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Soliton Characteristics at the Critical Density of Negative Ions in Ion-beam Plasmas

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Abstract

By using the reductive perturbation technique, ion–acoustic waves are studied in a generalised multicomponent plasma. The multiple ions modify drastically the characteristics of the solitary waves. In particular, the negative ions have a critical density at which the nonlinearity of the Korteweg–deVries (K–dV) equation vanishes and the ion–acoustic solitary wave is seen to be described by a modified K–dV (mK–dV) equation. Using higher order nonlinearities, the non-uniform transition of the K–dV equation to the mK–dV equation along with the conservation of the Sagdeev potential is described. Theoretical observations on the existence of the solitary waves, as expected, could be of interest in laboratory plasmas.

1. Introduction

The study of ion–acoustic solitons in plasmas is an area of active interest and recent reviewers have highlighted both theoretically and experimentally the various properties of the solitary waves. Many authors (Washimi and Taniuti 1966; Su and Gardner 1969; Jeffrey and Kakutani 1972; Taylor *et al.* 1972; Ikezi 1973; Tran and Hirt 1974; Das 1979; Lonngren 1983) have discussed extensively the evolution of ion–acoustic solitons in various plasmas through the derivation of a K–dV equation of the form

$$\frac{\partial \phi}{\partial \tau}+A\phi\frac{\partial \phi}{\partial \xi}+B\frac{\partial^3 \phi}{\partial \xi^3}=0, \tag{1}$$

where ϕ is the wave potential. Depending on the model considered, theoretical observations of the solitons show slightly different behaviour compared with those of the experiments. We may mention here the theoretical work of Das and Tagare (1975) and Das (1979) who studied the effects of negative ions on the solitons and showed that there exists a critical density of negative ions around the neighbourhood of which the amplitude of the solitary wave becomes infinitely large. Later, Watanabe (1984) studied the soliton's existence at the critical density at which the compressive and rarefactive solitons are observed. Furthermore, Raychaudhuri *et al.* (1985) and Verheest (1988) have made extensions to observe the interaction of negative ions with the solitons in the general multicomponent plasma. Studies on the soliton's behaviour in plasmas with the ionic species (Ar^+, F^-), (Ar^+, SF_6^-) have been made experimentally (Nakamura and Tsukabayashi 1985; Nakamura 1987) and show

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agreement with the earlier theoretical ones, especially the existence of the compressive and rarefactive solitons in the plasma. Very recently, Singh and Das (1989) have studied theoretically the existence of a critical density of negative ions in a generalised multicomponent plasma, with the ionic species (He^+, H^-), (Ar^+, F^-), (Ar^+, SF_6^-), (K^+, Cl^-) showing a matching relation with solitons observed experimentally.

In the present paper we take a plasma model with ions and multiple electron temperatures to show how ion beams and negative ions interact in exhibiting the fascinating features of the solitons at the critical density of negative ions. As expected, this type of soliton could be present in laboratory plasmas.

First of all, we consider a plasma with a small percentatge of non-isothermality and later, as a degenerate case, the solitons are studied in an isothermal plasma. We derive the K–dV equation to study the existence and behaviour of solitons at the critical density of negative ions, for which the nonlinear coefficient of the K–dV equation in an isothermal plasma vanishes, and requires the derivation of a mK–dV equation. Using an evolution equation involving higher order nonlinearities, the transformation of the K–dV equation to the mK–dV equation is also described and, finally, the salient features of the solitons are discussed in comparison with the results available to date.

2. Basic Equations and Derivation of the K–dV Equation

In order to study ion-acoustic waves of small amplitude, we consider a plasma consisting of positive ions, negative ions and ion beams with the multiple electron temperatures having the high and low values T_{eh} and T_{el}. The basic normalised equations governing the plasma dynamics in a unidirection are the equations of continuity

$$\frac{\partial \bar{n}_\alpha}{\partial \bar{t}} + \frac{\partial}{\partial \bar{x}}(\bar{n}_\alpha \bar{v}_\alpha) = 0, \tag{2}$$

the equations of motion

$$\frac{\partial \bar{v}_\alpha}{\partial \bar{t}} + \bar{v}_\alpha \frac{\partial \bar{v}_\alpha}{\partial \bar{x}} + q_\alpha \mu_\alpha \frac{\partial \bar{\phi}}{\partial \bar{x}} = 0, \tag{3}$$

supplemented by Poisson's equation

$$\frac{\partial^2 \bar{\phi}}{\partial \bar{x}^2} = \bar{n}_{\mathrm{el}} + \bar{n}_{\mathrm{eh}} - \sum_\alpha q_\alpha \bar{n}_\alpha, \tag{4}$$

where each of the normalised plasma parameters are defined as

$$\begin{gathered} \bar{n}_\alpha = n_\alpha/n_0, \quad \bar{n}_{\mathrm{el,h}} = n_{\mathrm{el,h}}/n_0, \quad \mu_\alpha = m_i/m_\alpha, \\ \bar{v}_\alpha = v_\alpha (KT_{\mathrm{ef}}/m_\alpha)^{-\frac{1}{2}}, \qquad T_{\mathrm{ef}} = T_{\mathrm{el}}\, T_{\mathrm{eh}}/(\mu T_{\mathrm{eh}} + \nu T_{\mathrm{el}}), \\ \bar{x} = x(KT_{\mathrm{ef}}/4\pi e^2 n_0)^{-\frac{1}{2}}, \qquad \bar{t} = t(4\pi e^2 n_0/m_\alpha)^{\frac{1}{2}}. \end{gathered} \tag{5}$$

Here μ and ν are the initial densities of the low and high temperature electron components, K is Boltzmann's constant, $\alpha = i, j, b$ stand for positive ions, negative ions and ion beams respectively, m_α is the mass of the αth type particle moving with the velocity v_α and having the density n_α, n_{el} and n_{eh} are the densities of the low and high temperature electrons normalised later to the background plasma density n_0, and $q_\alpha = 1$ when $\alpha = i, b$ and $q_\alpha = -1$ when $\alpha = j$.

