega_p^2. \quad (54)$$

Similar arguments lead to the result

$$D \approx (\omega_{pe}^2/\Omega_e - \omega_{pp}^2/\Omega_p)/\Omega_p = 0. \quad (55)$$

Certainly then $S \gg D$ which suggests that D may be neglected wherever it

appears in (52). This is indeed true, provided

$$S' = S - (k_{\parallel} + k'_{\parallel})^2 c^2 / \omega^2 > \max(k_{\perp}^2 c^2 / \omega^2). \quad (56)$$

In the special case where $k'_{\perp} = k_{\perp}$, $\psi' = -\psi$, and (56) is satisfied, the expression for μ^{LF} given by (52) simplifies to

$$\mu^{LF} \approx \left[\frac{-1}{S'}, \frac{i}{S' - N_{\perp}^2 \cos^2 \psi}, 0 \right]. \quad (57)$$

where $N_{\perp}^2 = 4k_{\perp}^2 c^2 / \omega^2$.

It can be shown (Ball 1988) that when equation (57) is substituted into (46), along with equation (36), the function $\eta(\psi)$ may be approximated in the low-frequency limit by

$$\eta^{LF}(\psi) \approx \frac{\omega_{pp}^2}{\Omega_p^2} \left[\frac{a_1 N_{\perp}^2 \cos^5 \psi + (a_2 N_{\perp}^2 + a_3 S') \cos^3 \psi + a_4 S' \cos \psi}{S'(S' - N_{\perp}^2 \cos^2 \psi)} \right] \quad (58)$$

where

$$\begin{aligned} a_1 &= 2 \mathcal{A}^3 A_1 A_2, & a_2 &= \mathcal{A}^3 (1 + 8 \mathcal{A}^2) A_1 A_2^2, \\ a_3 &= -2 \mathcal{A}^3 (1 + \mathcal{A}) A_1 A_2, & a_4 &= -\mathcal{A}^3 (1 + \mathcal{A}) (1 + 2 \mathcal{A})^2 A_1 A_2^2. \end{aligned} \quad (59)$$

Given equations (48) and (58) it is relatively simple to calculate $\langle |\eta^{LF}(\psi)|^2 \rangle$ and $2\langle F(\psi) \text{Re}[\eta^{LF}(\psi)] \rangle$ numerically for any given set of plasma parameters $\mathcal{A}$, S and wave parameters $k_{\perp}$, $k_{\parallel}$, $k'_{\parallel}$ which satisfy the assumptions made so far. Alternatively, if $\mathcal{A}^2 \gg 1$, as is certainly the case for O^+ ions ($\mathcal{A} = 16$), (59) may be approximated by:

$$\begin{aligned} a_1 &\approx 1/2 \mathcal{A}; & a_2 &\approx -1/2 \mathcal{A}; \\ a_3 &\approx -1/2; & a_4 &\approx 1/4. \end{aligned} \quad (60)$$

So $|a_3|$ and a_4 are comparable and are much larger than a_1 and $|a_2|$, and since it has already been assumed that $S' > N_{\perp}^2$ it is clear from (58) that the terms proportional to S' will dominate the numerator of $\eta^{LF}(\psi)$. Thus

$$\eta^{LF}(\psi) \approx \frac{\omega_{pp}^2}{\Omega_p^2} \frac{a_3 \cos^3 \psi + a_4 \cos \psi}{S' - N_{\perp}^2 \cos^2 \psi} \quad (61)$$

where a_3 and a_4 are given by (60). If (56) is strengthened by requiring $S' \gg N_{\perp}^2$ then the term $N_{\perp}^2 \cos^2 \psi$ in the denominator of (61) may be neglected, and thus

$$\langle |\eta^{LF}(\psi)|^2 \rangle \approx \frac{\omega_{pp}^4}{\Omega_p^4} \frac{\langle (2 \cos^3 \psi - \cos \psi)^2 \rangle}{16(S')^2} = \frac{\omega_{pp}^4}{\Omega_p^4} \frac{1}{64(S')^2}. \quad (62)$$

Since $\eta^{LF}(\psi)$ is real the mixed term of (47) is

$$2\langle F(\psi)\eta^{LF}(\psi)\rangle \approx -\frac{\omega_{pp}^2}{\Omega_p^2} \frac{\langle F(\psi)(2\cos^3\psi - \cos\psi)\rangle}{2S'} = \frac{\omega_{pp}^2}{\Omega_p^2} \frac{1}{8S'}. \quad (63)$$

(e) *The Electrostatic Approximation*

An alternative to the low-frequency approximation follows from the assumption that the shielding fields of the absorbing ions are approximately electrostatic (Melrose and Sy 1972; Melrose 1980b, equation (10.111); Melrose 1986, equation (6.49)). In this case

$$\frac{\lambda_{lm}(k^+)}{\Lambda(k^+)} = \frac{k_i^+ k_m^+}{|\mathbf{k}^+|^2} \frac{1}{K^L(k^+)} \quad (64)$$

with

$$K^L(k^+) = \frac{k_i^+ k_j^+}{|\mathbf{k}^+|^2} \kappa_{ij}(k^+). \quad (65)$$

An indication of how (64) arises is given in Appendix A. Assuming that the shielding fields are electrostatic is equivalent to assuming that the absorption proceeds via predominantly longitudinal virtual waves (Melrose and Sy 1972; Tsytovich and Shvartzburg 1966, 1967; Tsytovich 1966, 1970). The quantity $K^L(k^+)$ is referred to as the longitudinal part of the dielectric tensor.

