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General Dispersion Relation for Surface Waves on a Plasma–Vacuum Interface: Application to Magnetised Plasmas

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Abstract

The general dispersion relation for electromagnetic surface waves on a plasma–vacuum interface, recently derived by Rowe (1991), is applied to the case of a cold magnetised plasma bounded by a vacuum. It is illustrated how the dispersion relation and the surface wave fields may be determined in practice, and some general results are given. It is remarked that a plasma of this type satisfies the consistency conditions which were derived for the general theory by Rowe. These general results are then used to reproduce the dispersion relation of Cramer and Donnelly (1983) for low frequency surface waves in an electron–ion plasma. This example illustrates the general principles of the theory. A major difference between the derivation in their paper and the calculation of this paper is that in the former the plasma was assumed to be infinitely conducting whereas here the plasma is strictly assumed to have finite conductivity. The transition to infinite conductivity, which involves a slight extension of the general theory to include surface currents, is thus also discussed.

1. Introduction

It is well known that a bounded homogeneous plasma can support two distinct types of electromagnetic wave mode. One type of wave mode is characterised by a wavevector $\mathbf{k}$ which is real. These waves can propagate throughout the plasma, that is, they are not localised in space, and are thus commonly referred to as bulk (or body) waves. The second type of wave mode supported is characterised by a wavevector of the form $\mathbf{k} = k_n\hat{\mathbf{n}} + \mathbf{k}_s$, where $\mathbf{k}_s$ is real but k_n is imaginary. Here, $\hat{\mathbf{n}}$ is a unit vector normal to the bounding surface and $\mathbf{k}_s$ is the component of the wavevector parallel to the surface. These wave modes are localised at the surface, that is, they propagate along the bounding interface and decay exponentially in amplitude into both the plasma and the bounding medium. As such, they are referred to as surface waves.

A standard method for treating bulk waves in plasmas, based on Fourier transform theory, has existed for over thirty years (e.g. Stix 1962). This approach is useful as it allows one to describe the properties of bulk waves in a general way, that is, in a way which does not restrict the dielectric properties of the plasma. The properties of specific bulk wave modes, propagating in specific plasmas of interest, can then be obtained from within the framework of this more general theory by making the relevant approximations to the dielectric tensor which describes the plasma. The derivation of a corresponding theory for surface waves is more complicated due to the requirement that the wave fields satisfy

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the relevant electromagnetic boundary conditions at the interface. Any theory of electromagnetic surface waves is necessarily limited by the approximations made to simplify these boundary conditions. Nevertheless, a general theory of surface waves is desirable, and a result can be derived which is subject only to the suitability of the chosen boundary conditions.

A general dispersion relation for surface waves in an isotropic plasma bounded by a vacuum was derived by Barr and Boyd (1972), using an image plasma approach which is closely analogous to the method of images used extensively in electrostatic theory (e.g. Cheng 1983). This work is a generalisation of that of Guernsey (1969) and Fuse and Ichimaru (1975) in the sense that it is not restricted to waves which are electrostatic (potential) in character. Although it is applicable to both cold and thermal plasmas, the assumption of an isotropic plasma restricts its use to unmagnetised plasmas. This theory has only recently been extended to magnetised plasmas by Rowe (1991), who derived a general dispersion relation for electromagnetic surface waves propagating on a sharp plasma–vacuum interface. This new theory enables both unmagnetised and magnetised plasmas to be treated in the same formal way. Previously, surface waves in magnetised plasmas were treated in a less formal way (e.g. Wallis 1982; Cramer and Donnelly 1983), in the sense that the results were not derived from a more general theory. Additional features of the new theory are that it is a natural extension of the plasma response theory used to describe bulk waves in a homogeneous plasma, it is a dynamic (electromagnetic) analogue of the method of images used extensively to solve electrostatic problems, and the general dispersion relation is written in a more compact form than other less general results (e.g. Wallis 1982).

In view of the important contribution of this theory to the literature on surface waves in plasmas and electromagnetic theory in general, as well as the fact that it is entirely new in the context of magnetised plasmas, it is important to consider how this general theory may be applied to surface waves in a magnetised plasma in practice. In this paper we address this issue by applying the theory to a cold magnetised plasma, and by illustrating how the theory can be used to rederive an established result. In particular we show how the dispersion relation of Cramer and Donnelly (1983) is reproduced in the low frequency ($\omega \ll \Omega_e,\ \omega_{pe}$) limit of an electron–ion plasma. This is an important example as these low frequency waves (sometimes known as Alfvén surface waves) are likely to occur in laboratory and astrophysical plasmas of interest, and have been invoked in theories attempting to explain phenomena such as the heating of the solar corona (e.g. Gordon and Hollweg 1983; Assis and Busnardo-Neto 1987).

A significant difference between the calculation of Cramer and Donnelly (1983) and that presented herein (apart from the different theories used) is that they have made the assumption that the plasma is infinitely conducting (an idealised situation), whereas here the plasma is strictly assumed to have finite conductivity. These different assumptions lead to different interpretations of the wave fields associated with the surface waves. In comparing the two calculations it is thus also important to consider how the transition to infinite conductivity may be achieved in the context of this general theory.

This paper is structured as follows. In Section 2 the relevant bulk wave theory of plasmas is briefly discussed and we review the general theory of Rowe (1991) for surface waves, summarising the important results. In Section 3 the formalism of the surface wave theory is further developed. In particular, the method of evaluation of the dispersion relation is considered in general, as is the formal approach to calculating the wave fields associated with surface waves. These extensions were not dealt with by Rowe (1991). In Section 4 the general theory is applied to a cold magnetised plasma and some general results which have not previously been presented in the literature are given. These results are used in Section 5 when we treat the low frequency limit and derive the dispersion relation of Cramer and Donnelly (1983). Section 6 considers the transition to infinite conductivity.

2. Review of the General Theory

It is convenient to summarise the main results of the surface wave theory developed by Rowe (1991). To do this it is also useful to briefly discuss some relevant results of the standard bulk wave theory, for which we shall follow closely the notation of Melrose (1986). The definitions presented herein are required throughout this paper.

(a) Bulk Wave Theory

The wave equation for bulk waves in a homogeneous plasma can be written in the general form

$$[n^2(\boldsymbol{\kappa}\boldsymbol{\kappa}-\boldsymbol{\delta})+\boldsymbol{\delta}]\mathbf{E}(\omega,\mathbf{k}) = -i\left(\frac{\mu_0 c^2}{\omega}\right)\mathbf{J}(\omega,\mathbf{k})\,, \tag{1}$$

where $\mathbf{E}(\omega,\mathbf{k})$ is the Fourier transformed electric field of the wave with frequency ω and wavevector $\mathbf{k}$, $n = ck/\omega$ is the refractive index, $\boldsymbol{\kappa} = \mathbf{k}/k$ is the unit vector in the direction of propagation and $\boldsymbol{\delta}$ is the unit matrix. In the absence of extraneous source terms the Fourier transformed current density $\mathbf{J}(\omega,\mathbf{k})$ is identified as the induced current density

$$\mathbf{J}_{\mathrm{ind}}(\omega,\mathbf{k}) = \boldsymbol{\sigma}(\omega,\mathbf{k})\mathbf{E}(\omega,\mathbf{k})\,, \tag{2}$$

where the conductivity tensor $\boldsymbol{\sigma}(\omega,\mathbf{k})$ completely describes the linear electromagnetic response of the plasma. In this case the wave equation is known as the homogeneous wave equation (no source terms), and can be written as

$$\boldsymbol{\Lambda}(\omega,\mathbf{k})\mathbf{E}(\omega,\mathbf{k}) = 0\,, \tag{3}$$

where the response tensor $\boldsymbol{\Lambda}(\omega,\mathbf{k})$ is given by

$$\boldsymbol{\Lambda}(\omega,\mathbf{k}) = n^2(\boldsymbol{\kappa}\boldsymbol{\kappa}-\boldsymbol{\delta})+\mathbf{K}(\omega,\mathbf{k})\,, \tag{4}$$

and the equivalent dielectric tensor $\mathbf{K}(\omega,\mathbf{k})$ is related to $\boldsymbol{\sigma}(\omega,\mathbf{k})$ through

$$\mathbf{K}(\omega,\mathbf{k}) = \boldsymbol{\delta}+\frac{i}{\omega\varepsilon_0}\boldsymbol{\sigma}(\omega,\mathbf{k})\,. \tag{5}$$

The homogeneous wave equation determines the solutions for $\mathbf{E}(\omega, \mathbf{k})$ which describe the normal bulk modes of the medium. The condition for non-trivial solutions is that the determinant of $\mathbf{\Lambda}(\omega, \mathbf{k})$ vanish, which we write as

$$\Lambda(\omega, \mathbf{k}) = 0\,. \tag{6}$$

The zeros of (6) give the dispersion relations $\omega = \omega_M(\mathbf{k})$ of the bulk wave modes (labelled by M) in the plasma.