We assume the following boundary conditions at $|x| \to \infty$ (omitting the bars hereafter):

$$\begin{aligned}
&\text{(i)}\quad v_i \to 0, \quad v_j \to 0, \quad v_b \to v_b^{(0)};\\
&\text{(ii)}\quad n_\alpha \to n_\alpha^{(0)}, \quad n_{\text{el}} \to \mu, \quad n_{\text{eh}} \to \nu; \qquad (6)\\
&\text{(iii)}\quad \phi \to 0,
\end{aligned}$$

and (iv) the overall charge neutrality condition is maintained throughout the plasma and given by

$$\sum_\alpha q_\alpha n_\alpha^{(0)} = \mu + \nu. \tag{7}$$

Following Das *et al.* (1986), we consider a plasma with a small percentage of non-isothermality, introduced through the electron densities in the form

$$\begin{aligned}
n_{\text{el}} &= \mu\left\{\exp\left(\frac{\phi}{\mu+\nu\beta}\right) - \tfrac{4}{3} b_{\text{l}}\, \epsilon^{\frac{1}{2}} \left(\frac{\phi}{\mu+\nu\beta}\right)^{\frac{3}{2}}\right\},\\
n_{\text{eh}} &= \nu\left\{\exp\left(\frac{\beta\phi}{\mu+\nu\beta}\right) - \tfrac{4}{3} b_{\text{h}}\, \epsilon^{\frac{1}{2}} \left(\frac{\beta\phi}{\mu+\nu\beta}\right)^{\frac{3}{2}}\right\},
\end{aligned} \tag{8}$$

where b_{l} and b_{h} are arbitrary constants depending on the electron temperatures and $\beta = T_{\text{el}}/T_{\text{eh}}$.

In order to derive the K–dV equation, we introduce the stretched space–time coordinates ξ and τ as

$$\xi = \epsilon^{\frac{1}{2}}(x - \lambda t), \qquad \tau = \epsilon^{\frac{3}{2}} t, \tag{9}$$

where λ is the phase velocity of the ion–acoustic wave and ϵ measures the size of the perturbation.

Furthermore, all plasma parameters are expanded asymptotically as a power series in ϵ about the equilibrium state as

$$\begin{pmatrix} n_\alpha \\ v_\alpha \\ \phi \end{pmatrix} = \sum_{s=0}^{\infty} \epsilon^s \begin{pmatrix} n_\alpha^{(s)} \\ v_\alpha^{(s)} \\ \phi^{(s)} \end{pmatrix}, \tag{10}$$

along with

$$v_i^{(0)} = v_j^{(0)} = 0, \qquad \phi^{(0)} = 0. \tag{11}$$

Substituting the relations (8)–(11) into the basic equations (2)–(4) and using the boundary conditions (6), the first order in ϵ gives the following relations:

$$(\lambda-v_\alpha^{(0)})^2 n_\alpha^{(1)} = q_\alpha \mu_\alpha n_\alpha^{(0)} \phi^{(1)}, \tag{12}$$

$$(\lambda-v_\alpha^{(0)}) v_\alpha^{(1)} = q_\alpha \mu_\alpha \phi^{(1)}, \tag{13}$$

$$\sum_\alpha q_\alpha n_\alpha^{(1)} = \phi^{(1)}. \tag{14}$$

The evaluation of the perturbed parameters from (12)–(14) yields the phase velocity λ in the form

$$\frac{n_i^{(0)} + \mu_j n_j^{(0)}}{\lambda^2} + \frac{\mu_b n_b^{(0)}}{(\lambda-v_b^{(0)})^2} = 1, \tag{15}$$

from which one can get a fourth order equation in λ and if all the roots are real, each root would indicate a possible solitary wave. It is well known that $v_b^{(0)}$ equal to zero of a small value gives the stability of the waves. One can easily take $v_b^{(0)} = 0$ to study the stable ion–acoustic wave, as discussed by Das *et al.* (1989). Instability also arises due to the reflection of ions and ion beams which gives multiple streams in the plasma medium and is discussed later. However, in order to get the proper ion–acoustic solitary waves in ion-beam plasmas, we have chosen the initial velocity of the ion beam, functionally depending on the plasma parameters, in such a way that the instability of the waves does not play a part. For mathematical simplicity, and also to consider the possible stability of the waves, we have taken $v_b^{(0)} = 2\lambda$ (Karmakar *et al.* 1988) to get the phase velocity of the ion–acoustic wave explicitly from the relation:

$$\lambda^2 = \sum_\alpha \mu_\alpha n_\alpha^{(0)}. \tag{16}$$

Thus, for studying the solitary wave solitons in our plasma, we choose the initial ion-beam velocity as

$$v_b^{(0)} = 2\lambda. \tag{17}$$

The variation of the phase velocity λ is plotted in Figs 1 and 2. Fig. 1 shows that the phase velocity λ decreases as the concentration ratio of negative ions and ion beams, $n_j^{(0)}/n_b^{(0)}$, increases, whereas Fig. 2 shows that λ increases with the concentration of negative ions, $n_j^{(0)}$, for fixed ion-beam plasma parameters. Comparing Figs 1 and 2 we may conclude that in the plasma without ion beams, the phase velocity always increases with the addition of negative ions, but in the presence of ion beams the phase velocity behaves with opposite characteristics.