When $\kappa_{ij}(k^+)$ is given by the cold plasma approximation substitution of (64) and (65) into (35), with (26), gives

$$\mu_I^{ES} = \frac{-e^{-i(\psi+\psi')}(k_\perp \cos\psi + k'_\perp \cos\psi')}{S(k_\perp \cos\psi + k'_\perp \cos\psi')^2 + P(k_\parallel + k'_\parallel)^2} k_I^+ \quad (66)$$

where the superscript *ES* denotes the electrostatic approximation. In the special case where $k'_\perp = k_\perp$ and $\psi' = -\psi$, equation (66) reduces to

$$\mu_I^{ES} = \frac{-2k_\perp \cos\psi}{4Sk_\perp^2 \cos^2\psi + P(k_\parallel + k'_\parallel)^2} k_I^+ \quad (67)$$

where, from (24), $\mathbf{k}^+ = (2k_\perp \cos\psi, 0, k_\parallel + k'_\parallel)$.

When equation (67) is substituted into (46), along with equation (36), the electrostatic approximation to the function $\eta(\psi)$ can be written in the form

$$\eta^{ES}(\psi) = \eta_r^{ES}(\psi) + i\eta_i^{ES}(\psi) \quad (68)$$

with

$$\eta_r^{ES}(\psi) = \zeta\{b_1 k_\perp^2 \cos^5\psi + [b_2 k_\perp^2 + b_3(k_\parallel + k'_\parallel)^2] \cos^3\psi + b_4(k_\parallel + k'_\parallel)^2 \cos\psi\},$$

$$\eta_i^{ES}(\psi) = -\zeta 4\mathcal{A}^4 A_1 A_2 (k_\parallel^2 - k'^2_\parallel) \cos^2\psi \sin\psi, \quad (69)$$

where

$$\begin{aligned} b_1 &= -8 \mathcal{A}^3 A_1 A_2, & b_2 &= -4 \mathcal{A}^3 A_1 A_2^2 (1 + 8 \mathcal{A}^2), \\ b_3 &= -2 \mathcal{A}^3 A_1 A_2, & b_4 &= \mathcal{A}^3 A_1 A_2 \end{aligned} \quad (70)$$

and

$$\zeta = \frac{\omega_{pp}^2}{\Omega_p^2} \frac{1}{4Sk_{\perp}^2 \cos^2 \psi + P(k_{\parallel} + k'_{\parallel})^2}. \quad (71)$$

The details of this derivation are given in Ball (1988) and are omitted here.

The quantities $\langle |\eta^{ES}(\psi)|^2 \rangle$ and $2\langle F(\psi) \text{Re}[\eta(\psi)] \rangle$ can be calculated numerically from (48) and (68) *et seq.* for any given set of parameters. Alternatively if $\mathcal{A} \gg 1$ then (70) may be approximated by:

$$b_1 \approx -2/\mathcal{A}; \quad b_2 \approx 2/\mathcal{A}; \quad b_3 \approx -1/2\mathcal{A}; \quad b_4 \approx 1/4\mathcal{A}, \quad (72)$$

and the coefficient of the imaginary part $\eta_i^{ES}(\psi)$ is $4\mathcal{A}^4 A_1 A_2 \approx 1$. The quantities $|b_1|$, b_2 , $|b_3|$, b_4 are all comparable, but since $k_{\perp} \gg k_{\parallel}$ and $k'_{\parallel}$, it is clear that the terms proportional to b_1 and b_2 will dominate $\eta_r^{ES}(\psi)$. If the additional assumption

$$|P| \gg 4Sk_{\perp}^2/(k_{\parallel} + k'_{\parallel})^2 \quad (73)$$

is made, the term proportional to S in the denominator of (71) does not contribute. Combining these results gives

$$\begin{aligned} \eta^{ES}(\psi) \approx & -\frac{\omega_{pp}^2}{\Omega_p^2} \frac{1}{\mathcal{A}P(k_{\parallel} + k'_{\parallel})^2} [2k_{\perp}^2 (\cos^5 \psi - \cos^3 \psi) + \\ & + i \mathcal{A}(k_{\parallel}^2 - k'_{\parallel}{}^2) \cos^2 \psi \sin \psi], \end{aligned} \quad (74)$$

and thus

$$\langle |\eta^{ES}(\psi)|^2 \rangle \approx \frac{\omega_{pp}^4}{\Omega_p^4} \frac{3k_{\perp}^4 + 4\mathcal{A}^2(k_{\parallel}^2 - k'_{\parallel}{}^2)^2}{64\mathcal{A}^2P^2(k_{\parallel} + k'_{\parallel})^4}. \quad (75)$$

If $k'_{\parallel}$ is comparable to $k_{\parallel}$ the term proportional to $k_{\perp}^4$ dominates the numerator of (75) and hence

$$\langle |\eta^{ES}(\psi)|^2 \rangle \approx \frac{\omega_{pp}^4}{\Omega_p^4} \frac{3}{64\mathcal{A}^2P^2} \frac{k_{\perp}^4}{(k_{\parallel} + k'_{\parallel})^4}. \quad (76)$$