(b) Surface Wave Theory

For surface waves the situation is rather more complex. Consider a physical system consisting of a semi-infinite homogeneous plasma ($x < 0$) bounded by a vacuum ($x > 0$) with an infinitesimal (sharp) interface defined by the yz plane. A general dispersion relation for surface waves in such a system can be derived by appealing to the image plasma approach. This approach involves writing down a wave equation which describes a second system consisting of the physical plasma ($x < 0$) and an image plasma ($x > 0$). This system is known as the real–image plasma system. The wave equation for this system is

$$\mathbf{\Lambda}(\omega, \mathbf{k})\mathbf{E}(\omega, \mathbf{k}) = -i\left(\frac{\mu_0 c^2}{\omega}\right)\mathbf{M}_s(\omega, \mathbf{k}_s)\,, \tag{7}$$

where the source term $\mathbf{M}_s(\omega, \mathbf{k}_s)$ represents a surface current at the interface between the physical and image plasmas and $\mathbf{k}_s = (k_y, k_z)$ is the surface wavevector. At this point we stress that the image system surface current density $\mathbf{M}_s(\omega, \mathbf{k}_s)$, which is related to the discontinuity in the tangential components of the real and image plasma magnetic fields across the real–image system interface, is quite distinct from the real system surface current $\mathbf{J}_s(\omega, \mathbf{k}_s)$ which is related to the discontinuity in the tangential components of the real plasma and vacuum magnetic fields across the physical system interface. This difference is emphasised by the fact that for surface waves $\mathbf{M}_s(\omega, \mathbf{k}_s) \neq 0$, even though $\mathbf{J}_s(\omega, \mathbf{k}_s) = 0$ in many cases of interest. Compare, for instance, equations (91) and (18).

Note that equation (7) is the same as the homogeneous wave equation (3) except that it has a source term and the electric field $\mathbf{E}(\omega, \mathbf{k})$ now describes both the real and image plasma fields. The source term $\mathbf{M}_s(\omega, \mathbf{k}_s)$ is an essential feature of the image theory as applied to surface waves and is determined by imposing the relevant electromagnetic boundary conditions at the physical plasma–vacuum interface (see equation 18). In this way the problem of satisfying the boundary conditions of the physical system is replaced by the equivalent problem of solving an equation for $\mathbf{M}_s(\omega, \mathbf{k}_s)$, which in turn determines the nature of the image plasma fields and charges. Thus, as was pointed out in Section 1 and in Rowe (1991), the image approach is closely analogous to the method of images used in electrostatic theory. When $\mathbf{M}_s(\omega, \mathbf{k}_s) = 0$ equation (7) is equivalent to (3) and describes only bulk wave modes which are decoupled from the surface.

Equation (7) may be decomposed into two equations, using the fact that the x component of $\mathbf{M}_s(\omega, \mathbf{k}_s)$ is zero (as it is a surface vector). The first equation determines the x component of the electric field in the real–image plasma system in terms of the surface component $\mathbf{E}_s(\omega, \mathbf{k})$

$$E_x(\omega, \mathbf{k}) = -\frac{[\hat{\mathbf{x}}_n \cdot \mathbf{\Lambda}(\omega, \mathbf{k})]_s}{\Lambda_{xx}(\omega, \mathbf{k})} \cdot \mathbf{E}_s(\omega, \mathbf{k}), \tag{8}$$

and the second determines $\mathbf{E}_s(\omega, \mathbf{k})$ in terms of the source $\mathbf{M}_s(\omega, \mathbf{k}_s)$

$$\mathbf{\Gamma}_s(\omega, \mathbf{k})\mathbf{E}_s(\omega, \mathbf{k}) = -i\left(\frac{\mu_0 c^2}{\omega}\right)\mathbf{M}_s(\omega, \mathbf{k}_s), \tag{9}$$

where the 2×2 matrix $\mathbf{\Gamma}_s(\omega, \mathbf{k})$ is related to the response of the bulk plasma through

$$\mathbf{\Gamma}_s(\omega, \mathbf{k}) = \mathbf{\Lambda}_s(\omega, \mathbf{k}) - \frac{[\mathbf{\Lambda}(\omega, \mathbf{k}) \cdot \hat{\mathbf{x}}_n]_s [\hat{\mathbf{x}}_n \cdot \mathbf{\Lambda}(\omega, \mathbf{k})]_s}{\Lambda_{xx}(\omega, \mathbf{k})}. \tag{10}$$

In the above equations, the subscript s refers to the 2 dimensional yz subspace of the surface, and $\hat{\mathbf{x}}_n$ is the unit vector normal to the surface and pointing in the positive x direction. Hence,

$$\mathbf{\Lambda}_s(\omega, \mathbf{k}) = \begin{pmatrix} \Lambda_{yy} & \Lambda_{yz} \\ \Lambda_{zy} & \Lambda_{zz} \end{pmatrix}, \tag{11}$$

and

$$[\mathbf{\Lambda}(\omega, \mathbf{k}) \cdot \hat{\mathbf{x}}_n]_s = \begin{pmatrix} \Lambda_{yx} \\ \Lambda_{zx} \end{pmatrix}, \quad [\hat{\mathbf{x}}_n \cdot \mathbf{\Lambda}(\omega, \mathbf{k})]_s = (\Lambda_{xy}, \quad \Lambda_{xz}). \tag{12}$$

The surface component of the plasma electric field is written down from (9) in the convenient form

$$\mathbf{E}_s(\omega, \mathbf{k}) = -i\left(\frac{\mu_0 c^2}{\omega}\right)\mathbf{Q}_s(\omega, \mathbf{k})\mathbf{M}_s(\omega, \mathbf{k}_s), \tag{13}$$

where we introduce the surface field propagator

$$\mathbf{Q}_s(\omega, \mathbf{k}) = \frac{\gamma_s(\omega, \mathbf{k})}{\Gamma_s(\omega, \mathbf{k})}, \tag{14}$$

with $\gamma_s(\omega, \mathbf{k})$ and $\Gamma_s(\omega, \mathbf{k})$ the matrix of cofactors and the determinant of $\mathbf{\Gamma}_s(\omega, \mathbf{k})$ respectively.

The x dependence of the surface electric field (which is required for the application of the boundary conditions) is found by inverting the Fourier transform in (13) over k_x. We have

$$\mathbf{E}_s(\omega, x, \mathbf{k}_s) = -i\mu_0 c n_s \mathbf{Q}_s(\omega, x, \mathbf{k}_s)\mathbf{M}_s(\omega, \mathbf{k}_s), \tag{15}$$

where (:= denotes a definition)

$$\mathbf{Q}_s(\omega, x, \mathbf{k}_s) := \frac{1}{k_s}\int \frac{dk_x}{2\pi} e^{ik_x x}\mathbf{Q}_s(\omega, \mathbf{k}) \tag{16}$$

is defined to be dimensionless and $n_s := ck_s/\omega$ is the surface refractive index with $k_s := |\mathbf{k}_s|$.

The integral in (16) can generally be evaluated using contour integration. For the physical plasma fields ($x < 0$) the contour must be closed in the lower half of the complex k_x plane, while for the image fields ($x > 0$) the contour must be closed in the upper half plane. This ensures that the contribution from the semi-circle of the contour vanishes in each case, and that the integral can be written as a sum of residues of the poles in the integrand. Noting that

$$\Gamma_s(\omega, \mathbf{k}) = \frac{\Lambda(\omega, \mathbf{k})}{\Lambda_{xx}(\omega, \mathbf{k})}, \tag{17}$$

the poles of the surface field propagator $\mathbf{Q}_s(\omega, \mathbf{k})$ are immediately identified with the zeros of equation (6), except in this context solutions of the form $k_x = k_{xM}(\omega, \mathbf{k}_s)$ are sought. In this sense the total electric field of the surface wave is said to be determined by the sum of contributions from each of the bulk wave modes of the plasma.

Only the poles in the lower half of the complex plane generally contribute to the physical plasma fields (while poles in the upper half plane contribute to the image plasma fields) because of the way in which the contour of integration is closed. As a result, the wave fields spatially decay into the physical (and image) plasma, which is consistent with the definition of a surface wave. It is stressed that (15) applies only to the real plasma fields ($x < 0$) or the image plasma fields ($x > 0$). The vacuum fields of the physical system take the standard form as in Rowe (1991), and we will not reproduce them here.