The next higher order terms in λ give the relations

$$\frac{\partial n_\alpha^{(1)}}{\partial \tau} - (\lambda-v_\alpha^{(0)})\frac{\partial n_\alpha^{(2)}}{\partial \xi} + n_\alpha^{(0)}\frac{\partial v_\alpha^{(2)}}{\partial \xi} + \frac{\partial}{\partial \xi}(n_\alpha^{(1)} v_\alpha^{(1)}) = 0, \tag{18}$$

$$\frac{\partial v_\alpha^{(1)}}{\partial \tau} - (\lambda-v_\alpha^{(0)})\frac{\partial v_\alpha^{(2)}}{\partial \xi} + v_\alpha^{(1)}\frac{\partial v_\alpha^{(1)}}{\partial \xi} + q_\alpha \mu_\alpha \frac{\partial \phi^{(2)}}{\partial \xi} = 0, \tag{19}$$

$$\sum_\alpha q_\alpha n_\alpha^{(2)} = \phi^{(2)} - \frac{\partial^2 \phi^{(1)}}{\partial \xi^2} - \frac{4}{3}\frac{(\mu b_{\mathrm{l}} + \nu b_{\mathrm{h}} \beta^{\frac{3}{2}})(\phi^{(1)})^{\frac{3}{2}}}{(\mu+\nu\beta)^{\frac{3}{2}}} + \frac{(\mu+\nu\beta^2)(\phi^{(1)})^2}{2(\mu+\nu\beta)^2}, \tag{20}$$

and with the use of the first order results, the elimination of $v_\alpha^{(2)}$, $n_\alpha^{(2)}$ and $\phi^{(2)}$ from (18)–(20) gives after a straightforward mathematical manipulation the desired K–dV equation:

$$\frac{\partial\phi^{(1)}}{\partial\tau}+\{\alpha_1\phi^{(1)}+\alpha_2(\phi^{(1)})^{\frac{1}{2}}\}\frac{\partial\phi^{(1)}}{\partial\xi}+\alpha_3\frac{\partial^3\phi^{(1)}}{\partial\xi^3}=0, \tag{21}$$

where

$$\alpha_1 = \frac{3(\mu+\nu\beta)^2\Sigma_\alpha q_\alpha\mu_\alpha^2 n_\alpha^{(0)}-\lambda^4(\mu+\nu\beta^2)}{2\lambda(\mu+\nu\beta)^2(n_i^{(0)}+\mu_j n_j^{(0)}-\mu_b n_b^{(0)})},$$

$$\alpha_2 = \frac{\lambda^3(\mu b_1+\nu b_h\beta^{\frac{3}{2}})}{(\mu+\nu\beta)^{\frac{3}{2}}(n_i^{(0)}+\mu_j n_j^{(0)}-\mu_b n_b^{(0)})},$$

$$\alpha_3 = \frac{\lambda^3}{2(n_i^{(0)}+\mu_j n_j^{(0)}-\mu_b n_b^{(0)})}. \tag{22}$$

Following Das *et al.* (1986), the K–dV equation (21) admits a solution in the form

$$\phi^{(1)}=\left\{\frac{4\alpha_2}{15U}+\left(\frac{16\alpha_2^2}{225U^2}+\frac{\alpha_1}{3U}\right)^{\frac{1}{2}}\cosh(\chi/\delta_1)\right\}^{-2}, \tag{23}$$

where $\delta_1=2(\alpha_3/U)^{\frac{1}{2}}$ is the soliton width and $\chi=\xi-U\tau$. From which, under the condition $\alpha_1\gg\alpha_2$, the soliton solution of a simple K–dV equation derived in isothermal plasma is obtained as

$$\phi^{(1)}=(3U/\alpha_1)\,\mathrm{sech}^2(\chi/\delta_1). \tag{24}$$

On the other hand, when non-isothermality is dominant, i.e. $\alpha_1\ll\alpha_2$, the solution is given as

$$\phi^{(1)}=\frac{225U^2}{64\alpha_2^2}\,\mathrm{sech}^4(\chi/2\delta_1). \tag{25}$$

Now for a discussion on K–dV solitons, we must have a clear picture of the nonlinear coefficients α_1 and α_2. First, α_1 depends on the masses, densities of the charged particles and the phase velocity. The variation of α_1 with the density ratio, $n_j^{(0)}/n_b^{(0)}$, is shown in Fig. 1, where it is seen that α_1 increases as the ratio increases, whereas it shows a different feature (Fig. 2) with negative ion concentration $n_j^{(0)}$ for a fixed ion-beam concentration. Thus from Figs 1 and 2, we observe that α_1 may be positive, zero or negative depending on the various plasma parameters and, thereby, different features are exhibited in the form of compressive and rarefactive solitons. When the K–dV equation has the nonlinear coefficient α_1, in isolation, the case of an isothermal plasma arises and, correspondingly, the K–dV soliton solution (24) is obtained. A comparison of α_1 from Figs 1 and 2 shows that, due to the presence of ion beams and negative ions in the plasma, $\alpha_1>0$ or $\alpha_1<0$ corresponds to the compressive or rarefactive soliton. When α_1 increases, the amplitude of the ion–acoustic wave decreases yielding an increase in soliton width. In the case of infinitely large amplitude, the ion-beam presence leads to the breaking up of the soliton into multiple solitons.