The electrostatic approximation to the mixed term is, from (48) and (74),

$$\begin{aligned} 2\langle F(\psi) \text{Re}[\eta^{ES}(\psi)] \rangle & \approx -\frac{\omega_{pp}^2}{\Omega_p^2} \frac{4k_{\perp}^2}{\mathcal{A}P(k_{\parallel} + k'_{\parallel})^2} \langle F(\psi) (\cos^5 \psi - \cos^3 \psi) \rangle \\ & = -\frac{\omega_{pp}^2}{\Omega_p^2} \frac{5}{16\mathcal{A}P} \frac{k_{\perp}^2}{(k_{\parallel} + k'_{\parallel})^2}. \end{aligned} \quad (77)$$

4. Rate of Perpendicular Acceleration

The perpendicular energy of a classical particle with momentum $\mathbf{p}$ is $\varepsilon_{\perp} = p_{\perp}^2/2m$ which may be integrated over the particle distribution function $f(\mathbf{p})$ to give the average perpendicular energy per particle $\bar{\varepsilon}_{\perp} = [\int d^3\mathbf{p} \varepsilon_{\perp} f(\mathbf{p})]/n$. The 'rate of perpendicular acceleration' may then be defined to be the time rate of change of $\bar{\varepsilon}_{\perp}$, and may be calculated from

$$\frac{d\bar{\varepsilon}_{\perp}}{dt} = \frac{1}{n} \int d^3\mathbf{p} \varepsilon_{\perp} \frac{df(\mathbf{p})}{dt}. \quad (78)$$

The evolution of the particle distribution function due to double absorption and emission processes is given by equation (2) which is quite a general result; its derivation did not involve any approximations other than those implicit in the random phase approximation. However, in the application of primary interest here $k_{\perp} \gg k_{\parallel}$ and $k'_{\parallel}$ so the parallel and mixed 'diffusion' coefficients $A_{\parallel}$, $D_{\parallel\parallel}$ and $D_{\perp\parallel}$ are much smaller than the perpendicular coefficients $A_{\perp}$ and $D_{\perp\perp}$. The parallel and mixed terms in (2) may therefore be neglected and the relevant equation for the evolution of $f(\mathbf{p})$ is thus

$$\frac{df(\mathbf{p})}{dt} = \frac{1}{p_{\perp}} \frac{\partial}{\partial p_{\perp}} \left\{ p_{\perp} \left[A_{\perp} f(\mathbf{p}) + D_{\perp\perp} \frac{\partial f(\mathbf{p})}{\partial p_{\perp}} \right] \right\}. \quad (79)$$

The expressions for $A_{\perp}$ (equation 3) and $D_{\perp\perp}$ (equation 4) include only the effects of double absorption and double emission of waves at frequencies ω and ω' . The first term in (79) describes the effects of double emission stimulated by the presence of waves at either ω or ω' , but not both. The second term describes the effects of double absorption, and of double emission stimulated by the presence of waves at both ω and ω' . It is therefore appropriate to refer to these terms as being due to 'singly-induced' and 'doubly-induced' processes respectively.

After substitution of (79) into (78) the acceleration rate may be written in the form

$$\frac{d\bar{\varepsilon}_{\perp}}{dt} = \left[\frac{d\bar{\varepsilon}_{\perp}}{dt} \right]_1 + \left[\frac{d\bar{\varepsilon}_{\perp}}{dt} \right]_2 \quad (80)$$

with

$$\left[\frac{d\bar{\varepsilon}_{\perp}}{dt} \right]_1 = \frac{-1}{n} \int d^3\mathbf{p} \frac{p_{\perp} A_{\perp}}{m} f(\mathbf{p}) \quad (81)$$

and

$$\left[\frac{d\bar{\varepsilon}_{\perp}}{dt} \right]_2 = \frac{1}{n} \int d^3\mathbf{p} \frac{1}{mp_{\perp}} \frac{\partial(p_{\perp}^2 D_{\perp\perp})}{\partial p_{\perp}} f(\mathbf{p}) \quad (82)$$

where the subscripts 1 and 2 denote the singly and doubly-induced contributions respectively.

The contribution to $d\bar{\varepsilon}_{\perp}/dt$ due to singly-induced emission is systematic and not random; whilst $[d\bar{\varepsilon}_{\perp}/dt]_1$ may well be positive, the increase in $\bar{\varepsilon}_{\perp}$ is

not due to an increase in the random perpendicular motion of the ions. On the other hand the contribution to the acceleration rate due to the doubly-induced processes is a truly diffusive term and so leads to an increase in the random perpendicular motion of the ions.

The diffusion coefficient $D_{\perp\perp}$ is given by

$$D_{\perp\perp} = (\hbar\Omega/\nu_{\perp})^2 \int \frac{d^3\mathbf{k}}{(2\pi)^3} \int \frac{d^3\mathbf{k}'}{(2\pi)^3} N_M(\mathbf{k}) N_{M'}(\mathbf{k}') w_{MM'}(\mathbf{k}, \mathbf{k}', \mathbf{p}; 1) \quad (83)$$

where only the $s = 1$ contribution to (4) has been retained. Substitution of (44) into (83) gives the term $(d^3\mathbf{k}/(2\pi)^3) N_M(\mathbf{k}) R_M(\mathbf{k}) (d^3\mathbf{k}'/(2\pi)^3) N_{M'}(\mathbf{k}') R_{M'}(\mathbf{k}')$ which is to be integrated over the wave distributions. However since it has been assumed in §3 that the waves all have the same perpendicular wave number and are distributed in frequency space, it is convenient to reinterpret the integrals in terms of quantities which depend on ω and ω' . This may be done by considering the energetics of the waves.

