Nonlinear Wave Number Shift and Modulational Instability for Large Amplitude Waves in a Relativistic Magnetised Plasma

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Abstract

Properties of large amplitude waves in a relativistic magnetised plasma are studied using the method of reductive perturbation. The plasma under consideration consists of warm adiabatic ions and isothermal warm electrons, under the influence of a magnetic field. A consideration of large amplitude waves demands study of the relativistic situation. In the present case we consider both the electrons and ions to be relativistic. A KdV equation is derived from which a nonlinear Schrödinger equation is deduced by further scaling. Lastly we derive an expression for nonlinear wave number shift, critical angle of propagation and the condition for modulational instability. Our analysis is applicable to both laboratory and space plasmas.

1. Introduction

Amongst various nonlinear waves sustained in a plasma, ion–acoustic waves are a typical example, one which has been exhaustively studied by perturbation theory. The first such attempt was by Washimi and Taniuti (1966). Later, this formulation was extended to encompass situations with greater complexity (Tagare 1973). A completely different class of events occurs when the plasma is magnetised. Such an analysis for the hydromagnetic wave was initiated by Gardner and Morikawa (1960). On the other hand, it has been observed that it is impossible for an electromagnetic wave to penetrate a dense plasma unless the electrons become relativistic; that is, it becomes imperative to consider the mass variation of the electrons. Also, electromagnetic waves with frequency less than the electron plasma frequency cannot propagate in an unmagnetised plasma. The relativistic effect results in a downshift of the electron plasma frequency. Such a phenomenon is quite common in the radar-induced modification of the ionosphere (Shukla *et al.* 1986) so the mass variation of electrons must be considered.

On the other hand, recently the study of large amplitude waves has gained momentum. This class of waves is important in the light of phenomena such as Wakefield excitation (Kruer 1990), the beat wave accelerator (Chen 1990) and laser–plasma interactions (Bingham *et al.* 1989). Understanding the behaviour of large amplitude plasma waves is important for such practical situations. These involve a rich interplay between nonlinear and dispersive effects which may limit the amplitude of the wave generated and also the frequency shift. Some work has already been done by Kaw and Dawson (1970) and Chakraborty *et al.* (1984).

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In astrophysical situations it has been observed that particles are ejected at high velocities during solar bursts and pulsar radiation (Kaplan *et al.* 1973). Experimental evidence for the simultaneous acceleration of electrons and protons to relativistic energies (20 GeV for protons and 100 MeV for electrons) was obtained from the SMM and Hinotan spacecraft (Tanaka 1987). The hard X-ray and γ-ray periodic bursts on 7 June 1980 (from a solar flare) presented the first evidence of simultaneous acceleration of protons and electrons within 2 s. After the impulsive phase the flare continued for at least 1000 s with extended emission. During this phase strong acceleration of ions takes place. In this respect we can mention the work of Tsytovich who considered both the electrons and ions to be relativistic. Very recently, it has been observed that both electrons and ions can attain very high velocity for a plasma having a high Alfven speed $(V_{\mathrm{A}} \geq cm_{\mathrm{e}}/m_{\mathrm{i}})^{1/2}$ (Stenflo *et al.* 1970). Furthermore, ion heating and acceleration has been studied through computer simulation by Lembege *et al.* (1983).

Our discussion above gives sufficient motivation to study a magnetised plasma with both relativistic electrons and ions and for analysing the properties of large amplitude waves in the reductive perturbation framework. Already people have observed that relativistic effects influence the amplitude and width of the solitary wave in some simple systems. In this respect we can mention the work of Das and Paul (1985), Roy Chowdhury *et al.* (1988) and Nejoh (1987).

2. Formulation

We consider a plasma that has weakly relativistic ions and electrons. Some simplifying assumptions that we make will be spelled out in the course of our discussion. We also assume the plasma to be collisionless but, if the ions are adiabatic and the electrons are warm, the thermal speed of the electrons will be much greater than the wave speed of the hydromagnetic wave. Hence, the effect of resonance particles may be small and the variation of the electron distribution function in the velocity space will be quite small so that the effect of Landau damping may be neglected. Thus, we assume that we can formulate our problem using the two-fluid model following Kakutani *et al.* (1967). We denote by n_{i}, n_{e} the ion and electron density and V_{i}, V_{e} the corresponding velocities, with p_{i} the ion pressure and (B_x, B_y, B_z) the components of the magnetic field in the (x, y, z) directions. Then, the equations describing the plasma are written as