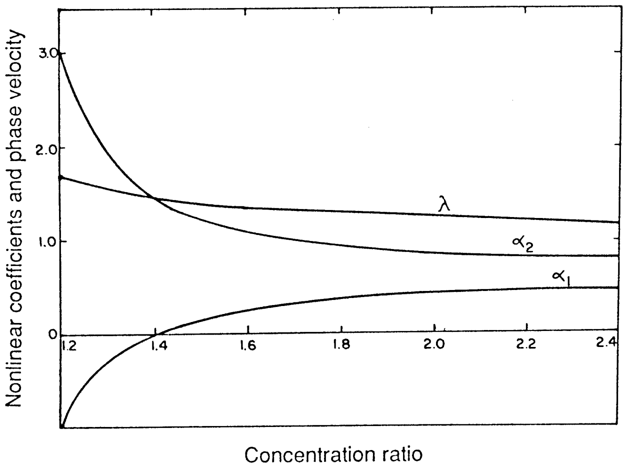

Fig. 1. Variation of the nonlinear coefficients α_1, α_2 and phase velocity λ with the density ratio $n_j^{(0)}/n_b^{(0)}$ for the typical values $\mu = 0 \cdot 1$, $\nu = 0 \cdot 9$, $\beta = 0 \cdot 5$, $\mu_b = 1 \cdot 0$, $\mu_j = 0 \cdot 5$, $b_l = 0 \cdot 2$ and $b_h = 0 \cdot 6$.

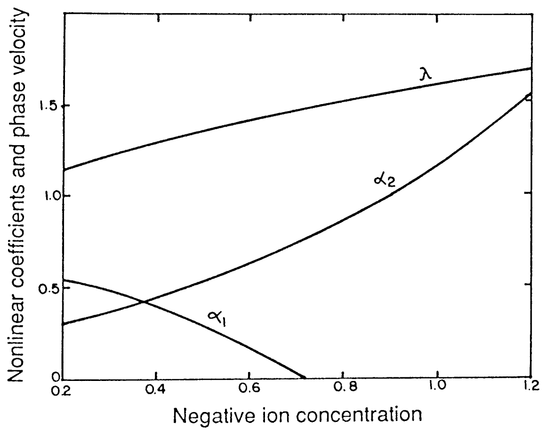

Fig. 2. Variation of the nonlinear coefficients α_1, α_2 and phase velocity λ with the negative ion concentration $n_j^{(0)}$ for the same values of parameters as in Fig. 1.

We show the numerical variation of α_2 and the dispersive effect α_3 with the density ratio $n_j^{(0)}/n_b^{(0)}$ in Figs 1 and 3 respectively. As α_2 and α_3 depend directly on λ, the nature of all three are similar. Thus, as in the case of λ, the nonlinear and dispersive coefficients decrease gradually as the concentration of density ratio increases. However, in a plasma with fixed ion-beam concentration, Fig. 2 shows that the variation of λ, α_2 and hence α_3 are opposite in nature. Thus, the introduction of negative ions along with ion beams modifies the existence and behaviour of ion–acoustic solitons compared with those obtained earlier in simpler plasmas.

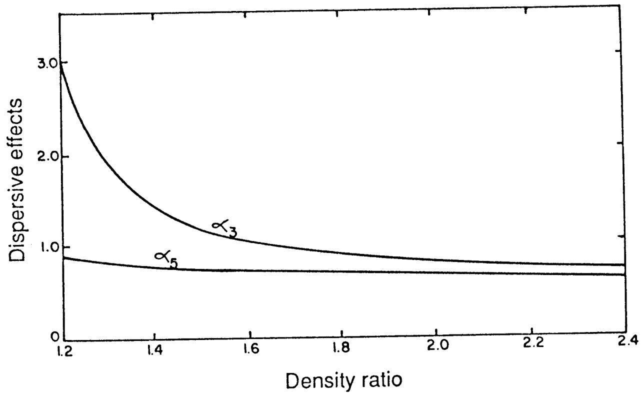

Fig. 3. Dispersive effects α_3 and α_5 plotted as functions of the density ratio $n_j^{(0)}/n_b^{(0)}$ for the typical values $\mu = 0 \cdot 1$, $\nu = 0 \cdot 9$, $\beta = 0 \cdot 5$, with $\mu_b = 1 \cdot 0$, $\mu_j = 0 \cdot 5$ for α_3; and $\mu_b = 1 \cdot 1$, $\mu_j = 2 \cdot 5$ for α_5.

We have already observed that, in the neighbourhood of the critical density of negative ions, the soliton amplitude can be large which emphasises the non-applicability of the reductive perturbation technique. In practice, it should not be so and thus a proper analysis of the existence of solitons in the plasma is required. Following White *et al.* (1972), we consider the solution of the basic equations in the form $f = f(x-Ut)$ with respect to a shock rest frame moving with the velocity U and, from the conservation laws, we have

$$n_\alpha v_\alpha = N_\alpha U, \tag{26}$$

$$v_\alpha^2 + 2q_\alpha \mu_\alpha \phi = U^2, \tag{27}$$

where N_α is the total density of the α-type charged particles. Eliminating v_α we get

$$n_\alpha = N_\alpha\left(1 - \frac{2q_\alpha \mu_\alpha \phi}{U^2}\right)^{-\frac{1}{2}} \tag{28}$$