The wave occupation number $N_M(\mathbf{k})$ was defined in terms of the energy density $W_M(\mathbf{k})$ in equation (1). The energy density $W_M(\mathbf{k})$ is related to the total wave energy W_M in a volume V by

$$W_M = \int \frac{d^3\mathbf{k}}{(2\pi)^3} W_M(\mathbf{k}). \quad (84)$$

If W_M^E is the total electric energy of the waves in a volume V , then $W_M^E(\mathbf{k})$ may be defined to be the electric energy density such that

$$W_M^E = \int \frac{d^3\mathbf{k}}{(2\pi)^3} W_M^E(\mathbf{k}). \quad (85)$$

The formal definition of the energy ratio $R_M(\mathbf{k})$ is then

$$R_M(\mathbf{k}) = W_M^E(\mathbf{k})/W_M(\mathbf{k}). \quad (86)$$

Substitution of (1) into (86) leads to $W_M^E(\mathbf{k}) = \hbar\omega_M(\mathbf{k}) V N_M(\mathbf{k}) R_M(\mathbf{k})$ which may be substituted into (85) to get

$$W_M^E = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \hbar\omega_M(\mathbf{k}) V N_M(\mathbf{k}) R_M(\mathbf{k}). \quad (87)$$

Since the waves are assumed to be distributed in frequency space, the energy density $W_M^E(\omega)$ may be defined such that

$$W_M^E = \int d\omega W_M^E(\omega). \quad (88)$$

Comparison of (87) and (88) suggests that $(d^3\mathbf{k}/(2\pi)^3) N_M(\mathbf{k}) R_M(\mathbf{k})$ may be identified with $d\omega W_M^E(\omega)/\hbar\omega V$. Reinterpreting the integrals of (83) in this fashion leads to the following expression for the diffusion coefficient:

$$D_{\perp\perp} = \frac{2\pi q^4}{m^2 \epsilon_0^2} \frac{k_{\perp}^2 G}{V^2} \int d\omega \int d\omega' \frac{W_M^E(\omega) W_M^E(\omega')}{\omega^2 \omega'^2} \delta(\omega + \omega' - \Omega). \quad (89)$$

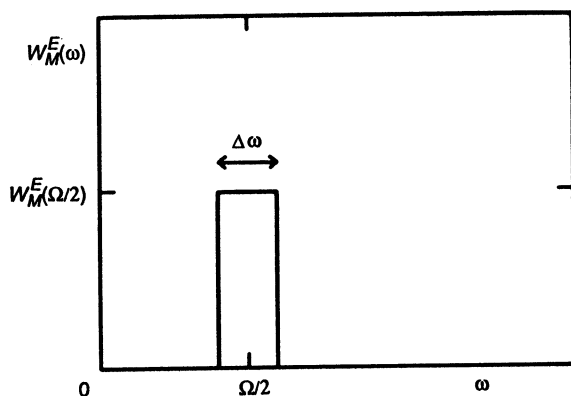


Fig. 1. Plot of the electric energy density, as a function of frequency, in the assumed wave distribution.

The integration over the δ -function is trivial and the integral in (89) becomes

$$\int \frac{W_M^E(\omega)W_M^E(\Omega - \omega)}{\omega^2(\Omega - \omega)^2} d\omega. \quad (90)$$

In §3 repeated references were made to “the special case where $\omega = \omega' = \Omega/2$ ”. Stated formally, it has been supposed that the wave frequencies span only a very narrow frequency band near $\Omega/2$. Hereafter, it is assumed that the wave distribution is such that the wave energy is distributed uniformly over a very narrow frequency band of width $\Delta\omega$, centred on $\Omega/2$. The distribution of electric wave energy is shown in Fig. 1. In this case (90) reduces to $16\Delta\omega[W_M^E(\Omega/2)]^2\Omega^4$ and replacing the integral in (89) with this expression yields

$$D_{\perp\perp} \approx \frac{32\pi q^4}{m^2\epsilon_0^2\Omega^4} \frac{k_{\perp}^2 G}{V^2} \Delta\omega [W_M^E(\Omega/2)]^2. \quad (91)$$

For the distribution of wave energy shown in Fig. 1, the total electric energy in the waves is

$$W_M^E = \int_{\Delta\omega} d\omega W_M^E(\Omega/2) = \Delta\omega W_M^E(\Omega/2). \quad (92)$$