$$
\begin{aligned}
\frac{\partial n_{\mathrm{i}}}{\partial t} + \nabla . (n_{\mathrm{i}} \boldsymbol{V}_{\mathrm{i}}) &= 0 , \\
\frac{\partial n_{\mathrm{e}}}{\partial t} + \nabla . (n_{\mathrm{e}} \boldsymbol{V}_{\mathrm{e}}) &= 0 , \\
\frac{\partial V_{\mathrm{i}\alpha}}{\partial t} + (\boldsymbol{V}_{\mathrm{i}} . \nabla) \boldsymbol{V}_{\mathrm{i}\alpha} &= R_{\mathrm{i}} (\boldsymbol{E} + \boldsymbol{V}_{\mathrm{i}} \times \boldsymbol{B}) - \frac{\sigma}{M^2 n_{\mathrm{i}}} \nabla p_{\mathrm{i}} , \\
\frac{\partial V_{\mathrm{e}\alpha}}{\partial t} + (\boldsymbol{V}_{\mathrm{e}} . \nabla) \boldsymbol{V}_{\mathrm{e}\alpha} &= R_{\mathrm{e}} (\boldsymbol{E} + \boldsymbol{V}_{\mathrm{e}} \times \boldsymbol{B}) - \frac{1}{M^2 n_{\mathrm{e}}} \nabla n_{\mathrm{e}} , \\
\frac{\partial p_{\mathrm{i}}}{\partial t} + (\boldsymbol{V}_{\mathrm{i}} . \nabla p_{\mathrm{i}}) + 3 p_{\mathrm{i}} (\nabla . \boldsymbol{V}_{\mathrm{i}\alpha}) &= 0 ,
\end{aligned}
$$

$$
\begin{aligned}
\nabla \times \boldsymbol{B} &= \left(\frac{U_0^*}{c}\right)\frac{\partial \boldsymbol{E}}{\partial t} + \frac{M_{\mathrm{A}}^2\, R_{\mathrm{i}}\, R_{\mathrm{e}}}{R_{\mathrm{i}} + R_{\mathrm{e}}}(n_{\mathrm{i}}\,\boldsymbol{V}_{\mathrm{i}} - n_{\mathrm{e}}\,\boldsymbol{V}_{\mathrm{e}})\,,\\
\nabla \times \boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t}, \qquad \nabla\boldsymbol{.}\boldsymbol{B} = 0\,,\\
\nabla\boldsymbol{.}\boldsymbol{E} &= \frac{R_{\mathrm{pe}}^2}{R_{\mathrm{e}}}(n_{\mathrm{i}} - n_{\mathrm{e}})\,,
\end{aligned}
\tag{1}
$$

where $\boldsymbol{V}_{\mathrm{i}\alpha} = \boldsymbol{V}_{\mathrm{i}}/(1-u^2/c^2)^{1/2}$. In writing these equations we have neglected the contribution of the displacement current since $u_0^*/c \ll 1$ or $R_{\mathrm{pe}} \gg 1$. Some specific constants used above are: R_{e}, the ratio of the electron cyclotron frequency Ω_{e} to the characteristic frequency ω_0^*; R_{i}, the ratio of the ion cyclotron frequency Ω_{i} to ω_0^*; u_0^*, the characteristic speed along the corresponding magnetic field; R_{pe}, the ratio of electron plasma frequency ω_{pe} to ω_0^*; V_{A}, the Alfven speed; M_{A}, Alfven Mach number; M, the usual Mach number; and n_0^*, the characteristic number density. Further, we write

$$
\begin{aligned}
\Omega_{\mathrm{i}} &= \frac{B_0^*}{m_{\mathrm{i}}\, c}, \qquad \Omega_{\mathrm{e}} = \frac{eB_0^*}{m_{\mathrm{e}}\, c}\,,\\
\omega_{\mathrm{pe}}^2 &= \frac{4\pi n_0^*\, e^2}{m_{\mathrm{e}}}, \qquad M_{\mathrm{A}} = \frac{u_0^*}{V_{\mathrm{A}}}\,,\\
V_{\mathrm{A}} &= B_0^*\{4\pi n_0^*(m_{\mathrm{i}} + m_{\mathrm{e}})\}^{1/2}\,,\\
E_0 &= u_0^*\, B_0^*\,,\\
V_{\mathrm{i}\alpha} &\approx V_{\mathrm{i}}(1 + V_{\mathrm{i}}^2/2c^2)\,.
\end{aligned}
\tag{2}
$$

Actually we have used the same units as those of Kakutani *et al.* (1967). The characteristic ion pressure is of the form $n_0^*\, kT_{\mathrm{i}}{}^{\sigma}$, σ being the ratio of ion temperature T_{i} to electron temperature T_{e}. We now assume the condition for a quasi-neutral plasma, that is $n_{\mathrm{i}} \approx n_{\mathrm{e}} \approx n$, and eliminate V_{e} in terms of the variables n, V_{i}, B, p_{i} to get

$$
\frac{\partial n}{\partial t} + \nabla\boldsymbol{.}(n\boldsymbol{V}_{\mathrm{i}}) = 0\,,
\tag{3}
$$