Applying the boundary conditions (in the physical plasma–vacuum system) of continuity of the surface electric and magnetic fields at $x = 0$ yields an equation for $\mathbf{M}_s(\omega, \mathbf{k}_s)$ of the form

$$\mathbf{Z}_s(\omega, \mathbf{k}_s)\mathbf{M}_s(\omega, \mathbf{k}_s) = 0, \tag{18}$$

where the 2×2 matrix $\mathbf{Z}_s(\omega, \mathbf{k}_s)$ may be regarded as the response tensor of the plasma–vacuum system. The response tensor is given explicitly by

$$\mathbf{Z}_s(\omega, \mathbf{k}_s) = \int \frac{dr_x}{2\pi} \mathbf{\Delta}_s(\omega, \mathbf{k})\mathbf{Q}_s(\omega, \mathbf{k}), \tag{19}$$

where

$$\int \frac{dr_x}{2\pi} := \lim_{x \to 0^-} \int \frac{dr_x}{2\pi} e^{ik_x x} \tag{20}$$

and $r_x := k_x/k_s$. The kernel matrix $\mathbf{\Delta}_s(\omega, \mathbf{k})$ is related to the bulk plasma response through

$$\mathbf{\Delta}_s(\omega, \mathbf{k}) = \mathbf{r}_s\mathbf{r}_s + r_x^v(r_x^v - r_x)\delta - r_x^v\mathbf{r}_s \frac{[\hat{\mathbf{x}}_n \cdot \mathbf{\Lambda}(\omega, \mathbf{k})]_s}{\Lambda_{xx}(\omega, \mathbf{k})}, \tag{21}$$

where $\mathbf{r}_s := \mathbf{k}_s/k_s$ and $r_x^v := k_x^v/k_s$ are the dimensionless surface wavevector and vacuum wavenumber respectively. The vacuum wavenumber is given in terms of the surface refractive index n_s by

$$r_x^v = \sqrt{\frac{1 - n_s^2}{n_s^2}}, \tag{22}$$

where the sign of the square root is chosen so that $\operatorname{Im} r_x^v > 0$ and the wave fields decay into the vacuum ($x > 0$).

Trivial solutions to (18) correspond to bulk wave modes which are not coupled to the surface while the non-trivial solutions correspond to surface waves. The condition for non-trivial solutions is in analogy to (6)

$$Z_s(\omega, \mathbf{k}_s) = 0\,, \tag{23}$$

where $Z_s(\omega, \mathbf{k}_s)$ is the determinant of $\mathbf{Z}_s(\omega, \mathbf{k}_s)$, and the zeros of this equation yield the surface mode dispersion relations $\omega = \omega_s(\mathbf{k}_s)$.

3. Extension of Formalism

It is useful to further develop the formalism of the general theory of surface waves given by Rowe (1991) and reviewed in Section 2. These extensions are needed for Sections 5 and 6.

(a) Evaluation of the Dispersion Relation

In order to determine the dispersion relation for specific surface wave modes one needs to evaluate the response tensor of the plasma–vacuum system. In general, as was the case for the integral in (16), the response tensor $\mathbf{Z}_s(\omega, \mathbf{k}_s)$ can be evaluated via contour integration. The result may thus be written as a sum

$$\mathbf{Z}_s(\omega, \mathbf{k}_s) = \sum_M \mathbf{Z}_{sM}(\omega, \mathbf{k}_s)\,, \tag{24}$$

over all contributions (labelled with M) corresponding to the poles $r_x = r_M$ (or $k_x = k_M$) of the integrand (note that we now drop the subscript x from r_{xM} and k_{xM} for brevity). The poles are identified as those of the surface field propagator $\mathbf{Q}_s(\omega, \mathbf{k})$, which we discussed in the previous section [the kernel matrix $\boldsymbol{\Delta}_s(\omega, \mathbf{k})$ generally has no poles of its own], and the contributions $\mathbf{Z}_{sM}(\omega, \mathbf{k}_s)$ regarded as contributions due to the bulk modes of the plasma. As was the case when we considered the x dependence of the physical plasma electric fields, the contour of integration must be closed in the lower half of the complex plane and only modes which spatially decay into the physical plasma (poles with $\operatorname{Im} r_M < 0$) contribute to the response of the system.

In identifying the contributions it is useful to adopt the convention

$$\mathbf{Z}_{sM}(\omega, \mathbf{k}_s) := \boldsymbol{\Delta}_{sM}(\omega, \mathbf{k}_s)\mathbf{Q}_{sM}(\omega, \mathbf{k}_s) \tag{25}$$

with

$$\boldsymbol{\Delta}_{sM}(\omega, \mathbf{k}_s) := \boldsymbol{\Delta}_s(\omega, k_M, \mathbf{k}_s)\,, \tag{26}$$

$$\mathbf{Q}_{sM}(\omega, \mathbf{k}_s) := -i \lim_{r_x \to r_M} [(r_x - r_M)\mathbf{Q}_s(\omega, \mathbf{k})]\,. \tag{27}$$

The dimensionless matrices $\mathbf{Q}_{sM}(\omega, \mathbf{k}_s)$ are then related to the inverse Fourier transform of $\mathbf{Q}_s(\omega, \mathbf{k})$ which appeared in (16) by

$$\mathbf{Q}_s(\omega, x, \mathbf{k}_s) = \sum_M e^{ik_M x}\mathbf{Q}_{sM}(\omega, \mathbf{k}_s)\,, \tag{28}$$

which indicates that the definition of (27) is a natural one.

For a magnetised plasma (and anisotropic media in general) the matrices $\mathbf{Q}_{sM}(\omega, \mathbf{k}_s)$ have the useful property

$$Q_{sM}(\omega, \mathbf{k}_s) \equiv 0\,, \tag{29}$$

that is, their determinants vanish identically. This is related to the fact that the bulk wave modes are distinct (non–degenerate), that is, $r_M \neq r_N$ for $M \neq N$, and is proven formally in the Appendix. An immediate consequence of (29) is that the determinants $Z_{sM}(\omega, \mathbf{k}_s) \equiv 0$, and the dispersion relation (23) may be written in the simplified form (Tr[ ] denotes trace)

$$Z_s(\omega, \mathbf{k}_s) = \sum_{M<N} \mathrm{Tr}[\mathbf{Z}_{sM}(\omega, \mathbf{k}_s)\zeta_{sN}(\omega, \mathbf{k}_s)] = 0\,, \tag{30}$$

where $M,\ N = 1, 2, 3, ...$ label the contributing bulk modes and $\zeta_{sN}(\omega, \mathbf{k}_s)$ is the matrix of cofactors of $\mathbf{Z}_{sN}(\omega, \mathbf{k}_s)$. This is derived from the more general result

$$\mathrm{Det}\left[\sum_i \mathbf{A}_i\right] = \sum_i \mathrm{Det}[\mathbf{A}_i] + \sum_{i<j} \mathrm{Tr}[\mathbf{A}_i \mathbf{a}_j]\,, \tag{31}$$

where $\mathbf{A}_i$ is a 2×2 matrix and $\mathbf{a}_i$ is its matrix of cofactors.

It is noted here that (29) is not satisfied for an isotropic plasma due to the degeneracy of the transverse bulk wave mode. In this case a different simplification to (30) applies as discussed by Rowe (1991).

(b) Determination of the Wave Fields

In determining the field properties of the surface waves we assume that we can ignore the antihermitian part of the response tensor $\mathbf{Z}_s(\omega, \mathbf{k}_s)$. This corresponds to the assumption that $\mathbf{\Lambda}(\omega, \mathbf{k})$ is hermitian in bulk wave theory. Implicit in this approach is that the surface waves are weakly damped ($\gamma \ll \omega_s$, where γ is the damping rate of the waves) as it is the antihermitian part which determines the dissipative part of the response of the system.