Alternatively

$$W_M^E = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \int_V d^3\mathbf{x} \frac{\epsilon_0 |\mathbf{E}(t, \mathbf{x})|^2}{2} = \frac{V\epsilon_0 E_0^2}{2} \quad (93)$$

where E_0 is the (space and time independent) RMS electric amplitude of the waves. Comparison of (92) and (93) leads to

$$W_M^E(\Omega/2) = \frac{V\epsilon_0 E_0^2}{2\Delta\omega} \quad (94)$$

and substitution of (94) into (91) yields

$$D_{\perp\perp} = \frac{8\pi m^2 k_{\perp}^2 G}{\Delta\omega} \left(\frac{E_0}{B}\right)^4 \quad (95)$$

where B is the magnitude of the ambient magnetic field. Substitution of (95) into (82) yields

$$\left[\frac{d\bar{\varepsilon}_{\perp}}{dt}\right]_2 = \frac{16\pi m k_{\perp}^2 G}{\Delta\omega} \left(\frac{E_0}{B}\right)^4. \quad (96)$$

The acceleration rate due to singly-induced emission scales according to

$$A_{\perp} = \hbar\Omega/\nu_{\perp} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \int \frac{d^3\mathbf{k}'}{(2\pi)^3} [N_M(\mathbf{k}) + N_{M'}(\mathbf{k}')] w_{MM'}(\mathbf{k}, \mathbf{k}', \mathbf{p}; 1), \quad (97)$$

where only the $s=1$ contribution to (3) has been retained. Comparison of (83) and (97) shows that while $D_{\perp\perp}$ is given by an integral of the *product* of $N_M(\mathbf{k})$ and $N_{M'}(\mathbf{k}')$, $A_{\perp}$ is given by an integral of the *sum* of $N_M(\mathbf{k})$ and $N_{M'}(\mathbf{k}')$. Since $N_M(\mathbf{k})$ is inversely proportional to frequency, the occupation numbers for very low frequency waves at reasonable power levels are generally very large. Therefore, it is logical to expect that

$$\left[\frac{d\bar{\varepsilon}_{\perp}}{dt}\right]_2 \gg \left[\frac{d\bar{\varepsilon}_{\perp}}{dt}\right]_1 \quad (98)$$

for the application to ion conic production where the ion gyrofrequency is typically just a few hertz. Arguments detailed in Appendix B indicate that (98) is true provided

$$\frac{16\pi^2 c^2 \epsilon_0 E_0^2}{m\nu_{\perp}^2} \frac{\nu_A}{\Delta\omega\Omega_{0+}^2} \gg 1 \quad (99)$$

where ν_A is the Alfvén speed and $\nu_{\perp}^2$ has been averaged over $f(\mathbf{p})$. Thus when (99) is satisfied the perpendicular acceleration rate is dominated by the doubly-induced processes and hence, from (80) and (96),

$$\frac{d\bar{\varepsilon}_{\perp}}{dt} = \frac{16\pi m k_{\perp}^2 G}{\Delta\omega} \left(\frac{E_0}{B}\right)^4. \quad (100)$$

The corresponding equation for the 'rate of average perpendicular energy gain' derived by Temerin and Roth (1986) is

$$\frac{d\bar{\varepsilon}_{\perp}}{dt} = \frac{20\pi m k_{\perp}^2}{9\Delta\omega} \left(\frac{E_0}{B}\right)^4. \quad (101)$$

Note that (101) differs from equation (4) of Temerin and Roth (1986) by a factor of two because the definition of the bandwidth $\Delta\omega$ used in this work corresponds to a bandwidth $\Delta\omega/2$ as used by Temerin and Roth.

Clearly then, the expression for the perpendicular acceleration rate derived here from the semiclassical description of double-cyclotron absorption, is functionally identical to the result of Temerin and Roth (1986). Any difference between the two results must be contained in the multiplicative factor G .

The comparison between these two results, which have been derived using totally different methods, is not considered further in this paper. A detailed comparison requires an evaluation of the factor G which is only possible in the context of a specific application for which the relevant plasma and wave parameters can be estimated. The comparison is considered in detail for the production of O^+ ion conics by Ball (1989).

5. The Assumptions

A number of simplifying assumptions have been introduced as and when required in §3 and §4. Those assumptions are discussed in this section.

(a) Wave Distribution

The process of perpendicular acceleration by double-cyclotron absorption was suggested as a source mechanism for O^+ -conics by Temerin (1986) because of observations of waves at frequencies below the oxygen gyrofrequency Ω_{O^+} . Since the waves propagate nearly perpendicular to the ambient magnetic field $\mathbf{B}$ it has been assumed throughout this work that $k_{\perp} \gg k_{\parallel}$. A very simple, idealised wave distribution (as considered by Temerin and Roth 1986) has been considered here in order to simplify the analysis as much as possible. Thus it has been assumed that the waves are distributed in:

- (1) frequency space with uniform energy density $W_M^E(\Omega/2)$ over a narrow range of frequencies (width $\Delta\omega$), centred about $\Omega/2$;
- (2) $\mathbf{k}$ -space such that the perpendicular wave number $k_{\perp}$ is the same for each wave, and such that the distribution is axially symmetric about $\mathbf{B}$.

(b) Background Plasma and Oxygen Ions

The first assumption made which falls into this category refers to the velocity of the ion which interacts with the waves. Introduced in the derivation of (15), the assumption is that the parallel speed $v_{\parallel}$ is at most of the same order as the perpendicular speed $v_{\perp}$. This assumption is invalid if for some reason the initial distribution of the absorbing ions is strongly field-aligned, i.e. $|v_{\parallel}| \gg v_{\perp}$.