$$
\begin{aligned}
\frac{\partial \boldsymbol{V}_{\mathrm{i}\alpha}}{\partial t} &= \frac{1}{n}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \frac{1}{R_{\mathrm{e}}}\left[\left(\frac{1}{n}(\nabla \times \boldsymbol{B})\boldsymbol{.}\nabla\right)\boldsymbol{V}_{\mathrm{i}}\right.\\
&\quad \left. + \frac{\mathrm{d}}{\mathrm{d}t}\left\{\left(1 + \frac{V_{\mathrm{i}}^2}{2c^2}\right)\frac{1}{n}(\nabla \times \boldsymbol{B})\right\}\right] - \frac{1}{M_{\mathrm{A}}^2\, R_{\mathrm{i}}\, R_{\mathrm{e}}}\left(1 + \frac{m_{\mathrm{e}}}{m_{\mathrm{i}}}\right)\\
&\quad \times \left(\frac{1}{n}(\nabla \times \boldsymbol{B})\boldsymbol{.}\nabla\right)\left(\frac{1}{n}(\nabla \times \boldsymbol{B})\right)\left(1 + \frac{V_{\mathrm{i}}^2}{2c^2}\right)\\
&\quad + \left(\frac{M_{\mathrm{A}}}{M}\right)^2\left\{\frac{\sigma}{n}\nabla p_{\mathrm{i}} + \frac{1}{n}\nabla n\right\},
\end{aligned}
\tag{4}
$$

$$
\frac{\partial p_{\mathrm{i}}}{\partial t} + (\boldsymbol{V}_{\mathrm{i}}\boldsymbol{.}\nabla p_{\mathrm{i}}) + 3p_{\mathrm{i}}(\nabla\boldsymbol{.}\boldsymbol{V}_{\mathrm{i}\alpha}) = 0\,,
\tag{5}
$$

$$
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\boldsymbol{V}_{\mathrm{i}} \times \boldsymbol{B}) - \frac{1}{R_{\mathrm{i}}}\left\{\nabla \times \left(\frac{\mathrm{d}\boldsymbol{V}_{\mathrm{i}\alpha}}{\mathrm{d}t}\right)\right\}.
\tag{6}
$$

In each of the above equations $\partial/\partial t$ stands for $\partial/\partial t + (\boldsymbol{V}_{\mathrm{i}} \cdot \nabla)$.

We now consider one-dimensional plane waves, and the magnetic field is assumed to have the components $B_0(\cos\theta, \sin\theta, 0)$. Our further assumption is that the relativistic effect is important only in the x-direction, parallel to that of the ion pressure gradient. We denote the components of velocity as (u, v, ω); so we have

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(n\, u) = 0\,, \tag{7}$$

$$M_{\mathrm{A}}^2 \frac{\mathrm{d}u_\alpha}{\mathrm{d}t} + \left(\frac{M_{\mathrm{A}}}{\mu}\right)^2 \left(\frac{\sigma}{n}\frac{\partial p_{\mathrm{i}}}{\partial x} + \frac{1}{n}\frac{\partial n}{\partial x}\right) + \tfrac{1}{2}\frac{\partial}{\partial x} \times \{\tfrac{1}{2}(By^2 + B_z^2)\} = 0\,, \tag{8}$$

$$M_{\mathrm{A}}^2 \frac{\mathrm{d}v_\alpha}{\mathrm{d}t} - \frac{B_x}{n}\frac{\partial B_y}{\partial x} + \frac{1}{R_{\mathrm{e}}}\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(1 + \frac{u^2}{2c^2}\right)\frac{1}{n}\frac{\partial B_z}{\partial x}\right] = 0\,, \tag{9}$$

$$M_{\mathrm{A}}^2 \frac{\mathrm{d}\omega_\alpha}{\mathrm{d}t} = \frac{B_x}{n}\frac{\partial B_z}{\partial x} + \frac{1}{M_{\mathrm{A}}^2}\frac{1}{R_{\mathrm{e}}}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{n}\left(1 + \frac{u^2}{2c^2}\right)\frac{\partial B_y}{\partial x}\right], \tag{10}$$

$$\frac{\partial p_{\mathrm{i}}}{\partial t} + u\frac{\partial p_{\mathrm{i}}}{\partial x} + 3p_{\mathrm{i}}\frac{\partial u_\alpha}{\partial x} = 0\,, \tag{11}$$

$$\frac{\mathrm{d}B_y}{\mathrm{d}t} - B_x\frac{\partial V}{\partial x} + B_y\frac{\partial u}{\partial x} - \frac{1}{R_{\mathrm{i}}}\frac{\partial}{\partial x}\left(\frac{\mathrm{d}\omega_\alpha}{\mathrm{d}t}\right) = 0\,, \tag{12}$$

$$\frac{\mathrm{d}B_z}{\mathrm{d}t} - B_x\frac{\partial \omega}{\partial x} + B_z\frac{\partial u}{\partial x} + \frac{1}{R_{\mathrm{i}}}\frac{\partial}{\partial x}\left(\frac{\mathrm{d}V_\alpha}{\mathrm{d}t}\right) = 0\,, \tag{13}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\,,$$

$$V_\alpha = \frac{V}{(1 - u^2/c^2)^{1/2}}\,, \qquad \omega_\alpha = \frac{\omega}{(1 - u^2/c^2)^{1/2}}\,. \tag{14}$$