Consider equation (18). Formally, one can write down a solution for $\mathbf{M}_s(\omega, \mathbf{k}_s)$ in the form

$$\mathbf{M}_s(\omega, \mathbf{k}_s) := M_s(\mathbf{k}_s)\mathbf{m}_s(\mathbf{k}_s)2\pi\delta(\omega - \omega_s(\mathbf{k}_s))\,, \tag{32}$$

where $M_s(\mathbf{k}_s)$ is the amplitude of the field and $\mathbf{m}_s(\mathbf{k}_s)$ is a unimodular vector ($*$ denotes complex conjugation)

$$\mathbf{m}_s(\mathbf{k}_s) \cdot \mathbf{m}_s^*(\mathbf{k}_s) = 1\,. \tag{33}$$

Negative frequency solutions are included in the definition (32) through the convention $\omega_s(-\mathbf{k}_s) = -\omega_s(\mathbf{k}_s)$ which, together with the conditions $M_s^*(\mathbf{k}_s) = M_s(-\mathbf{k}_s)$ and $\mathbf{m}_s^*(\mathbf{k}_s) = \mathbf{m}_s(-\mathbf{k}_s)$, ensures that the reality condition on the Fourier transform is satisfied. The vector $\mathbf{m}_s(\mathbf{k}_s)$ can be constructed by noting that it must be proportional to both columns of $\boldsymbol{\zeta}_s(\omega_s(\mathbf{k}_s), \mathbf{k}_s)$, the matrix of cofactors of $\mathbf{Z}_s(\omega_s(\mathbf{k}_s), \mathbf{k}_s)$. This is expressed in a formal sense by the equation

$$\boldsymbol{\zeta}_s(\omega_s(\mathbf{k}_s), \mathbf{k}_s) = \mathrm{Tr}[\boldsymbol{\zeta}_s(\omega_s(\mathbf{k}_s), \mathbf{k}_s)]\mathbf{m}_s(\mathbf{k}_s)\mathbf{m}_s^*(\mathbf{k}_s)\,. \tag{34}$$

The surface electric field of the wave in the physical plasma ($x < 0$) can be written down from (15) with (28) in the form

$$\mathbf{E}_s(\omega, x, \mathbf{k}_s) = \sum_M e^{ik_M x}\mathbf{E}_{sM}(\omega, \mathbf{k}_s)\,, \tag{35}$$

where

$$\mathbf{E}_{sM}(\omega, \mathbf{k}_s) = -i\mu_0 c n_s \mathbf{Q}_{sM}(\omega, \mathbf{k}_s)\mathbf{M}_s(\omega, \mathbf{k}_s) \tag{36}$$

is the electric field of mode M in the plasma at the surface ($x = 0^-$). Using (32) we define

$$\mathbf{E}_{sM}(\omega, \mathbf{k}_s) := E_{sM}(\mathbf{k}_s)\mathbf{e}_{sM}(\mathbf{k}_s)2\pi\delta(\omega - \omega_s(\mathbf{k}_s))\,, \tag{37}$$

with

$$E_{sM}(\mathbf{k}_s)\mathbf{e}_{sM}(\mathbf{k}_s) = -i\mu_0 c n_s M_s(\mathbf{k}_s)\mathbf{Q}_{sM}(\omega_s(\mathbf{k}_s), \mathbf{k}_s)\mathbf{m}_s(\mathbf{k}_s)\,, \tag{38}$$

and where for each mode $E_{sM}(\mathbf{k}_s)$ is the amplitude of the field (at $x = 0^-$) and $\mathbf{e}_{sM}(\mathbf{k}_s)$ is the unimodular surface-polarisation vector. The conditions for reality are analogous to those for $\mathbf{M}_s(\omega, \mathbf{k}_s)$ except that from (35) we see that we also require the condition $k_M^*(\omega_s(\mathbf{k}_s), \mathbf{k}_s) = -k_M(\omega_s(-\mathbf{k}_s), -\mathbf{k}_s)$.

In principle, equation (38) can be used to determine $\mathbf{e}_{sM}(\mathbf{k}_s)$ given $\mathbf{m}_s(\mathbf{k}_s)$. We can calculate $\mathbf{e}_{sM}(\mathbf{k}_s)$ in an alternative and more direct way by noting that (29) implies that it must be proportional to both columns of $\mathbf{Q}_{sM}(\omega_s(\mathbf{k}_s), \mathbf{k}_s)$. In analogy to (34) this may be written as

$$\mathbf{Q}_{sM}(\omega_s(\mathbf{k}_s), \mathbf{k}_s) = \mathrm{Tr}[\mathbf{Q}_{sM}(\omega_s(\mathbf{k}_s), \mathbf{k}_s)]\mathbf{e}_{sM}(\mathbf{k}_s)\mathbf{e}_{sM}^*(\mathbf{k}_s)\,. \tag{39}$$

We may then identify from (38) the field amplitude

$$E_{sM}(\mathbf{k}_s) = -i\mu_0 c n_s M_s(\mathbf{k}_s)\mathrm{Tr}[\mathbf{Q}_{sM}(\omega_s(\mathbf{k}_s), \mathbf{k}_s)]\mathbf{e}_{sM}^*(\mathbf{k}_s)\cdot\mathbf{m}_s(\mathbf{k}_s)\,. \tag{40}$$

This expression may be used to calculate the ratio of the amplitudes of any two given modes M and N

$$\frac{E_{sM}(\mathbf{k}_s)}{E_{sN}(\mathbf{k}_s)} = \frac{\mathrm{Tr}[\mathbf{Q}_{sM}(\omega_s(\mathbf{k}_s), \mathbf{k}_s)]\mathbf{e}_{sM}^*(\mathbf{k}_s)\cdot\mathbf{m}_s(\mathbf{k}_s)}{\mathrm{Tr}[\mathbf{Q}_{sN}(\omega_s(\mathbf{k}_s), \mathbf{k}_s)]\mathbf{e}_{sN}^*(\mathbf{k}_s)\cdot\mathbf{m}_s(\mathbf{k}_s)}\,. \tag{41}$$

In this theory, the x component of the electric field is treated as a subsidiary field as is the magnetic field. It is, however, relevant to note the following results. The x component for a particular mode is (from equation 8)

$$E_{xM}(\mathbf{k}_s) = -E_{sM}(\mathbf{k}_s)\frac{[\hat{\mathbf{x}}_n\cdot\boldsymbol{\Lambda}_M(\omega_s(\mathbf{k}_s), \mathbf{k}_s)]_s}{\Lambda_{xxM}(\omega_s(\mathbf{k}_s), \mathbf{k}_s)}\cdot\mathbf{e}_{sM}(\mathbf{k}_s)\,, \tag{42}$$

where $\mathbf{\Lambda}_M(\omega, \mathbf{k}_s)$ and $\Lambda_{xxM}(\omega, \mathbf{k}_s)$ are defined in analogy with (26). The magnetic field of each mode can then be determined from Faraday's law

$$\mathbf{B}(\omega, \mathbf{k}) = \frac{\mathbf{k} \times \mathbf{E}(\omega, \mathbf{k})}{\omega}, \tag{43}$$

and in particular the surface component may be written in analogy with (35), with

$$\mathbf{B}_{sM}(\omega, \mathbf{k}_s) := B_{sM}(\mathbf{k}_s)\mathbf{b}_{sM}(\mathbf{k}_s)2\pi\delta(\omega - \omega_s(\mathbf{k}_s)), \tag{44}$$

and with amplitude $B_{sM}(\mathbf{k}_s)$ (at $x = 0^-$) and unimodular vector $\mathbf{b}_{sM}(\mathbf{k}_s)$. These results will be used in sections 5 and 6.

Finally we note that the surface components of the vacuum electric and magnetic fields can be calculated directly from the above results using the fact that the surface components of the electric and magnetic fields were assumed to be continuous at the boundary. The x components are then obtained from $\mathbf{k}^v \cdot \mathbf{B}^v = 0$ and the usual vacuum relation $\mathbf{k}^v \cdot \mathbf{E}^v = 0$, where the superscript v indicates the vacuum fields and wavenumbers.

4. Cold Magnetised Plasma

We now consider some general results for a cold magnetised plasma. These results have not previously appeared in the literature and will be used in Section 5.

Assuming that the ambient magnetic field $\mathbf{B}_0$ is directed along the z-axis (and is thus parallel to the surface) we have the well known dielectric tensor (Stix 1962; Melrose 1986)

$$\mathbf{K}(\omega) = \begin{pmatrix} S(\omega) & -iD(\omega) & 0 \\ iD(\omega) & S(\omega) & 0 \\ 0 & 0 & P(\omega) \end{pmatrix}, \tag{45}$$

where

$$\begin{aligned} S(\omega) &= 1 - \sum \frac{\omega_p^2}{(\omega^2 - \Omega^2)}, \\ D(\omega) &= \sum \frac{\epsilon\omega_p^2\Omega}{\omega(\omega^2 - \Omega^2)}, \\ P(\omega) &= 1 - \sum \frac{\omega_p^2}{\omega^2}, \end{aligned} \tag{46}$$

and the sums are over all species of particle with plasma and cyclotron frequencies

$$\omega_p = \sqrt{\frac{nq^2}{\varepsilon_0 m}}, \quad \Omega = \frac{|q|B_0}{m}. \tag{47}$$

Here, $\epsilon = q/|q|$ is the sign of charge q of a given species of mass m and number density n.