Poisson's equation then takes the form

$$\frac{d^2\phi}{dx^2} = n_{el} + n_{eh} - \sum_\alpha q_\alpha N_\alpha\left(1 - \frac{2q_\alpha \mu_\alpha \phi}{U^2}\right)^{-\frac{1}{2}} \tag{29}$$

Equation (29) shows that the kinetic energy of the positive ion introduces a barrier at $U = \surd 2\phi$. The ion–acoustic wave propagates with a velocity smaller than $\surd 2\phi$ and is reflected from this barrier introduced by the positive ions. Similarly, due to the ion beams, the ion–acoustic wave reflects again from the barrier at $U = \surd 2\mu_b\phi$. When $\mu_b > 1$, the wave is reflected from the barrier at

$U=\sqrt{2\phi}$, which separates the plasma into two entirely different regions, and the solitons can be observed only in the region $U<\sqrt{2\phi}$. Similarly, the case $\mu_b<1$ implies that the barrier at $U=\sqrt{2\mu_b\phi}$, introduced by the ion beams, relects the wave but earlier than the barrier at $U=\sqrt{2\phi}$. Thus, the barriers introduced by positive ions and ion beams form different plasma regions having multiple non-symmetrical wave propagations due to which the possible instability plays a part. In this type of reflection phenomena, the waves get reflected before attaining infinitely large amplitude. To continue the discussion, we consider Poisson's equation which, by using (26) and (27), is expressed in terms of the potential energy $V(\phi)$ as

$$\frac{d^2\phi}{dx^2}=n_{\rm el}+n_{\rm eh}-\sum_\alpha q_\alpha N_\alpha\left(1-\frac{2q_\alpha\mu_\alpha\phi}{U^2}\right)^{-\frac{1}{2}}\equiv\frac{dV}{d\phi}, \tag{30}$$

which gives the solution for $V(\phi)$ as

$$\begin{aligned}-V(\phi) &= \mu(\mu+\nu\beta)\left\{\exp\left(\frac{\phi}{\mu+\nu\beta}\right)-\tfrac{8}{15}b_{\rm l}\,\epsilon^{\frac{1}{2}}\left(\frac{\phi}{\mu+\nu\beta}\right)^{\frac{5}{2}}\right\}\\ &+\frac{\nu(\mu+\nu\beta)}{\beta}\left\{\exp\left(\frac{\beta\phi}{\mu+\nu\beta}\right)-\tfrac{8}{15}b_{\rm h}\,\epsilon^{\frac{1}{2}}\left(\frac{\beta\phi}{\mu+\nu\beta}\right)^{\frac{5}{2}}\right\}\\ &+\sum_\alpha\frac{U^2N_\alpha}{\mu_\alpha}\left(1-\frac{2q_\alpha\mu_\alpha\phi}{U^2}\right)^{\frac{1}{2}}+\text{constant}.\end{aligned} \tag{31}$$

Now, for a fixed value of U, the variation of $V(\phi)$ gives a maximum value at the nonzero roots of $V(\phi)$ given by (31). To study the existence of the ion–acoustic wave requires a determination of the quantity $d^2V/d\phi^2$ for $\phi=0$. A necessary condition for the existence of the soliton wave is when this quantity is less than zero, whereas the quantity greater than zero predicts the formation of a shock in the plasma. From (30) we have

$$\left.\frac{d^2V}{d\phi^2}\right|_{\phi=0}=-\frac{1}{U^2}(U^2-\lambda^2). \tag{32}$$

This shows that stable solitons will exist only when $U>\lambda$, whereas $U<\lambda$ ensures the non-existence of ion–acoustic waves. Hence, we conclude that the existence of waves requires a necessary condition depending on the shock front velocity U and the phase velocity λ.

So far we have not shown the existence and behaviour of the solitons at the critical density of the negative ions for which the nonlinear coefficient α_1 of the K–dV equation derived in isothermal plasma is zero. In order to gain a deeper insight, we consider the higher order nonlinearities to derive a mK–dV equation. The critical density of the negative ions at which α_1 vanishes in isolation gives the relation

$$\lambda^4=\frac{3(\mu+\nu\beta)^2}{\mu+\nu\beta^2}\sum_\alpha q_\alpha\mu_\alpha^2 n_\alpha^{(0)}. \tag{33}$$

However, it is not sufficient to describe the ion–acoustic solitary wave by the K–dV equation or by the mK–dV equation in the vicinity of the critical density. One has to take into account quadratic and cubic nonlinearities in the evolution of the mK–dV equation. Such a choice of the nonlinearity leads us in Section 4 to a study of the transition of the K–dV equation to the mK–dV equation, as well as the conservation of the Sagdeev potential.

3. Derivation of the mK–dV Equation

To derive the mK–dV equation we use the stretched space–time coordinates ξ and τ (Watanabe 1984)

$$\xi = \epsilon(x-\lambda t), \qquad \tau = \epsilon^3 \lambda t. \tag{34}$$

Following Das *et al.* (1986) the isothermality of the plasma is introduced through the electron densities

$$n_{\mathrm{el}} = \mu \exp\left(\frac{\phi}{\mu+\nu\beta}\right), \qquad n_{\mathrm{eh}} = \nu \exp\left(\frac{\beta\phi}{\mu+\nu\beta}\right). \tag{35}$$

Using (10), (34) and (35) along with the boundary conditions (6), the basic equations (2)–(4) give, for the lowest order in ϵ,

$$(\lambda - v_\alpha^{(0)})^2 n_\alpha^{(1)} = q_\alpha \mu_\alpha n_\alpha^{(0)} \phi^{(1)}, \tag{36}$$

$$(\lambda - v_\alpha^{(0)}) v_\alpha^{(1)} = q_\alpha \mu_\alpha \phi^{(1)}, \tag{37}$$

$$\sum_\alpha q_\alpha n_\alpha^{(1)} = \phi^{(1)}. \tag{38}$$

From here the phase velocity λ of the ion–acoustic wave can be obtained, as given earlier by (16).