In the study of ordinary ion–acoustic waves the transformation of the coordinate variables pertaining to reductive perturbation was derived from a simple physical condition. In the present case the number of equations and dependent variables is large. To reduce the system we follow the same procedure as Kakutani (1974) by introducing stretched variables in the case of a magnetised plasma. The chief motivation is to derive a single nonlinear dispersive equation from the set (7)–(14). That is, if we go to a frame of reference moving with the solitary wave and assume the amplitude to be a small quantity of order ϵ, then the dispersion relation obtained from the linearised form of the equation of motion dictates that the required new coordinates are to be defined via the equations (Kakutani 1974)

$$\xi = \epsilon^{1/2}(x - \lambda t), \qquad \tau = \epsilon^{3/2} t\,. \tag{15}$$

To write the expansion formulae for the other physical quantities we consider the asymptotic conditions

$$\begin{Bmatrix} n \\ u \\ V \\ w \\ p_i \\ B_x \\ B_y \\ B_z \end{Bmatrix} \longrightarrow \begin{Bmatrix} 1 \\ u_0 \\ 0 \\ 0 \\ 1 \\ \cos\theta \\ \sin\theta \\ 0 \end{Bmatrix} \quad \text{as } x \to \infty . \tag{16}$$

Hence we set

$$\begin{aligned}
n &= 1 + \epsilon n_1 + \epsilon^2 n_2 + \dots, \qquad u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \dots, \\
V &= \epsilon V_1 + \epsilon^2 V_2 + \dots, \qquad W = \epsilon^{3/2} w_1 + \epsilon^{5/2} W_2 + \dots, \\
p_{\mathrm{i}} &= 1 + \epsilon p_{\mathrm{i}_1} + \epsilon^2 p_{\mathrm{i}_2} + \dots, \\
B_x &= \cos\theta, \qquad B_y = \sin\theta + \epsilon B_{y_1} + \epsilon^2 B_{y_2} + \dots, \\
B_z &= \epsilon^{3/2} B_{z_1} + \epsilon^{5/2} B_{z_2} + \dots .
\end{aligned} \tag{17}$$

Equating first powers of ϵ we get

$$-(\lambda - u_0)\frac{\partial n_1}{\partial \xi} + \frac{\partial u_1}{\partial \xi} = 0\,, \tag{18}$$

$$\begin{aligned}
-(\lambda - u_0)\left(1 + \frac{3u_0^2}{2c^2}\right)\frac{\partial u_1}{\partial \xi} &+ \frac{\sin\theta}{M_{\mathrm{A}}^2}\frac{\partial B_{y_1}}{\partial \xi} \\
&+ \frac{1}{\mu^2}\sigma\frac{\partial p_{\mathrm{i}_1}}{\partial \xi} + \frac{\partial n_1}{\partial \xi} = 0\,,
\end{aligned} \tag{19}$$

$$-(\lambda - u_0)\left(1 + \frac{u_0^2}{2c^2}\right)\frac{\partial V_1}{\partial \xi} - \frac{\cos\theta}{M_{\mathrm{A}}^2}\frac{\partial B_{y_1}}{\partial \xi} = 0\,, \tag{20}$$

$$-(\lambda - u_0)\frac{\partial p_{\mathrm{i}_1}}{\partial \xi} + 3\left(1 + \frac{3u_0^2}{2c^2}\right)\frac{\partial u_1}{\partial \xi} = 0\,, \tag{21}$$

$$-(\lambda - u_0)\frac{\partial B_{y_1}}{\partial \xi} + \sin\theta\frac{\partial u_1}{\partial \xi} - \cos\theta\frac{\partial u_1}{\partial \xi} = 0\,, \tag{22}$$

from which we get

$$\begin{aligned}
u_1 &= \frac{\sin\theta}{X_1 M_{\mathrm{A}}^2} B_{y_1}, \qquad V_1 = -\frac{\cos\theta}{X_2 M_{\mathrm{A}}^2} B_{y_1}\,, \\
n_1 &= \frac{\sin\theta}{X_1 M_{\mathrm{A}}^2 (\lambda - u_0)} B_{y_1}\,, \\
X_2 &= (\lambda - u_0)\left(1 + \frac{u_0^2}{2c^2}\right), \\
X_1 &= (\lambda - u_0)\left(1 + \frac{3u_0^2}{2c^2}\right) - \frac{1}{M^2}\left\{\frac{1}{\lambda - u_0} + \frac{3\sigma}{\lambda - u_0}\left(1 + \frac{3u_0^2}{2c^2}\right)\right\}.
\end{aligned} \tag{23}$$