The elements of the surface field propagator $\mathbf{Q}_s(\omega, \mathbf{k})$ corresponding to this dielectric tensor may be written down from (14) in the form (cf. notation of 11)

$$
\begin{aligned}
Q_{yy}(\omega, \mathbf{k}) &= \frac{(S-n_s^2)(P-n_s^2 r_y^2)-n_s^2 r_x^2(S-n_s^2 r_y^2)}{\Lambda(\omega, \mathbf{k})}, \\
Q_{yz}(\omega, \mathbf{k}) &= -\frac{\{[(S-n_s^2)-n_s^2 r_x^2] r_y - iDr_x\} n_s^2 r_z}{\Lambda(\omega, \mathbf{k})}, \\
Q_{zy}(\omega, \mathbf{k}) &= -\frac{\{[(S-n_s^2)-n_s^2 r_x^2] r_y + iDr_x\} n_s^2 r_z}{\Lambda(\omega, \mathbf{k})}, \\
Q_{zz}(\omega, \mathbf{k}) &= \frac{(S-n_s^2 r_z^2)[(S-n_s^2)-n_s^2 r_x^2]-D^2}{\Lambda(\omega, \mathbf{k})},
\end{aligned}
\tag{48}
$$

where all quantities are as defined previously. The corresponding elements of the kernel matrix $\mathbf{\Delta}_s(\omega, \mathbf{k})$ are from (21)

$$
\begin{aligned}
\Delta_{yy}(\omega, \mathbf{k}) &= \frac{(1-n_s^2 r_z^2)(S-n_s^2)-r_x r_x^v n_s^2(S-n_s^2 r_z^2)+iDr_x^v r_y n_s^2}{n_s^2(S-n_s^2)}, \\
\Delta_{yz}(\omega, \mathbf{k}) &= \frac{[(S-n_s^2)-n_s^2 r_x r_x^v] r_y r_z}{(S-n_s^2)}, \\
\Delta_{zy}(\omega, \mathbf{k}) &= \frac{[(S-n_s^2)-n_s^2 r_x r_x^v] r_y r_z + iDr_x^v r_z}{(S-n_s^2)}, \\
\Delta_{zz}(\omega, \mathbf{k}) &= \frac{(1-n_s^2 r_y^2)(S-n_s^2)-r_x r_x^v n_s^2(S-n_s^2 r_y^2)}{n_s^2(S-n_s^2)}.
\end{aligned}
\tag{49}
$$

According to Sections $2b$ and $3a$ we need to identify the poles of $\mathbf{Q}_s(\omega, \mathbf{k})$, which we have already noted involves solving equation (6) for solutions of the form $r_x = r_{xM}(\omega, \mathbf{k}_s)$. With this in mind, we write the determinant $\Lambda(\omega, \mathbf{k})$ as

$$
\Lambda(\omega, \mathbf{k}) = A(\omega, \mathbf{k}_s) r_\perp^4 - B(\omega, \mathbf{k}_s) r_\perp^2 + C(\omega, \mathbf{k}_s), \tag{50}
$$

where $r_\perp^2 = r_x^2 + r_y^2$, and the coefficients

$$
\begin{aligned}
A(\omega, \mathbf{k}_s) &= S n_s^4, \\
B(\omega, \mathbf{k}_s) &= n_s^2[(P+S)(S-n_s^2 r_z^2)-D^2], \\
C(\omega, \mathbf{k}_s) &= P[(S-n_s^2 r_z^2)^2-D^2],
\end{aligned}
\tag{51}
$$

are independent of k_x. It is then convenient to write

$$
\Lambda(\omega, \mathbf{k}) = A(\omega, \mathbf{k}_s)(r_x^2 - r_+^2)(r_x^2 - r_-^2), \tag{52}
$$

where the zeros $r_\pm$ (corresponding to $M = \pm$) are given by

$$
r_\pm^2 = \frac{B \pm \sqrt{B^2 - 4AC}}{2A} - r_y^2. \tag{53}
$$

Equation (6) with (52) has four solutions for r_x, namely $\pm r_+$ and $\pm r_-$, where we may arbitrarily define $r_\pm$ so that Im $r_\pm < 0$. Thus, in a cold magnetised plasma with $\mathbf{B}_0$ parallel to the surface there are at most two bulk modes which may contribute to the surface wave fields and hence the response of the system. The other two poles are always in the upper half of the complex plane and so may only contribute to the image plasma fields. It is relevant to note that on the basis of this observation it is possible to show that the consistency conditions derived by Rowe (1991), and which apply to the image approach, are always satisfied for surface waves in a plasma with the dielectric tensor as given in (45). This vindicates the use of the general theory of surface waves for cold magnetised plasmas.

Finally we note that the general expressions for the matrix elements and the poles presented here are too complicated to enable us to derive a specific dispersion relation without first applying some approximation. In the following section we treat such a specific case.

5. Low Frequency Waves

(a) Approximate Dielectric Tensor

The general results of the previous section may be simplified considerably by considering low frequency waves in an electron–ion plasma. Specifically, we consider frequencies well below the electron cyclotron and plasma frequencies ($\omega \ll \Omega_e,\ \omega_{pe}$). We also make the reasonable assumption that the mass density of electrons is much smaller than that of the ions ($n_e m_e \ll n_i m_i$), which may be written in the convenient form

$$\mu := \Omega_i/\Omega_e \ll 1 \tag{54}$$

using charge neutrality ($n_e|q_e| = n_i|q_i|$), and we assume that the Alfvén refractive index $n_A := c/v_A$ (where v_A is the Alfvén velocity) satisfies

$$n_A \gg 1\,. \tag{55}$$

Both of these assumptions have wide ranging validity, with $\mu < 10^{-3}$ and $n_A > 100$ typical of most plasmas of interest (such as the solar corona).

The dielectric tensor elements of the previous section (equation 45) then simplify to

$$S \approx \frac{n_A^2}{(1-f^2)},\quad D \approx -\frac{n_A^2 f}{(1-f^2)},\quad P \approx -\frac{n_A^2}{\mu f^2}\,, \tag{56}$$

where $f := \omega/\Omega_i$ is the normalised frequency and we have retained only dominant terms in both μ and n_A. The bulk wave modes which propagate in a plasma with this dielectric tensor are well known. They are the fast magneto-sound (FMS) and Alfvén (A) modes, also known as the compressional and shear Alfvén modes respectively. Note also that from the form of P we can deduce that (54) amounts to the assumption that the plasma is highly conducting. This may be seen from the fact that $\mu \ll 1$ implies that $|P|$ is very large, which in turn implies that the zz element of the conductivity tensor $\boldsymbol{\sigma}(\omega, \mathbf{k})$ is large in magnitude. This is distinct from the infinite conductivity case $\mu \equiv 0$, which we consider in Section 6.

(b) Simplification of Poles

Before we can approximate the general forms of $\mathbf{Q}_s(\omega, \mathbf{k})$ and $\mathbf{\Delta}_s(\omega, \mathbf{k})$ obtained in Section 4 we must first consider the approximate forms of the poles defined by (53). Retaining only dominant terms in μ (noting that $P \gg S$, D for $f \not\approx 1$) we may identify

$$r_+^2 \approx \frac{P(S - n_s^2 r_z^2)}{S n_s^2},$$
$$r_-^2 \approx \left(\frac{S - n_s^2}{n_s^2}\right) - \frac{D^2}{n_s^2(S - n_s^2 r_z^2)}, \tag{57}$$

assuming $(S - n_s^2 r_z^2) \not\approx 0$, i.e. that the frequency of the wave is far from the generalised Alfvén resonance frequency (Cramer and Donnelly 1992)

$$\omega^2 = v_A^2 k_z^2 / (1 + v_A^2 k_z^2 / \Omega_i^2)\,. \tag{58}$$

Note that these approximations are in fact still valid for $f \approx 1$ even though S and D may then be comparable in size with P.

The two modes here labelled simply $+$ and $-$ have been given the generic names short wavelength or quasi-electrostatic (QEW) and magnetohydrodynamic (MHD) modes respectively by Cramer and Donnelly (1992). In the very low frequency limit ($f \ll 1$) the latter may be identified more specifically as the FMS mode as then $D \approx 0$ and from (57)

$$r_-^2 \approx \left(\frac{S - n_s^2}{n_s^2}\right), \tag{59}$$

which corresponds to $\omega^2(k) = v_A^2 k^2$ (where $k^2 = k_s^2 + k_-^2$), i.e. the dispersion relation for FMS bulk waves. The $+$ mode is more tightly bound to the surface of the plasma than is the $-$ mode as indicated by the fact that $|r_+| \gg |r_-|$. The smaller skin depth as seen by the $+$ mode is a consequence of the high conductivity of the plasma. In the infinite conductivity case, considered latter, this leads to the presence of a surface current.