To the next higher order in ϵ we obtain

$$n_\alpha^{(2)} = \frac{3\mu_\alpha^2 n_\alpha^{(0)}}{2\lambda^4}(\phi^{(1)})^2 + \frac{q_\alpha \mu_\alpha n_\alpha^{(0)}}{\lambda^2}\phi^{(2)}, \tag{39}$$

$$v_\alpha^{(2)} = p_\alpha\left(\frac{\mu_\alpha^2}{2\lambda^3}(\phi^{(1)})^2 + \frac{q_\alpha \mu_\alpha}{\lambda}\phi^{(2)}\right), \tag{40}$$

where $p_\alpha = 1$ for $\alpha = i, j$ and $p_\alpha = -1$ for $\alpha = b$, and further

$$\sum_\alpha q_\alpha n_\alpha^{(2)} = \phi^{(2)} + \frac{\mu+\nu\beta^2}{2(\mu+\nu\beta)^2}(\phi^{(1)})^2, \tag{41}$$

from which we obtain

$$\left(\frac{1}{\lambda^2}\sum_\alpha \mu_\alpha n_\alpha^{(0)} - 1\right)\phi^{(2)} + \tfrac{1}{2}\left(\frac{3}{\lambda^4}\sum_\alpha q_\alpha \mu_\alpha^2 n_\alpha^{(0)} - (\mu+\nu\beta^2)/(\mu+\nu\beta)^2\right)(\phi^{(1)})^2 = 0. \tag{42}$$

Now, the coefficients of $\phi^{(2)}$ and $(\phi^{(1)})^2$ in (42) vanish by using (16) and the condition of the nonlinear coefficient α_1 to be zero given by (33), showing that Poisson's equation (41) is always satisfied when α_1 in the K–dV equation (21) is zero.

Furthermore, the next higher order in ϵ gives the relations

$$\lambda\frac{\partial n_\alpha^{(1)}}{\partial\tau}-(\lambda-v_\alpha^{(0)})\frac{\partial n_\alpha^{(3)}}{\partial\xi}+n_\alpha^{(0)}\frac{\partial v_\alpha^{(3)}}{\partial\xi}+\frac{\partial}{\partial\xi}(n_\alpha^{(1)}v_\alpha^{(2)})+\frac{\partial}{\partial\xi}(n_\alpha^{(2)}v_\alpha^{(1)})=0, \tag{43}$$

$$\lambda\frac{\partial v_\alpha^{(1)}}{\partial\tau}-(\lambda-v_\alpha^{(0)})\frac{\partial v_\alpha^{(3)}}{\partial\xi}+\frac{\partial}{\partial\xi}(v_\alpha^{(1)}v_\alpha^{(2)})+q_\alpha\mu_\alpha\frac{\partial\phi^{(3)}}{\partial\xi}=0, \tag{44}$$

$$\sum_\alpha q_\alpha n_\alpha^{(3)}=\phi^{(3)}+\frac{\mu+\nu\beta^2}{(\mu+\nu\beta)^2}(\phi^{(1)}\phi^{(2)})+\frac{\mu+\nu\beta^3}{6(\mu+\nu\beta)^3}(\phi^{(1)})^3-\frac{\partial^2\phi^{(1)}}{\partial\xi^2}, \tag{45}$$

which can be simplified using the earlier first and second order results in ϵ. The elimination of $v_\alpha^{(3)}$, followed by some usual mathematical manipulation, gives the desired mK–dV equation in the form

$$\frac{\partial\phi^{(1)}}{\partial\tau}+\alpha_4(\phi^{(1)})^2\frac{\partial\phi^{(1)}}{\partial\xi}+\alpha_5\frac{\partial^3\phi^{(1)}}{\partial\xi^3}=0, \tag{46}$$

where

$$\alpha_4=\frac{15(\mu+\nu\beta)^3\Sigma_\alpha\mu_\alpha^3 n_\alpha^{(0)}-\lambda^6(\mu+\nu\beta^3)}{4\lambda^4(\mu+\nu\beta)^3(n_i^{(0)}+\mu_j n_j^{(0)}-\mu_b n_b^{(0)})}, \tag{47a}$$

$$\alpha_5=\frac{\lambda^2}{2(n_i^{(0)}+\mu_j n_j^{(0)}-\mu_b n_b^{(0)})}. \tag{47b}$$

The variation of the nonlinear coefficient α_4 with negative ion density in relation to the ion-beam density is plotted in Fig. 4. This shows that α_4 increases as the density ratio increases in the ion-beam plasma. Again, we show in Fig. 5 the variation of α_4 with negative ion concentration for a fixed ion-beam concentration. It is seen that the nonlinear coefficient decreases gradually with concentration. Comparing Figs 4 and 5 we conclude that α_4 increases or decreases depending on negative ion and ion-beam concentrations. Thus the ion beams in the plasma play an important role in the existence and behaviour of the solitons. Again, Fig. 3 shows the variation of the dispersive effect α_5 in the plasma with density ratio. The overall variation shows that α_4 increases while the dispersive coefficient decreases with an increase in negative ion concentration. But in the absence of ion beams α_4 and α_5 behave in the opposite way compared with the variations discussed earlier.