The expression for the phase velocity is also found to be

$$\lambda^{\pm} - u_0 = (-B \pm \sqrt{B^2 - 4AC})^{1/2}/2A\,, \tag{24}$$

where

$$\begin{aligned} B &= -\left[\frac{M_{\mathrm{A}}^2}{M^2}\left(1+\frac{u_0^2}{2c^2}\right)\left\{1+3\sigma\left(1+\frac{3u_0^2}{2c^2}\right)\right\}\right.\\ &\qquad \left. + \left(1+\frac{u_0^2}{2c^2}\right) + \cos^2\theta\,\frac{u_0^2}{c^2}\right], \end{aligned} \tag{25}$$

$$A = M_{\mathrm{A}}^2\left(1+\frac{u_0^2}{2c^2}\right)\left(1+\frac{3u_0^2}{2c^2}\right), \tag{26}$$

$$C = \frac{\cos^2\theta}{M^2}\left\{1+3\sigma\left(1+\frac{3u_0^2}{2c^2}\right)\right\}, \tag{27}$$

$$M_{\mathrm{A}}^2 = 1 + \frac{\sin^2\theta}{M^2-1}\,.$$

From terms of higher order in ϵ we obtain

$$\frac{\partial B_{y_1}}{\partial \tau} + \frac{Q}{\Lambda} B_{y_1} \frac{\partial B_{y_1}}{\partial \xi} + \frac{P}{\Lambda}\frac{\partial^3 B_{y_1}}{\partial \xi^3} = 0\,, \tag{28}$$

whence other dependent variables have been eliminated. Equation (28) is the KdV equation governing the propagation of nonlinear waves within the plasma. Here P, Q and Λ are given by

$$P = (\lambda - u_0)\left(1+\frac{u_0^2}{2c^2}\right)\left(X_3\,\frac{R_{\mathrm{i}}-R_{\mathrm{e}}}{R_{\mathrm{i}}\,R_{\mathrm{e}}} + \frac{1}{R_{\mathrm{i}}^2\,M_{\mathrm{A}}^2}\right), \tag{29}$$

$$\begin{aligned} X_3 &= \left(1 - \frac{\cos^2\theta}{M_{\mathrm{A}}^2(\lambda-u_0)^2(1+u_0/2c^2)^2}\right)^{-1}\left(\frac{1}{M_{\mathrm{A}}^2\,R_{\mathrm{i}}}\right.\\ &\qquad \left. - \frac{\cos^2\theta}{M_{\mathrm{A}}^4\,R_{\mathrm{e}}(\lambda-u_0)^2(1+u_0^2/2c^2)}\right), \end{aligned} \tag{30}$$

$$\begin{aligned} Q &= \frac{\sin^3\theta}{X_1\,X_2\,M_{\mathrm{A}}^2}\left[\frac{1}{X_1}\left\{1+\frac{3u_0^2}{2c^2} + \frac{1}{M^2}\left(\frac{9\sigma}{(\lambda-u_0)^2}\left(1+\frac{3u_0^2}{2c^2}\right)^2 + \frac{1}{(\lambda-u_0)^2}\right)\right\}\right.\\ &\qquad \left. - \frac{1}{\lambda-u_0}\right] + \frac{\sin\theta}{X_1^2\,M_{\mathrm{A}}^2}\left[(\lambda-u_0)\left(1+\frac{3u_0^2}{2c^2}\right) + X_1 - \frac{1}{X_2\,M_{\mathrm{A}}^2}\right.\\ &\qquad \times\left(1+\frac{3u_0^2}{2c^2}\right) + \frac{X_1^2}{X_2} + \frac{X_1}{X_2\,M_{\mathrm{A}}^2(\lambda-u_0)} + \frac{1}{M^2}\left\{\frac{1}{\lambda-u_0}\right.\\ &\qquad \left.\left. + \frac{9\sigma(1+3u_0^2/2c^2)^2}{\lambda-u_0} - \frac{9\sigma(1+3u_0^2/2c^2)^2}{(\lambda-u_0)^2X_2\,M_{\mathrm{A}}^2}\right\}\right]\\ &\qquad + \frac{1}{\sin\theta}\left((\lambda-u_0) - \frac{1}{M_{\mathrm{A}}^2\,X_2}\right), \end{aligned} \tag{31}$$

$$\Lambda = 1 + \frac{\sin^2\theta}{M_A^2 X_1 X_2}\left[\left(1+\frac{3u_0^2}{2c^2}\right) + \frac{1}{M^2(\lambda-u_0)^2}\left\{1+3\sigma\left(1+\frac{3u_0^2}{2c^2}\right)\right\}\right]$$
$$+\cos^2\theta\,\frac{1+u_0^2/2c^2}{X_2^2 M_A^2} + \frac{(\lambda-u_0)(1+3u_0^2/2c^2)}{X_1} + \frac{1}{M^2 X_1}$$
$$\times\left(\frac{3\sigma(1+3u_0^2/2c^2)}{\lambda-u_0} + \frac{1}{\lambda-u_0} - \frac{3\sigma}{M_A^2 X_2}\,\frac{1+3u_0^2/2c^2}{(\lambda-u_0)^2}\right.$$
$$\left.-\frac{1}{M_A^2 X_2^2(\lambda-u_0)^2}\right) - \frac{1+3u_0^2/2c^2}{X_1 X_2 M_A^2}\,. \tag{32}$$