(c) Response Tensor

Having simplified the poles of the surface field propagator we now evaluate the response tensor of the plasma–vacuum system according to Section 3*a*. Note that in general the contributions (as defined by equation 25) may be expanded in powers of $\sqrt{\mu}$ and that the approximations of (56) and (57) are valid to $O(\sqrt{\mu})$. In this paper, however, only the zeroth order terms $O(1)$ will be retained. This corresponds to calculating the zeroth order dispersion relation [the first order correction may be subsequently obtained by retaining the $O(\sqrt{\mu})$ terms]. In the following it proves useful to introduce the notation $F = \mu P$ and $\bar{r}_+ = \sqrt{\mu} r_+$ so that all μ dependence is explicit.

For the $-$ mode we have the surface field propagator (from equations 27 and 28)

$$\mathbf{Q}_{s-}(\omega, \mathbf{k}_s) = \frac{iF(S - n_s^2)}{2 S n_s^4 r_- \bar{r}_+^2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + O(\mu) \tag{60}$$

and the kernel matrix (from equations 26 and 49)

$$\mathbf{\Delta}_{s-}(\omega,\mathbf{k}_s)=\begin{pmatrix}\Delta_{yy-} & \Delta_{yz-}\\ \Delta_{zy-} & \Delta_{zz-}\end{pmatrix}+O(\mu)\,, \tag{61}$$

where in view of the form of $\mathbf{Q}_{s-}(\omega,\mathbf{k}_s)$ we need only identify the elements

$$\begin{aligned}\Delta_{yy-} &= \frac{(1-n_s^2r_z^2)(S-n_s^2)-r_-r_x^vn_s^2(S-n_s^2r_z^2)+iDr_x^vr_yn_s^2}{n_s^2(S-n_s^2)}\,,\\ \Delta_{zy-} &= \frac{[(S-n_s^2)-n_s^2r_-r_x^v]r_yr_z+iDr_x^vr_z}{(S-n_s^2)}\,,\end{aligned} \tag{62}$$

in writing down the result

$$\mathbf{Z}_{s-}(\omega,\mathbf{k}_s)=\frac{iF(S-n_s^2)}{2Sn_s^4r_-\bar{r}_+^2}\begin{pmatrix}\Delta_{yy-} & 0\\ \Delta_{zy-} & 0\end{pmatrix}+O(\mu)\,. \tag{63}$$

For the + mode the corresponding expansions are best written in the form

$$\frac{\mathbf{Q}_{s+}(\omega,\mathbf{k}_s)}{\sqrt{\mu}}=$$

$$\frac{iF}{2S^2n_s^4\bar{r}_+^3}\begin{pmatrix}n_s^4r_y^2r_z^2 & -n_s^2(S-n_s^2r_z^2)r_yr_z\\ -n_s^2(S-n_s^2r_z^2)r_yr_z & (S-n_s^2r_z^2)^2\end{pmatrix}+O(\sqrt{\mu})\,, \tag{64}$$

$$\sqrt{\mu}\mathbf{\Delta}_{s+}(\omega,\mathbf{k}_s)=-\frac{\bar{r}_+r_x^v}{(S-n_s^2)}\begin{pmatrix}(S-n_s^2r_z^2) & n_s^2r_yr_z\\ n_s^2r_yr_z & (S-n_s^2r_y^2)\end{pmatrix}+O(\sqrt{\mu})\,, \tag{65}$$

and give

$$\mathbf{Z}_{s+}(\omega,\mathbf{k}_s)=\frac{iFr_x^v}{2Sn_s^4\bar{r}_+^2}\begin{pmatrix}0 & 0\\ n_s^2r_yr_z & -(S-n_s^2r_z^2)\end{pmatrix}+O(\sqrt{\mu})\,. \tag{66}$$

We note that both modes give a finite contribution of $O(1)$ to the total response tensor of the system, and that the determinants $Q_{s\pm}(\omega,\mathbf{k}_s)$ [and therefore $Z_{s\pm}(\omega,\mathbf{k}_s)$] are identically zero to lowest order as they must be to be consistent with (29).

(d) Dispersion Relation

The determinant of the response tensor is, according to (30) and using (57) to eliminate $\bar{r}_+$,

$$\begin{aligned}Z_s(\omega,\mathbf{k}_s) &= \left[\frac{(S-n_s^2)r_x^v}{4n_s^4(S-n_s^2r_z^2)r_-}\right]\Delta_{yy-} \\ &= \frac{r_x^v}{4n_s^6(S-n_s^2r_z^2)r_-}[(S-n_s^2)(1-n_s^2r_z^2)-r_x^vr_-n_s^2(S-n_s^2r_z^2)+iDn_s^2r_x^vr_y]\,,\end{aligned} \tag{67}$$

and the dispersion relation $Z_s(\omega, \mathbf{k}_s) = 0$ becomes

$$(S - n_s^2)(1 - n_s^2 r_z^2) - r_x^v r_- n_s^2 (S - n_s^2 r_z^2) + iDn_s^2 r_x^v r_y = 0 . \qquad (68)$$

Up to this point displacement current effects have been retained through the use of the exact expression for the vacuum wavenumber r_x^v as in (22). Noting that a solution of (68) will satisfy $n_s \sim n_A$ ($\gg$1 by equation 55) we may ignore these effects in the first instance by taking $r_x^v = +i$ (and $n_s^2 r_z^2 \gg 1$) in (68). Then the dispersion relation reduces to

$$(S - n_s^2) r_z^2 + Dr_y + ir_-(S - n_s^2 r_z^2) = 0 , \qquad (69)$$

which can be solved for analytic solutions to n_s^2. After further manipulation one obtains the positive frequency solutions

$$f = \alpha |r_z| [(\alpha^2 r_y^2 r_z^2 + 2 - r_z^2)^{1/2} + \alpha r_y |r_z|] , \qquad (70)$$

where we have defined the dimensionless parameter $\alpha := v_A k_s / \Omega_i$. Equation (70) is the solution of Cramer and Donnelly (1983), apart from notational differences, and is discussed in detail therein.

(e) Wave Fields

The electric fields of the two modes in the plasma may be determined according to Section 3*b*. As was pointed out, the surface-polarisation vectors may be identified directly from the columns of $Q_{s\pm}(\omega, \mathbf{k}_s)$ as given in (60) and (64). Thus, we have to $O(1)$

$$\begin{aligned} \mathbf{e}_{s-}(\mathbf{k}_s) &= (1, 0) = \hat{\mathbf{y}} , \\ \mathbf{e}_{s+}(\mathbf{k}_s) &= \frac{(-n_s^2 r_y r_z, (S - n_s^2 r_z^2))}{\sqrt{(S - n_s^2 r_z^2)^2 + n_s^4 r_y^2 r_z^2}} , \end{aligned} \qquad (71)$$

where $\hat{\mathbf{y}}$ is the unit vector in the positive y direction and it is understood that these expressions are evaluated at $\omega = \omega_s(\mathbf{k}_s)$. The ratio of the amplitudes of the two modes may then be determined from (41), which involves writing down a solution for the vector $\mathbf{M}_s(\omega, \mathbf{k}_s)$, or alternatively may be determined by writing

$$\boldsymbol{\Delta}_{s+}(\omega, \mathbf{k}_s) \mathbf{E}_{s+}(\omega, \mathbf{k}_s) + \boldsymbol{\Delta}_{s-}(\omega, \mathbf{k}_s) \mathbf{E}_{s-}(\omega, \mathbf{k}_s) = 0 , \qquad (72)$$

using (18) with (24), (25) and (36). Using (37) we write this in the form

$$E_{s+}(\mathbf{k}_s) \boldsymbol{\Delta}_{s+}(\omega_s(\mathbf{k}_s), \mathbf{k}_s) \mathbf{e}_{s+}(\mathbf{k}_s) + E_{s-}(\mathbf{k}_s) \boldsymbol{\Delta}_{s-}(\omega_s(\mathbf{k}_s), \mathbf{k}_s) \mathbf{e}_{s-}(\mathbf{k}_s) = 0 , \qquad (73)$$

and noting that the y component of this equation is identically zero [to $O(1)$], we take the z component to obtain the ratio of the amplitudes

$$\frac{E_{s-}(\mathbf{k}_s)}{E_{s+}(\mathbf{k}_s)} = - \left[\frac{\hat{\mathbf{z}} \cdot \boldsymbol{\Delta}_{s+}(\omega_s(\mathbf{k}_s), \mathbf{k}_s) \cdot \mathbf{e}_{s+}(\mathbf{k}_s)}{\hat{\mathbf{z}} \cdot \boldsymbol{\Delta}_{s-}(\omega_s(\mathbf{k}_s), \mathbf{k}_s) \cdot \mathbf{e}_{s-}(\mathbf{k}_s)} \right] , \qquad (74)$$

where $\hat{\mathbf{z}}$ is the unit vector in the positive z direction. Evaluating this expression using the results of (61), (65) and (71) we find

$$\frac{E_{s-}(\mathbf{k}_s)}{E_{s+}(\mathbf{k}_s)} = \frac{r_+ r_x^v S}{\Delta_{zy-}\sqrt{(S-n_s^2r_z^2)^2+n_s^4r_y^2r_z^2}}\,, \tag{75}$$

where we again note that the right side is to be evaluated at $\omega=\omega_s(\mathbf{k}_s)$ and where Δ_{zy-} as given by (62) may be simplified to $\Delta_{zy-}=r_x^v(r_- - r_x^v)r_z/r_y$.