The present analysis shows that at the critical density for which α_1 of the K–dV equation (21) vanishes the nonlinear and dispersive terms of the mK–dV equation (46) are always positive. From Wadati (1973), equation (46) has multiple soliton solutions.

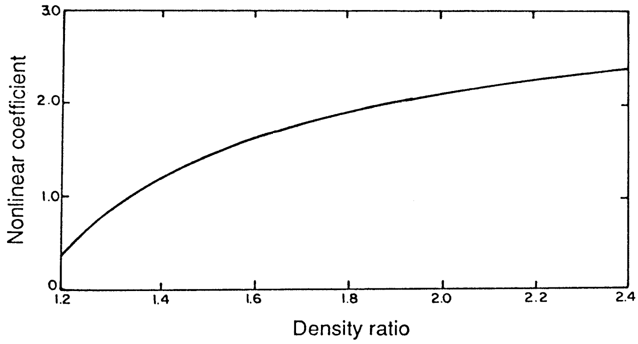

Fig. 4. Nonlinear coefficient α_4 as a function of the density ratio $n_j^{(0)}/n_b^{(0)}$ for typical values of $\mu = 0 \cdot 1$, $\nu = 0 \cdot 9$, $\beta = 0 \cdot 5$, $\mu_b = 1 \cdot 1$ and $\mu_j = 2 \cdot 5$.

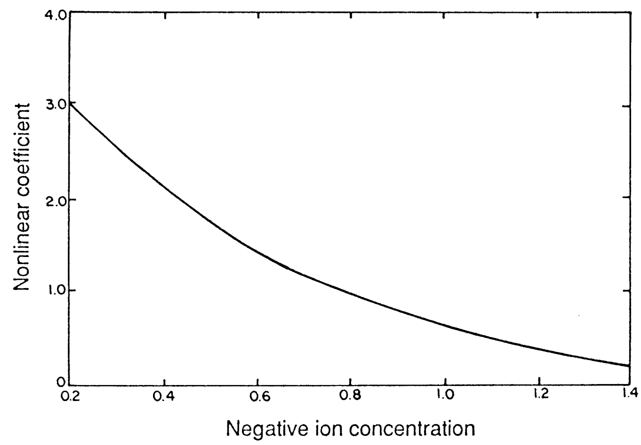

Fig. 5. Nonlinear coefficient α_4 as a function of the negative ion concentration $n_j^{(0)}$ for the same parameter values given in Fig. 4.

To obtain one solution we introduce the variable $\chi = \xi - U\tau$ defined earlier and, following Davidson (1972) and Das *et al.* (1986), the solution of equation (46) is read as

$$\phi^{(1)} = \phi_0 \operatorname{sech}(\chi/\delta_2), \tag{48}$$

where $\phi_0 = \pm(6U/\alpha_4)^{\frac{1}{2}}$ is the amplitude and $\delta_2 = (\alpha_5/U)^{\frac{1}{2}}$ the width of the ion-acoustic wave. The $\pm$ sign for the amplitude shows that for the mK–dV equation both the compressive and rarefactive solitons exist in the plasma. The solution (48) exhibits the characteristics of 'sech' shape solitary waves, differing from the K–dV solitons possessing the 'sech^2' shape or 'sech^4' shape solutions.

4. Derivation of an Evolution Equation near Critical Density

The derivation of the mK–dV equation (46) holds good only when the nonlinear coefficient α_1 of the K–dV equation (21) is zero at the critical density introduced by the negative ions. Keeping this in mind, we derive an equation in the vicinity of the critical density and later, analyse the transformation of the mK–dV soliton from the K–dV soliton. From Watanabe (1984), Poisson's equation on equating the terms of order ϵ^2 gives

$$\sum_\alpha q_\alpha n_\alpha^{(2)} - \phi^{(2)} - \frac{\mu+\nu\beta^2}{2(\mu+\nu\beta)^2}(\phi^{(1)})^2 = \left(\frac{3}{\lambda^4}\sum_\alpha q_\alpha \mu_\alpha^2 n_\alpha^{(0)} - \frac{\mu+\nu\beta^2}{(\mu+\nu\beta)^2}\right)\frac{(\phi^{(1)})^2}{2}. \tag{49}$$

We consider now the case where the coefficient of $(\phi^{(1)})^2$ on the right-hand side is nonzero but of order $O(\epsilon)$. This leads the right-hand side to be of order ϵ^3 and it can be taken to be zero in order ϵ^2. Equation (42) is satisfied and this will be used to find Poisson's equation of order ϵ^3. Finally, the equation takes the form

$$\frac{\partial^2\phi^{(1)}}{\partial\xi^2} = \left(\phi^{(3)} + \frac{\mu+\nu\beta^2}{(\mu+\nu\beta)^2}(\phi^{(1)}\phi^{(2)}) + \frac{\mu+\nu\beta^3}{6(\mu+\nu\beta)^3}(\phi^{(1)})^3 - \sum_\alpha q_\alpha n_\alpha^{(3)}\right) + \left(\phi^{(2)} + \frac{\mu+\nu\beta^2}{2(\mu+\nu\beta)^2}(\phi^{(1)})^2 - \sum_\alpha q_\alpha n_\alpha^{(2)}\right). \tag{50}$$

Thus instead of (46), we obtain another mK–dV equation:

$$\frac{\partial\phi^{(1)}}{\partial\tau} + \{\alpha_1\phi^{(1)} + \alpha_4(\phi^{(1)})^2\}\frac{\partial\phi^{(1)}}{\partial\xi} + \alpha_5\frac{\partial^3\phi^{(1)}}{\partial\xi^3} = 0, \tag{51}$$

where α_1, α_5 are defined in (22) and (47).