The other assumptions made about the background plasma and the absorbing ions were discussed at the beginning of §3*b*, and refer to the collective nonlinear response of the plasma. First it was assumed that the plasma consisted solely of electrons, protons and the absorbing ions. Contributions due to heavy ions other than the absorbing species were neglected. This is consistent with the relative densities of the different ionic components of the plasma in ion conic source regions. Next it was assumed that the cold plasma approximation could be used for the electron and proton contributions to the collective plasma response, and it was assumed that the absorbing ions were test particles and did not contribute to the plasma response at all. The use of the cold plasma approximation may be justified by consideration of the typical temperatures

of electrons and protons in the acceleration region. Quite high temperatures would be necessary to produce significant thermal effects at the frequencies of interest, $\Omega_{O^+}/2$ and Ω_{O^+} , which are far-removed from the singularities at Ω_e and Ω_p in the cold plasma approximations to the electron and proton contributions respectively. However the primary justification of the use of the cold plasma approximation, and of the test-particle approximation, is that they allow the analysis to proceed.

The assumption that the background plasma is charge-neutral is logical in the absence of evidence to the contrary. Given the test-particle approximation, charge neutrality is equivalent to requiring that the number densities n_e and n_p are equal.

(c) Wave Properties

Three assumptions have been made about the fundamental physical properties of the waves. The assumption that the wave electric fields are perpendicular to the ambient magnetic field may be justified on physical grounds. If this was not the case then the waves would produce electric fields along the magnetic field lines. The electrons in a plasma are free to move along ambient magnetic field lines on timescales of the order of $1/\omega_{pe}$. Thus for low-frequency waves with $\omega \ll \omega_{pe}$, the timescale of the wave disturbance greatly exceeds that of the electron motion along ambient magnetic field lines. The plasma electrons can therefore respond to, and cancel, the parallel electric fields much faster than the wave disturbance can produce them. The assumption that the waves are linearly polarised (with $t_M = 0$) corresponds to choosing the simplest non-trivial case. (TS-like double absorption and emission does not occur for right-circularly polarised waves.) This choice is equivalent to the assumption of Temerin and Roth (1986) that the perpendicular wave vector is parallel to the wave electric field. Finally, the assumption that the parallel wave numbers of any two waves are comparable essentially follows from the choice of wave distribution. The fact that any two waves in the distribution must have very similar frequencies (differing by at most $\Delta\omega$) and have the same perpendicular wave number $k_\perp$ implies that $k_\parallel \sim k'_\parallel$, provided the wave frequency is not near a resonance in the dispersion equation.

(d) Combined Plasma and Wave Properties

The assumption that the electric fields of the waves are perpendicular to $\mathbf{B}$ (discussed above) requires

$$\Omega \ll \omega_{pe} = \left(\frac{n_e e^2}{\epsilon_0 m_e} \right)^{1/2}. \quad (102)$$

This assumption, and the remaining assumptions which have not yet been discussed, are essentially 'empirical' in the sense that they can be tested by experimental data. The 'small-gyroradius approximation', introduced after equation (14), requires

$$k_\perp R = \frac{k_\perp v_\perp}{\Omega} \ll 1. \quad (103)$$

The next empirical assumption, which led to (19), involved neglecting the longitudinal Doppler-shift terms $k_{\parallel}v_{\parallel}$ and $k'_{\parallel}v_{\parallel}$ in the resonance condition. This approximation requires $k_{\parallel}v_{\parallel} \ll \Omega/2$ and follows automatically from the small-gyroradius approximation and the assumptions that $v_{\parallel} \sim v_{\perp}$ and $k_{\perp} \gg k_{\parallel}$.

Two alternative approximations to the 'nonlinear' contribution to the probability of double-cyclotron absorption were presented in §3. The discussion following equation (50) suggests that the low-frequency limit is valid when

$$|P| \gg N^{+2} \quad \text{and} \quad |P| \gg S \quad (104)$$

where P is given by (50), S may be approximated according to (54), and

$$N^{+2} = [4k_{\perp}^2 \cos^2 \psi + (k_{\parallel} + k'_{\parallel})^2] c^2 / \omega^2. \quad (105)$$

The final result of the low-frequency limit calculations followed from the assumption

$$S - (k_{\parallel} + k'_{\parallel})^2 c^2 / \omega^2 \gg 4k_{\perp}^2 c^2 / \omega^2. \quad (106)$$

The conditions for the validity of the electrostatic approximation are harder to quantify. However, Melrose and Sy (1972) suggested that the electrostatic approximation is valid provided N^{+2} greatly exceeds all components of the dielectric tensor, i.e. provided

$$N^{+2} \gg \max(S, D, |P|). \quad (107)$$

The final result for the electrostatic approximation followed from the assumption $k'_{\parallel} \sim k_{\parallel}$ (discussed above), and the requirement that (73) be satisfied.

In §4 it was assumed that the doubly-induced processes dominate the rate of perpendicular acceleration. This assumption is equivalent to the requirement that (99) be satisfied.

Estimates of the values of the wave parameters $k_{\parallel}$, $k_{\perp}$ and E_0 , and of the plasma parameters n_p , $v_{\parallel}$, $v_{\perp}$ and Ω_{O^+} , are sufficient to check all of the above requirements. These same parameters are also sufficient to evaluate the rates of perpendicular acceleration due to double-cyclotron absorption implied by (100) and (101). A discussion of the determination of these parameters from experimental data relevant to O^+ -conic production, of the applicability of the 'empirical assumptions', and of the numerical estimates of the O^+ acceleration rate, is given in Ball (1989).