The solitary wave solution is given as

$$B_{y_1} = \phi_0\,\mathrm{sech}^2(\xi - v\tau)\,;$$
$$\phi_0 = 12p/Q, \qquad v = 4p/\Lambda\,. \tag{33}$$

3. Observations

An important aspect of (28) is that even when $\theta = \pi/2$ the equation does not collapse. Actually, we get a simpler form of the equation. This may be due to the electron inertia taken into account by the term $1/R_e$, and hence the solitary wave exists even when $\theta = \pi/2$. Now if we consider the propagation of a plane wave and make an approximate linearisation we get

$$-\frac{\omega}{k} \approx (\lambda - u_0)(1 + \nu k^2 + ...)\,. \tag{34}$$

For non-dispersive waves ν should go to zero. In our case ν is given by

$$\nu = \left(1+\frac{u_0^2}{2c^2}\right)\frac{X_3}{R_i} + \frac{\cos^2\theta}{X^2}\,\frac{1}{M_A^2 R_e}\,\frac{1}{R_i}\,\frac{1}{X_2 M_A^2} - \frac{X_3}{\lambda - u_0}\,. \tag{35}$$

Actually the solution of the equation $\nu = 0$ yields the critical angle $\theta = \theta_c$, but in the present case the value of θ_c is too complicated to be reproduced here. However, we can demonstrate that in the proper nonrelativistic limit we get back the same value as Kakutani *et al.* (1967). For the nonrelativistic cold plasma $\lambda = 1/M_A$ and $\Lambda = 1$, so the equation $\nu = 0$ reduces to

$$\frac{1}{2M_A^2}\left[\frac{\cos^2\theta}{R_e}\left\{\frac{1}{R_i} - \frac{1}{\sin^2\theta}\left(\frac{1}{R_e} - \frac{\cos^2\theta}{R_i}\right)\right\} + \frac{1}{R_i\sin^2\theta}\right.$$
$$\left.\left(\frac{1}{R_e} - \frac{\cos^2\theta}{R_i}\right)\right] = 0\,, \tag{36}$$

which entails

$$\cot^2\theta_c\left(\sqrt{\frac{R_e}{R_i}} - \sqrt{\frac{R_i}{R_e}}\right)^2 - 1 = 0\,, \tag{37}$$

and this is the expression obtained by Kakutani *et al.*

4. Structure of the Soliton

We thus observe that the characteristics of the soliton in a magnetised relativistic plasma are very interesting and the soliton has an important role to play in various physical situations. We have therefore numerically estimated the width of the soliton in a plasma in the situations of a thermonuclear discharge and an atmosphere space plasma.

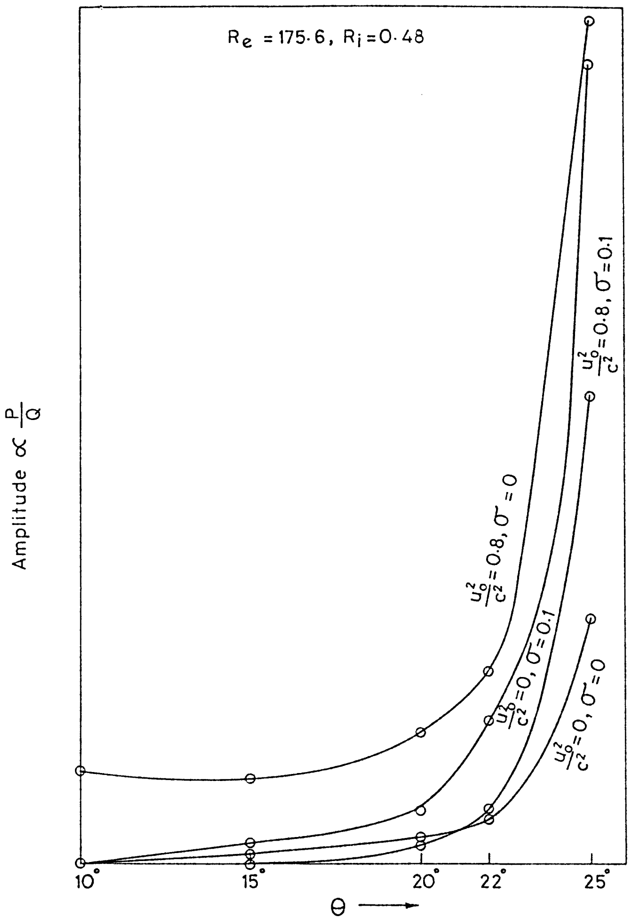

Fig. 1. Variation of the soliton amplitude in the case of a laboratory plasma for various values of u_0^2/c^2 and σ.