The corresponding results for the magnetic fields of the two modes are to $O(1)$

$$\begin{aligned}\mathbf{b}_{s-}(\mathbf{k}_s) &= \frac{(-n_s^2r_yr_z+iDr_z/r_-,(S-n_s^2r_z^2)-iDr_y/r_-)}{\sqrt{(Sr_y-iD/r_-)^2+(S-n_s^2)^2r_z^2}}\,,\\ \mathbf{b}_{s+}(\mathbf{k}_s) &= (1,0)=\hat{\mathbf{y}}\,,\end{aligned} \tag{76}$$

with the ratio of the magnetic field amplitudes given by

$$\frac{B_{s-}(\mathbf{k}_s)}{B_{s+}(\mathbf{k}_s)} = \frac{r_-r_y\sqrt{(Sr_y-iD/r_-)^2+(S-n_s^2)^2r_z^2}}{(r_x^v-r_-)(S-n_s^2)r_z}\,. \tag{77}$$

Equation (75) indicates that in the high conductivity limit (54) the surface component of the electric field of the $-$ mode dominates that of the $+$ mode [as $|r_+|\sim O(1/\sqrt{\mu})$], and the surface electric field is thus predominantly in the $\hat{\mathbf{y}}$ direction. In the case of the magnetic fields, however, (77) indicates that both modes are of comparable amplitude, and it is this which ensures the continuity of the surface magnetic field across the interface, as was assumed in the derivation of (18).

It is important to note that as the zeroth order dispersion relation (68) is independent of μ it may also be derived by replacing the inequality of (54) with the equality $\mu\equiv 0$. This corresponds to assuming that the plasma is infinitely conducting (rather than just highly conducting), as then $|P|\equiv\infty$ in (56), and is equivalent to the assumption used by Cramer and Donnelly (1983) that the electron mass can be neglected ($m_e\equiv 0$). We stress that this is only an idealisation which works for the $O(1)$ results but which is not valid when one is interested in the higher order terms.

In the $\mu\equiv 0$ case only the $-$ mode fields contribute to the plasma fields while the $+$ mode fields are effectively expelled from the plasma. This may be seen from the fact that when $\mu\equiv 0$, $r_+, k_+\equiv -i\infty$ and the exponential factor e^{ik_+x} associated with the $+$ mode wave fields vanishes for all $x\le 0^-$. In physical terms, the penetration depth of the plasma as seen by the $+$ mode becomes vanishingly small.

The removal of the $+$ mode electric field from the plasma means that in the infinite conductivity case the z component of the electric field vanishes (see equation 71). The removal of the $+$ mode magnetic field from the plasma gives rise to a discontinuity in the y component of the magnetic field across the surface layer, as can be seen from the fact that $\mathbf{b}_{s+}(\mathbf{k}_s)$ for the high conductivity case was directed in the $\hat{\mathbf{y}}$ direction. This discontinuity is associated with the excitation of a surface current along the z axis. These observations have been made by Cramer and Donnelly (1983) in their discussion of the plasma fields in the infinite conductivity case.

The theory developed by Rowe (1991) and summarised in Section 2 of this paper is inappropriate in the idealised infinite conductivity case as the surface magnetic field was assumed to be continuous from the outset. This condition may, however, be relaxed and it is of some formal interest to consider how the transition to infinite conductivity may be effected in this general theory. For completeness, this is considered in the following section.

6. Transition to Infinite Conductivity

We begin this discussion by considering the origin of the surface current which appears in the infinite conductivity case. Consider the induced current in the plasma which is given by (2). Using (8) to eliminate $E_x(\omega, \mathbf{k})$ the surface component may be written in the form

$$\mathbf{J}_{s\mathrm{ind}}(\omega, \mathbf{k}) = \mathbf{\Gamma}_s^\sigma(\omega, \mathbf{k})\mathbf{E}_s(\omega, \mathbf{k}), \tag{78}$$

where

$$\mathbf{\Gamma}_s^\sigma(\omega, \mathbf{k}) = \boldsymbol{\sigma}_s(\omega, \mathbf{k}) - \frac{[\boldsymbol{\sigma}(\omega, \mathbf{k}) \cdot \hat{\mathbf{x}}_n]_s[\hat{\mathbf{x}}_n \cdot \mathbf{\Lambda}(\omega, \mathbf{k})]_s}{\Lambda_{xx}(\omega, \mathbf{k})} \tag{79}$$

is a 2×2 matrix in analogy with $\mathbf{\Gamma}_s(\omega, \mathbf{k})$ of Section 2. Inverting the Fourier transform of (78) over k_x as in (15) and (28) we have (for $x < 0$)

$$\mathbf{J}_{s\mathrm{ind}}(\omega, x, \mathbf{k}_s) = \sum_M e^{ik_M x}\mathbf{\Gamma}_{sM}^\sigma(\omega, \mathbf{k}_s)\mathbf{E}_{sM}(\omega, \mathbf{k}_s), \tag{80}$$

where $\mathbf{\Gamma}_{sM}^\sigma(\omega, \mathbf{k}_s)$ is defined in analogy with (26) and $\mathbf{E}_{sM}(\omega, \mathbf{k}_s)$ is to be interpreted as in (37). We will only be interested in the + mode contribution to the current, which may be written as (using equation 37)

$$\mathbf{J}_{s\mathrm{ind}+}(\omega, x, \mathbf{k}_s) = \mathbf{J}_{s\mathrm{ind}+}(x, \mathbf{k}_s)2\pi\delta(\omega - \omega_s(\mathbf{k}_s)), \tag{81}$$

where

$$\mathbf{J}_{s\mathrm{ind}+}(x, \mathbf{k}_s) = E_{s+}(\mathbf{k}_s)e^{ik_+x}\mathbf{\Gamma}_{s+}^\sigma(\omega_s(\mathbf{k}_s), \mathbf{k}_s)\mathbf{e}_{s+}(\mathbf{k}_s). \tag{82}$$

For the conductivity tensor corresponding to $\mathbf{K}(\omega)$ of Section 4 we have

$$\mathbf{\Gamma}_s^\sigma(\omega, \mathbf{k}) = i\omega\varepsilon_0 \begin{pmatrix} (1-S) + \dfrac{iD(n_s^2 r_x r_y - iD)}{(S - n_s^2)} & \dfrac{iDn_s^2 r_x r_z}{(S - n_s^2)} \\ 0 & -P \end{pmatrix}, \tag{83}$$

so that in the high conductivity limit (retaining dominant terms in μ only)

$$\mathbf{\Gamma}_{s+}^\sigma(\omega, \mathbf{k}_s) \approx i\omega\varepsilon_0 \begin{pmatrix} \dfrac{iDn_s^2 r_+ r_y}{(S - n_s^2)} & \dfrac{iDn_s^2 r_+ r_z}{(S - n_s^2)} \\ 0 & 1-P \end{pmatrix}. \tag{84}$$

Using (71) to identify $\mathbf{e}_{s+}(\mathbf{k}_s)$ we obtain

$$\mathbf{J}_{s\mathrm{ind}+}(x, \mathbf{k}_s) = \frac{i\omega\varepsilon_0 E_{s+}(\mathbf{k}_s)e^{ik_+x}}{\sqrt{(S - n_s^2 r_z^2)^2 + n_s^4 r_y^2 r_z^2}} \begin{pmatrix} iDn_s^2 r_+ r_z \\ -P(S - n_s^2 r_z^2) \end{pmatrix}, \tag{85}$$

which can be written in terms of the $-$ mode amplitude $E_{s-}(\mathbf{k}_s)$ using (57) and (75)

$$\mathbf{J}_{s\mathrm{ind}+}(x,\mathbf{k}_s) = -\left(\frac{\varepsilon_0 c n_s \Delta_{zy-}}{r_x^v}\right) E_{s-}(\mathbf{k}_s) i k_+ e^{ik_+ x} \begin{pmatrix} O(\sqrt{\mu}) \\ 1 \end{pmatrix} . \tag{86}$$