6. Summary

Simple analytic approximations for the quantities which describe the process of perpendicular acceleration of heavy ions via double-cyclotron absorption have been presented. The assumptions made in this work have been discussed in detail in order to facilitate an accurate assessment of the applicability of the final results to the problem of O^+ -conic production.

Approximations to the probability of double-cyclotron absorption, which is the quantity of central importance in the semiclassical description of the double-cyclotron interaction, have been derived. The expression for the

absorption probability is particularly simple in the special case where the waves all have frequencies very close to half the gyrofrequency of the absorbing ions. Two alternative approximations for the contribution due to 'nonlinear' double emission and absorption have been discussed, both of which lead to the same general form for the absorption probability. The difference between these two approximations occurs only in a multiplicative factor (G). For the same special case where the wave distribution is quasimonochromatic, a very simple expression for the rate of increase of the average perpendicular energy of the absorbing ions due to double-cyclotron absorption has been derived. The expression for the perpendicular acceleration rate derived here has exactly the same functional form as that derived by Temerin and Roth (1986). Any differences between the two results are contained in the multiplicative factor G . A detailed comparison of the results of this work and the results of Temerin and Roth (1986), for the specific application to the production of O^+ -conics in the Earth's magnetosphere, is presented in Ball (1989).

The major advantage of the method of calculation of the rate of perpendicular acceleration presented in this paper, as opposed to the method of Temerin and Roth (1986), is that all the assumptions made in this work are explicit (with the exception of those inherent in the random-phase approximation). This means that each step in the calculation is open to scrutiny, and each assumption may be evaluated in the light of available experimental data. It also means that the details of the approximations are open to modification, which is particularly important in case new data become available which invalidate the assumptions or approximations used here. Alternatively, numerical methods could be invoked at a number of intermediate stages to avert the need for some of the cruder approximations and assumptions introduced here to allow the analysis to proceed.

No attempt has been made to consider the evolution of the waves due to absorption by ions. The general equations describing the evolution of the wave occupation numbers due to double absorption and emission were presented as part of the general semiclassical theory by Ball and Melrose (1989). Presumably, if new data serve to confirm the feasibility of this acceleration mechanism then numerical simulations would eventually have to treat the damping of the waves self-consistently. Such considerations are beyond the scope of this paper.

Acknowledgments

The author is deeply indebted to D. B. Melrose for suggesting the problem considered here as worthy of attention, for his initial work aimed at developing suitable approximations, for his assistance with many difficulties encountered along the way, and for his very helpful comments on the manuscript of this paper. This work formed a part of the authors PhD thesis; the author would like to thank his PhD supervisor R. G. Hewitt for his patient and valuable assistance in the preparation of the thesis manuscript.

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Appendix A: Electrostatic Form of λ_{ij}/Λ

The inhomogeneous wave equation relates the current due to the unperturbed (*up*) motion of the absorbing ion $\mathbf{J}^{up}$, to the shielding (*sh*) fields of the ion $\mathbf{E}^{sh}$:

$$\Lambda_{ij}(k^+)E_j^{sh}(k^+) = \frac{-i}{\epsilon_0\omega^+} J_i^{up}(k^+). \quad (\text{A1})$$

If (A1) is contracted with $\lambda_{li}(k^+)$ using the identity $\lambda_{li}(k^+)\Lambda_{ij}(k^+) = \Lambda(k^+)\delta_{ij}$, and the indices are relabelled appropriately, the following expression for the shielding field is obtained:

$$E_i^{sh}(k^+) = \frac{-i}{\epsilon_0\omega^+} \frac{\lambda_{ij}(k^+)}{\Lambda(k^+)} J_j^{up}(k^+). \quad (\text{A2})$$

If the shielding fields are electrostatic (the virtual waves are longitudinal) then $\mathbf{E}^{sh}$ and $\mathbf{k}^+$ are parallel or antiparallel so

$$E_j^{sh}(k^+) = \pm \mathbf{E}^{sh}(k^+) \frac{k_j^+}{|\mathbf{k}^+|}. \quad (\text{A3})$$

Contracting (A3) with $\Lambda_{ij}(k^+) = c^2(k_i^+k_j^+ - |\mathbf{k}^+|^2)/\omega^{+2} + \kappa_{ij}(k^+)$ gives

$$\Lambda_{ij}(k^+)E_j^{sh}(k^+) = \pm \mathbf{E}^{sh}(k^+) \kappa_{ij}(k^+) \frac{k_j^+}{|\mathbf{k}^+|}. \quad (\text{A4})$$

Now substitute (A4) into (A1)

$$\pm \mathbf{E}^{sh}(k^+) \kappa_{ij}(k^+) \frac{k_j^+}{|\mathbf{k}^+|} = \frac{-i}{\epsilon_0\omega^+} J_i^{up}(k^+) \quad (\text{A5})$$

and contract (A5) with $k_l^+ k_j^+ / (|\mathbf{k}^+|^2 K^L(k^+))$:

$$\left[\frac{k_l^+ k_j^+}{|\mathbf{k}^+|^2} \kappa_{ij}(k^+) \right] \frac{1}{K^L(k^+)} \left\{ \pm |\mathbf{E}^{sh}(k^+)| \frac{k_l^+}{|\mathbf{k}^+|} \right\} \\ = \frac{-i}{\epsilon_0 \omega^+} \frac{k_l^+ k_j^+}{|\mathbf{k}^+|^2} \frac{1}{K^L(k^+)} J_i^{up}(k^+). \quad (\text{A6})$$