In the case of thermonuclear phenomena we used the data of Denise and Delcroix: $n_e = n_i = 10^{15}$ cm^{-3}, $B_0 = 10^4$ G, and the corresponding ions are D_2 (deuterium) with $m = 3{\cdot}3 \times 10^{-24}$ g. Moreover, we assume the Mach number $M = 0{\cdot}9$ and $R_e = 175{\cdot}6$, $R_i = 0{\cdot}48$. Using these data we have plotted the amplitude of the soliton as a function of angle θ for various values of u_0^2/c^2 and σ. This actually corresponds to the situation observed in a laboratory. On the other hand, in the case of a space plasma corresponding to the ion N_2 we have

$R_e = 880$, $R_i = 0 \cdot 016$. In both these situations the behaviour of the amplitudes is depicted in Figs 1 and 2. While in the first case (i.e. laboratory plasma), the cold, nonrelativistic situation does not differ widely from the case with finite u_0^2/c^2 or σ, in the latter case of an atmospheric or space plasma the cold nonrelativistic situation differs considerably from the hot relativistic situation. Also, from the expressions for P, Q and Λ, one can ascertain some facts about the width of the soliton.

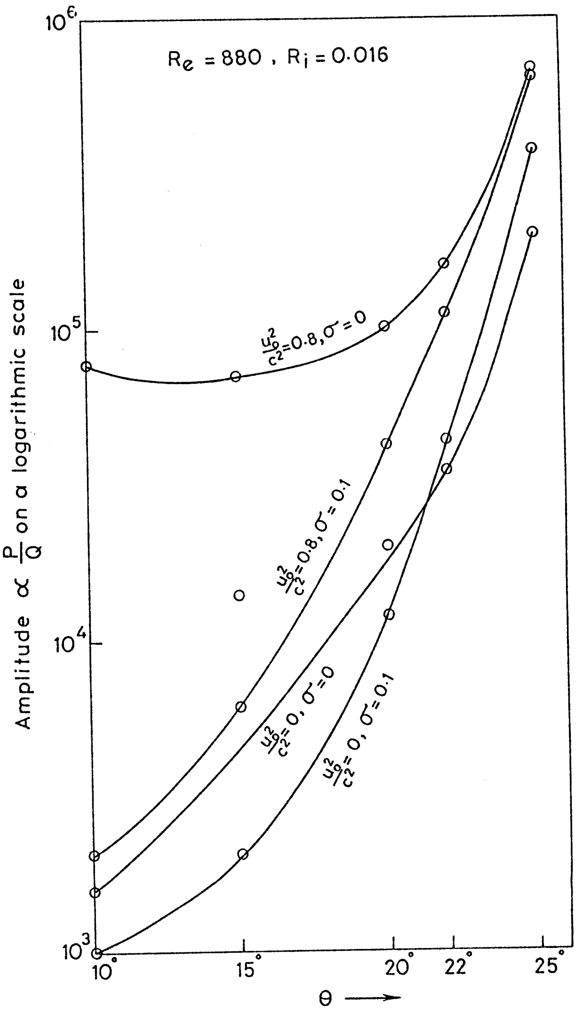

Fig. 2. Variation of the soliton amplitude in the case of a space plasma for various values of u_0^2/c^2 and σ.

The width remains almost unchanged for any propagation angle θ in a cold plasma ($\sigma \approx 0$), even in the case when relativistic effects are taken into account. But in a warm plasma ($\sigma \approx 1$) the width of the soliton increases as the propagation angle decreases. In the present case the width decreases sharply to zero for $\theta \approx \cos^{-1}(M) = 25 \cdot 80^\circ$ and that is why in Figs 1 and 2 the range of θ is restricted to a maximum value of 25°. It may be noted that, according to Tanaka (1987), though initially both electrons and ions are considered relativistic, subsequently the ions are treated as slow and nonrelativistic.

5. Nonlinear Wave Number Shift

To estimate the nonlinear frequency shift we first deduce a nonlinear Schrödinger equation from (28) via a second stage of reductive perturbation. We set

$$B_{y_1} = \sum_{n=0}^{\alpha} \mu^n \sum_{l=-\alpha}^{\alpha} B_1^{(n)}(\eta, \xi) \exp\{i\,\ell(K\xi - \delta\tau)\}\,, \tag{38}$$

where

$$\eta = \mu(\xi - x\tau), \qquad \xi = \mu^2\tau\,. \tag{39}$$

Substituting in (28) and equating coefficients of various powers of μ and different harmonics we get for $n = 1$ and $\ell = 1$

$$\delta = -P'K^3\,. \tag{40}$$

The coefficient of μ^2 and $\ell = 1$ yields

$$\chi = -3P'K^2\,. \tag{41}$$

The coefficient of μ^2 and $\ell = 2$ leads to

$$B_2^{(2)} = \frac{Q'}{6P'K^2}(B_1^{(1)})^2\,, \tag{42}$$

and the third-order term in μ with $\ell = 0$ requires

$$B_0^{(2)} = \frac{Q'}{\chi}\,\tfrac{1}{2}|\,B_1^{(1)}\,|^2 - C\,, \tag{43}$$

where C is a constant of integration.