As $|k_+| \sim O(1/\sqrt{\mu})$, the induced current associated with the $+$ mode is large and is confined to the edge of the plasma. Specifically, the skin depth of the plasma as seen by the $+$ mode is $d_{s+} = 1/|k_+| \sim O(\sqrt{\mu})$. In the infinite conductivity case $\mu \equiv 0$, $k_+ \equiv -i\infty$ as noted in the previous section and we reinterpret $ik_+ e^{ik_+ x}$ as $\delta(x)$, noting $x \leq 0$. The current then becomes

$$\mathbf{J}_{s\mathrm{ind}+}(x,\mathbf{k}_s) = -\left(\frac{\varepsilon_0 c n_s \Delta_{zy-}}{r_x^v}\right) E_{s-}(\mathbf{k}_s)\delta(x)\hat{\mathbf{z}} , \tag{87}$$

and we can reinterpret (81) as

$$\mathbf{J}_{s\mathrm{ind}+}(\omega, x,\mathbf{k}_s) = \mathbf{J}_{s\mathrm{ext}}(\omega,\mathbf{k}_s)\delta(x) , \tag{88}$$

where

$$\mathbf{J}_{s\mathrm{ext}}(\omega,\mathbf{k}_s) = J_{s\mathrm{ext}}(\mathbf{k}_s)\hat{\mathbf{z}} 2\pi\delta(\omega - \omega_s(\mathbf{k}_s)) \tag{89}$$

and

$$J_{s\mathrm{ext}}(\mathbf{k}_s) = -\left(\frac{\varepsilon_0 c n_s \Delta_{zy-}}{r_x^v}\right) E_{s-}(\mathbf{k}_s) . \tag{90}$$

The physical interpretation of (88) is that the induced bulk current due to the $+$ mode electric field has now become an extraneous surface current density $\mathbf{J}_{s\mathrm{ext}}(\omega,\mathbf{k}_s)$, corresponding to a vanishing skin depth $d_{s+} \equiv 0$.

Let us now consider the necessary generalisation of (18) to treat the infinite conductivity case. In the presence of the surface current $\mathbf{J}_{s\mathrm{ext}}(\omega,\mathbf{k}_s)$ we must replace our wave equation (18) with

$$\mathbf{Z}_s(\omega,\mathbf{k}_s)\mathbf{M}_s(\omega,\mathbf{k}_s) = -\left(\frac{i r_x^v}{n_s^2}\right)\mathbf{J}_{s\mathrm{ext}}(\omega,\mathbf{k}_s) , \tag{91}$$

that is, we add the extraneous surface current as a source term. This comes from applying the boundary condition on the surface components of the physical system magnetic fields, allowing for the existence of a surface current.

In the infinite conductivity case the pole $r_+ \equiv -i\infty$ formally lies outside our contour of integration (as the semi-circle can never be large enough to enclose the pole) and only the $-$ mode contributes to the response of the system, in line with the discussion at the end of Section 5e. We can thus write (91) explicitly in terms of the electric field $\mathbf{E}_{s-}(\omega,\mathbf{k}_s)$ in the plasma:

$$\mathbf{\Delta}_{s-}(\omega,\mathbf{k}_s)\mathbf{E}_{s-}(\omega,\mathbf{k}_s) = -\left(\frac{\mu_0 c r_x^v}{n_s}\right)\mathbf{J}_{s\mathrm{ext}}(\omega,\mathbf{k}_s) . \tag{92}$$

Considering the expressions for the electric field surface polarisation vector of the $-$ mode and the surface current from the $+$ mode (equations 71 and 89) we obtain

$$E_{s-}(\mathbf{k}_s)\mathbf{\Delta}_{s-}(\omega_s(\mathbf{k}_s),\mathbf{k}_s)\cdot\hat{\mathbf{y}} = -\left(\frac{\mu_0 c r_x^v}{n_s}\right) J_{s\mathrm{ext}}(\mathbf{k}_s)\hat{\mathbf{z}}\,. \tag{93}$$

The y component of this equation gives $\Delta_{yy-}(\omega_s(\mathbf{k}_s),\mathbf{k}_s)=0$ and the z component gives

$$E_{s-}(\mathbf{k}_s) = -\left(\frac{\mu_0 c r_x^v}{n_s}\right)\frac{J_{s\mathrm{ext}}(\mathbf{k}_s)}{\Delta_{zy-}(\omega_s(\mathbf{k}_s),\mathbf{k}_s)}\,, \tag{94}$$

which we note are entirely consistent with the surface wave dispersion relation (68) and the surface current given by (90) respectively, as they must be. We have now seen how the infinite conductivity case may be obtained in the context of the general surface wave theory.

7. Concluding Remarks

The main aim of this paper has been to show how the general dispersion relation of Rowe (1991) for surface waves at a sharp plasma–vacuum interface may be applied to magnetised plasmas in practice. As pointed out in Section 1, the dispersion relation of Cramer and Donnelly (1983) was chosen for this purpose primarily because low frequency surface waves of this type are of importance in applications to both space and laboratory plasmas. It is straightforward to generalise the calculation presented herein to a thermal plasma and to calculate the Cherenkov (Landau and Transit Time Magnetic) damping of these waves. This has application to theories which deal with the heating of the solar corona. In the same way, the cyclotron damping (which becomes important near the ion-cyclotron frequency Ω_i) may also be calculated, and may be of importance to theories of heating of laboratory plasmas. Both of these calculations are currently under investigation.

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Appendix

According to the definitions (27) and (14) the surface field propagator evaluated for a particular bulk mode M of the plasma is

$$\begin{aligned}\mathbf{Q}_{sM}(\omega,\mathbf{k}_s) &= -i\lim_{r_x\to r_M}\left[(r_x-r_M)\frac{\boldsymbol{\gamma}_s(\omega,\mathbf{k})}{\boldsymbol{\Gamma}_s(\omega,\mathbf{k})}\right]\\ &= -i\lim_{r_x\to r_M}\left[(r_x-r_M)\frac{\Lambda_{xx}(\omega,\mathbf{k})\boldsymbol{\gamma}_s(\omega,\mathbf{k})}{\Lambda(\omega,\mathbf{k})}\right],\end{aligned} \tag{A1}$$

where we have also made use of (17). Near non-degenerate poles we have by Taylor expansion

$$\Lambda(\omega,\mathbf{k}) \approx \left.\frac{\partial\Lambda(\omega,\mathbf{k})}{\partial r_x}\right|_{r_x=r_M}(r_x-r_M) \tag{A2}$$

so that (A1) becomes

$$\mathbf{Q}_{sM}(\omega,\mathbf{k}_s) = -i\left.\frac{\Lambda_{xx}(\omega,\mathbf{k})\boldsymbol{\gamma}_s(\omega,\mathbf{k})}{\partial\Lambda(\omega,\mathbf{k})/\partial r_x}\right|_{r_x=r_M}. \tag{A3}$$

The determinant of $\mathbf{Q}_{sM}(\omega,\mathbf{k}_s)$ is then

$$Q_{sM}(\omega,\mathbf{k}_s) = -\left[\frac{\Lambda_{xx}(\omega,\mathbf{k})}{\partial\Lambda(\omega,\mathbf{k})/\partial r_x}\right]^2_{r_x=r_M}\gamma_s(\omega,k_M,\mathbf{k}_s)\,, \tag{A4}$$

where $\gamma_s(\omega,k_M,\mathbf{k}_s)=\det[\boldsymbol{\gamma}_s(\omega,k_M,\mathbf{k}_s)]$. By definition, the matrix of cofactors $\boldsymbol{\gamma}_s(\omega,\mathbf{k})$ is related to $\boldsymbol{\Gamma}_s(\omega,\mathbf{k})$ through

$$\boldsymbol{\Gamma}_s(\omega,\mathbf{k})\boldsymbol{\gamma}_s(\omega,\mathbf{k}) = \Gamma_s(\omega,\mathbf{k})\boldsymbol{\delta} \tag{A5}$$

which, taking the determinant of both sides, yields

$$\gamma_s(\omega,\mathbf{k}) = \Gamma_s(\omega,\mathbf{k}) = \frac{\Lambda(\omega,\mathbf{k})}{\Lambda_{xx}(\omega,\mathbf{k})}\,. \tag{A6}$$

As a result, $\gamma_s(\omega,k_M,\mathbf{k}_s)\equiv 0$ [as $\Lambda(\omega,k_M,\mathbf{k}_s)\equiv 0$ by definition of k_M] and (29) immediately follows from (A4).

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