From Wadati (1975) the derived mK–dV equation (51) will have a multiple number of solitons. A parallel discussion on the nature of the solitons could be made but we will not do so here. To examine the soliton solution of (51) we consider the variable $\chi = \xi - U\tau$ as before and, following the usual procedure (Singh and Das 1989), we get

$$\frac{d^2\phi^{(1)}}{d\chi^2} = -\frac{1}{\alpha_5}\left(-U\phi^{(1)} + \frac{\alpha_1}{2}(\phi^{(1)})^2 + \frac{\alpha_4}{3}(\phi^{(1)})^3\right). \tag{52}$$

The relation between the effective potential P and the perturbed wave potential $\phi^{(1)}$ is

$$\frac{d^2\phi^{(1)}}{d\chi^2} = -\frac{dP}{d\phi^{(1)}} \tag{53}$$

which, along with (52), gives the potential P as

$$P = \frac{1}{2\alpha_5}\left(-U(\phi^{(1)})^2 + \frac{\alpha_1}{3}(\phi^{(1)})^3 + \frac{\alpha_4}{6}(\phi^{(1)})^4\right). \tag{54}$$

To compare (54) with the exact stationary solution of the basic equations (2)–(4) we, following Watanabe (1984) and Sagdeev (1966), introduce the variable

$X = x - Mt$ (M being the velocity of a localised wave travelling without changing the wave form) in equations (2)–(4) to read

$$\frac{d^2\phi}{dX^2} = -\frac{dS}{d\phi}, \tag{55}$$

where S is the Sagdeev potential.

Using $M = \lambda(1+U)$ and assuming $S \to 0$ when $\phi \to 0$, we get for the small wave amplitude ϕ together with $U \leq 1$

$$S \approx \frac{1}{2\alpha_5}\left(-U\phi^2 + \frac{\alpha_1}{3}\phi^3 + \frac{\alpha_4}{6}\phi^4\right). \tag{56}$$

Comparing the identical relations (54) and (56) we observe that (54) is true not only in the vicinity of the critical density of negative ions but is applicable over the whole range of negative ion concentration. Furthermore, our present plasma model, though it differs from that of Watanabe (1984), does not affect the nature of the transition of the K–dV equation to the mK–dV equation and vice versa.

The nature of the nonlinear coefficient α_4 shown in Figs 4 and 5 depends on the concentration of negative ions and ion beams. The presence of negative ions ensures the occurrence of compressive and rarefactive solitons, whereas positive ions and ion beams modify the behaviour of the solitons giving only compressive solitons. Furthermore, the combination of multiple ions of both kinds, ion beams and multiple electron temperatures in the plasma leads to a slower exhibition of the critical density of negative ion concentrations and, consequently, exhibition of the rarefactive soliton in the plasma occurs later.

5. Conclusions

Emerging from the present study on the ion-acoustic solitary waves in a generalised multicomponent plasma are results which show the main interaction of the negative ions, ion beams and the solitons at the critical density observed in the plasma. In a plasma with positive ions, ion beams and electrons the K–dV equation always gives compressive solitons, whereas the presence of negative ions gives either the compressive or rarefactive soliton depending on the sign of the nonlinear coefficient α_1. As the concentration of negative ions increases, α_1 correspondingly decreases (Fig. 2) and thereby the amplitude of the ion–acoustic wave increases considerably. With a higher concentration of negative ions, a situation may arise in which the soliton amplitude is infinitely large and the charge separation providing the dispersive effect will not be sufficient to prevent the steepening of the wave, as well as the breaking up of the soliton. However, it has been shown that the waves get reflected from barriers introduced by positive ions and ion beams before attaining infinitely large amplitudes. The barriers at $U = \sqrt{2\mu_\alpha\phi}$ introduced by positive ions and ion beams depend on the mass ratio μ_α. In the case $\mu_b > 1$, i.e. lighter ion-beam mass, the wave gets reflected from the barrier at $U = \sqrt{2\phi}$ introduced by the positive ion, whereas for $\mu_b < 1$ the reflection occurs from the barrier at $U = \sqrt{2\mu_b\phi}$ introduced by the ion beam, but earlier than the barrier at $U = \sqrt{2\phi}$. Furthermore, it has been shown that Poisson's equation yields a condition for the existence of the stable soliton and requires the condition $U > \lambda$, otherwise the stable solitons will not be in the plasma.

The study of the mK–dV equation at the critical density of negative ions yields both the compressive and rarefactive solitons, depending on the sign of the soliton amplitude. In the vicinity of the critical concentration of negative ions, the transition of the K–dV to the mK–dV equation and vice versa has been described with the help of another mK–dV equation (51) involving higher order nonlinearities. Moreover, the discussion and conservation of the Sagdeev potential show that the mK–dV equation (51) holds good for all values of negative ion concentration. Finally, we conclude that the presence of negative ions and ion beams along with multiple electrons in the plasma not only drastically modifies the characteristics of the ion–acoustic waves, but also predicts a slower exhibition of the critical density of negative ions compared with a plasma without ion beams. Consequently, rarefactive solitons appear later in relation to negative ions. The present study calls for further work in order to clarify the significance of these results for laboratory plasmas.

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