Lastly, we get

$$\frac{\partial B_1^{(1)}}{\partial \xi} + 3P'\,\frac{\partial^2 B_1^{(1)}}{\partial \eta^2} + B_0^{(2)}\,B_1^{(1)}\,ik = 0\,. \tag{44}$$

Substituting the value of $B_0^{(2)}$ we get

$$i\,\frac{\partial B_1^{(1)}}{\partial \xi} + 3P'K\,\frac{\partial^2 B_1^{(1)}}{\partial \eta^2} + \frac{Q'^2}{12P'K}\,|\,B_1^{(1)}\,|^2 B_1^{(1)} + \frac{Q'^2}{P'K}\,CB_1^{(1)} = 0\,. \tag{45}$$

The solitary wave solution of (45) is

$$B_1^{(1)} = A \operatorname{sech} b(\eta - V\xi)\exp(\mathrm{i}\,\Theta)\,, \tag{46}$$

with
$$A^2 = 36\,\frac{P'^2K^2b^2}{Q^2}\,, \qquad K = \frac{V}{6PK}\,,$$

$$\Theta = K\eta - \left(\Omega + \frac{Q'^2C}{6P'K}\right)\xi\,, \tag{47}$$

$$\Omega = 3P'K(b^2 - K^2)\,,$$

so that the amount of nonlinear wave number shift is $Q'^2C/6P'K$.

6. Stability Analysis

The nonlinear Schrödinger equation deduced above can be written as

$$\mathrm{i}\,\frac{\partial B_1^{(1)}}{\partial \xi} + \alpha\,\frac{\partial^2 B_1^{(1)}}{\partial \eta^2} = -\beta\,|\,B_1^{(1)}\,|^2 B_1^{(1)} - B_1^{(1)}\,, \tag{48}$$

where

$$\alpha = 3P'K, \quad \beta = Q'^2/12P'K\,, \quad \gamma = Q'^2C/6P'K\,, \tag{49}$$

which can be deduced from the Lagrangian

$$\alpha = -\tfrac{1}{2}\{\mathrm{i}\,(B_{1\xi}^{(1)*}\,B_1^{(1)} - B_{1\xi}^{(1)}\,B_1^{(1)*}) + \alpha\,|\,B_{1\eta}^{(1)}\,|^2 + \beta\,|\,B_1^{(1)}\,|^4 + \gamma\,|\,B_1^{(1)}\,|^2\}\,. \tag{50}$$

It is easily seen that the momentum and number of solitons are

$$P = \tfrac{1}{2}\mathrm{i}(B_{1\eta}^{(1)*}\,B_1^{(1)} - B_{1\eta}^{(1)}\,B_1^{(1)*})\,,$$

$$N = |\,B_1^{(1)}\,|^2\,. \tag{51}$$

If we now consider a variation of the corresponding Hamiltonian, subject to the condition that the total number and momentum of the solitons are fixed *a priori*, then we can demand that

$$\delta \int (H - \lambda P - \mu N) = 0\,, \tag{52}$$

in accordance with the constrained variational principle. For arbitrary variation, the Lagrange multipliers $(\tilde{\lambda},\ \tilde{\mu})$ are found to be

$$\tilde{\lambda} = \alpha K\,,$$

$$\tilde{\mu} = -\frac{\alpha}{2}(b^2 + K^2) + \beta A^2 + \frac{\gamma}{2}\,, \tag{53}$$

and the stability is ascertained by the nature of the eigenvalues of the following linear differential operator:

$$\left[\alpha\partial^2_{\eta\eta} - \left(\frac{Q^2C}{6P'K} + 3P'Kb^2\right) - 6\beta A^2\,\mathrm{sech}^2 b(\eta - V\tau)\right]\Psi = \lambda\Psi, \tag{54}$$

or

$$[\alpha\partial^2_{\eta\eta} - 6\beta A^2\,\mathrm{sech}^2 b(\eta - V\tau)]\Psi = \lambda'\Psi, \tag{55}$$

$$\lambda' = \lambda + \frac{Q^2C}{6P'K} + 3P'Kb^2 .$$

The left-hand side of (55) is known to possess a single negative discrete eigenvalue, corresponding to the one soliton case, which immediately leads to a restriction on the admissible values for λ' and yields the condition for stability.

7. Discussion

In our analysis we have investigated the formation and propagation of solitary waves in a plasma, in which both the electrons and ions have been considered relativistic. Our analysis encompasses cases of both laboratory and space plasmas. Conditions for the modulational stability of envelope solitons have been investigated by a variational procedure. Lastly, it is demonstrated that the expression for the critical angle reduces to that of Kakutani *et al.* (1967).

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