By (65) the quantity in [] in (A6) is just $K^L(k^+)$, and by (A3) the quantity in { } is just $E_i^{sh}(k^+)$ so (A6) may be rewritten as

$$E_i^{sh}(k^+) = \frac{-i}{\epsilon_0 \omega^+} \frac{k_l^+ k_j^+}{|\mathbf{k}^+|^2} \frac{1}{K^L(k^+)} J_j^{up}(k^+) \quad (\text{A7})$$

where the indices have been renamed $i \rightarrow j$ and $l \rightarrow i$. Comparison of (A2) and (A7) leads immediately to (64).

Appendix B: Singly-induced versus Doubly-induced Acceleration

Comparison of (81) and (82), together with the fact that $D_{\perp\perp}$ is independent of $p_{\perp}$, implies

$$\frac{[d\bar{\epsilon}_{\perp}/dt]_2}{[d\bar{\epsilon}_{\perp}/dt]_1} \sim \frac{2D_{\perp\perp}}{p_{\perp} A_{\perp}} \quad (\text{B1})$$

where it is understood that $p_{\perp} A_{\perp}$ has been averaged over $f(\mathbf{p})$. Substitution of (83) and (97) into (B1), together with (44), yields

$$\frac{[d\bar{\epsilon}_{\perp}/dt]_2}{[d\bar{\epsilon}_{\perp}/dt]_1} \sim \frac{h\Omega}{mv_{\perp}^2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} N_M(\mathbf{k}) R_M(\mathbf{k}) \bigg/ \int \frac{d^3\mathbf{k}}{(2\pi)^3} R_M(\mathbf{k}), \quad (\text{B2})$$

where $v_{\perp}^2$ is understood to have been averaged over $f(\mathbf{p})$. The numerator of (B2) may be evaluated by replacing the integrand $(d^3\mathbf{k}/(2\pi)^3) N_M(\mathbf{k}) R_M(\mathbf{k})$ by $d\omega W_M^E(\omega)/h\omega V$ (see after (88)). Then, for the distribution of wave power shown in Fig. 1,

$$\int d\omega \frac{W_M^E(\omega)}{h\omega V} \approx \frac{2\Delta\omega W_M^E(\Omega/2)}{h\Omega V} \quad (\text{B3})$$

and replacing $W_M^E(\Omega/2)$ according to (94) gives

$$\int \frac{d^3\mathbf{k}}{(2\pi)^3} N_M(\mathbf{k}) R_M(\mathbf{k}) \approx \frac{\epsilon_0 E_0^2}{h\Omega}. \quad (\text{B4})$$

The denominator of (B2) may be approximated by supposing that the waves are Alfvén waves, in which case $R_M(\mathbf{k}) = v_A^2/2c^2$ where v_A is the Alfvén speed. Then

$$\int \frac{d^3\mathbf{k}}{(2\pi)^3} R_M(\mathbf{k}) = \frac{v_A^2}{2c^2} \frac{1}{V_c} \quad (\text{B5})$$

where the quantity V_c , called the coherence volume of the wave distribution, is given by

$$V_c = \left[\int_w \frac{d^3 \mathbf{k}}{(2\pi)^3} \right]^{-1}. \quad (\text{B6})$$

The subscript w in (B6) indicates that the integral is over only that portion of $\mathbf{k}$ -space in which waves are present. Substitution of (B4) and (B5) into (B2) yields

$$\frac{[d\bar{\epsilon}_\perp/dt]_2}{[d\bar{\epsilon}_\perp/dt]_1} \approx V_c \left(\frac{\epsilon_0 E_0^2}{v_A^2/2c^2} \right) \frac{1}{mv_\perp^2}. \quad (\text{B7})$$

Note that $\epsilon_0 E_0^2/2$ is the electric energy per unit volume in the waves, so the quantity in large parentheses in (B7) is the total wave energy per unit volume, and the product with V_c gives a measure of the total wave energy. On the other hand, $mv_\perp^2/2$ is the average perpendicular energy per particle, so the right-hand side of (B7) is essentially the ratio of wave energy to particle energy.

A very rough estimate of V_c may be obtained by writing

$$\int_w \frac{d^3 \mathbf{k}}{(2\pi)^3} \approx \frac{(\Delta\Omega)k^2\Delta k}{(2\pi)^3} \quad (\text{B8})$$

where k is a typical wave number ($\approx \omega/v_A$), $(\Delta\Omega)$ is the solid angle occupied by the wave distribution ($\leq 4\pi$), and $\Delta k \approx \Delta\omega/(\partial\omega/\partial k)$ ($\partial\omega/\partial k \approx v_A$). Collecting these results in (B8), and replacing ω by $\Omega/2$, gives

$$V_c \approx \frac{4(2\pi v_A)^3}{\Omega^2 \Delta\omega(\Delta\Omega)}. \quad (\text{B9})$$

Substitution of (B9) into (B7) leads immediately to (99).

