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Anisotropic Dispersion of a Charged Particle Swarm in a Turbulent Gas in an Electrostatic Field

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Abstract

This paper generalises an earlier result of Saffman (1960) to account for cross effects between turbulent and molecular diffusion for charged particle swarms in a gas in the presence of an electrostatic field. It is shown that turbulence enhances the anisotropic character of diffusion. The desirability of using a full kinetic theory analysis as against a limited hydrodynamic description of the swarm is discussed, and one possible tractable approach pointed out.

1. Introduction

Swarm experiments (Huxley and Crompton 1974; Mason and McDaniel 1988) are normally carried out in drift tubes containing quiescent nonturbulent gases whose properties are uniform and static. Thus a swarm of charged particles released into the gas from a source drifts under the influence of an applied electrostatic field E and simultaneously diffuses by virtue of collisions with gas molecules. In the so-called hydrodynamic regime (Kumar *et al.* 1980), the swarm has relaxed to a state where its density $n(\boldsymbol{r}, t)$ varies only slowly over distances of the order of the mean free path between collisions and in times of the order of the mean free time between collisions. Under these circumstances, Fick's law of diffusion [see equation (7) below] holds and, in the language of fluid mechanics (Monin and Yaglom 1971), the swarm is said to be a 'passive additive', which is characterised by 'molecular' transport coefficients K (mobility) and $\mathbf{D}$ (diffusion tensor) respectively. If the gas is turbulent, however, the behaviour of the swarm is quite different, as particles are now advected with the bulk motion of the gas and generally diffuse by turbulent action at a much greater rate. In the case of molecular diffusion, particulate properties (e.g. collision cross sections) control the swarm behaviour, while for turbulent diffusion the fluid properties of the gas are evident. It is usual to think of the two processes as operating on quite different length and time scales, allowing for separate treatment of the effects. If, however, the distinction is not clear cut, as it could well be for light swarm particles (electrons, positrons, muons, etc.) where the mean free path for *energy* transfer is the relevant microscopic scale length and may be comparable with macroscopic dimensions (Robson 1976), the possibility of significant 'cross effects' arises. The present investigation is along these lines. The problem for an uncharged passive additive has been considered by Townsend (1954) and

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by Saffman (1960). In this paper we consider charged particle swarms, where diffusion is anisotropic in the presence of an electric field.

Section 2 outlines the theory, starting from a prescription of the turbulent properties of the gas and proceeding through the solution of the diffusion equation, from which dispersion in directions along and transverse to the applied field respectively is calculated. It is found that the presence of turbulence *enhances* the anisotropic character of the diffusion.

In Section 3 the results are discussed and a procedure outlined for a more rigorous analysis of cross effects, starting from the kinetic Boltzmann equation for the swarm particle phase space distribution function, and avoiding the density gradient expansion implicit in the hydrodynamic theory of Section 2.

2. Theory

(a) Turbulent Gas

Properties of the gas, including the statistical properties of the turbulence, are assumed known. The swarm behaves as a passive additive, i.e. it does not affect gas properties in any significant way. The aim is to express swarm characteristics in terms of this given information.

Let the gas have a turbulently fluctuating velocity field $\boldsymbol{u}(\boldsymbol{r},t)$ with zero mean, i.e.

$$\langle\, \boldsymbol{u}(\boldsymbol{r},t)\,\rangle = 0\,, \tag{1}$$

where $\langle \dots \rangle$ denotes an average over fluctuations. If the turbulence is homogeneous and stationary, then the autocorrelation function has the property

$$\langle\, u_i(\boldsymbol{r},t)\, u_j(\boldsymbol{r}',t')\,\rangle = B_{ij}(\boldsymbol{r}'-\boldsymbol{r},t'-t) \qquad (i,j=1,2,3)\,, \tag{2}$$

for any two points $\boldsymbol{r},\boldsymbol{r}'$ and times t,t'. The tensor $\mathbf{B}(\boldsymbol{\rho},t)$ and/or its Fourier transform,

$$\boldsymbol{\Phi}(\boldsymbol{k},\tau) = \int \exp(-i\boldsymbol{k}\,.\,\boldsymbol{\rho})\,\mathbf{B}(\boldsymbol{\rho},\tau)\,\mathrm{d}\boldsymbol{\rho}\,, \tag{3}$$

are assumed given, although it is not necessary to know the explicit form in the following analysis. We also assume that the gas behaves like an incompressible fluid, i.e.

$$\nabla\,.\,\boldsymbol{u} = 0\,, \tag{4}$$

and that the turbulence is isotropic. The latter assumption implies the following tensor structure for the transformed autocorrelation function:

$$\boldsymbol{\Phi}(\boldsymbol{k},\tau) = E(k,\tau)\,(\mathbf{1}-\hat{\boldsymbol{k}}\hat{\boldsymbol{k}})\,, \tag{5}$$

where $E(k,\tau)$ is the spectral energy density in $\boldsymbol{k}$-space, $\hat{\boldsymbol{k}}$ is a unit vector and $\mathbf{1}$ is the unit tensor. These results are well known (Monin and Yaglom 1971; Tatarski 1967). Fluctuations in the gas density N are assumed negligible.

(b) Swarm Particles

It is assumed that swarm particles are very sparse, with density $n \ll N$, and that mutual interaction between them is negligible. Initially, a pulse of n_0 particles is released from the origin, i.e.

$$n(\boldsymbol{r},0) = n_0\,\delta(\boldsymbol{r})\,. \tag{6}$$

Thus, in the first stage of development (the 'kinetic regime') the swarm particle density gradient is large and time evolution is rapid. If the gas were quiescent, the pulse would smooth out rapidly in a few collision times and in the 'hydrodynamic regime' Fick's law would hold:

$$n(\boldsymbol{v} - \boldsymbol{u}) = K\boldsymbol{E} - \mathbf{D}\,.\nabla n\,. \tag{7}$$

Here $\boldsymbol{v}(\boldsymbol{r},t)$ is the velocity of the swarm fluid, K is the mobility coefficient and

$$\mathbf{D} = D_{\perp}\mathbf{1} + (D_{||} - D_{\perp})\hat{\boldsymbol{E}}\hat{\boldsymbol{E}} \tag{8}$$

denotes the diffusion tensor. The usual supposition (implicit rather than explicit) is that even if the gas is turbulent, the hydrodynamic representation (7) holds. This is correct as long as the turbulent length and time scales are large compared with their microscopic counterparts. If not, the swarm is perpetually constrained to remain in the kinetic regime and an alternative analysis must be sought. This question is addressed further in Section 3.

The equation of continuity for the swarm is

$$\frac{\partial n}{\partial t} + \nabla\,.(n\boldsymbol{v}) = 0 \tag{9}$$

and, together with (7), this yields the diffusion equation,

$$\frac{\partial n}{\partial t} + \boldsymbol{u}\,.\nabla n = -\boldsymbol{v}_{\mathrm{dr}}\,.\nabla n + \mathbf{D} : \nabla\nabla n\,, \tag{10}$$

where

$$\boldsymbol{v}_{\mathrm{dr}} = K\mathbf{E} \tag{11}$$

is the drift velocity. Our task is to solve (10), given the velocity field $\boldsymbol{u}(\boldsymbol{r},t)$, the initial condition (6) and appropriate boundary conditions. In this work we take an infinite medium, with n and its derivatives all vanishing at large distances from the origin.

(c) Solution of Diffusion Equation

The Fourier–Laplace transform of $n(\boldsymbol{r},t)$ is

$$\bar{n}(\boldsymbol{k},\omega) = \int_0^{\infty} \mathrm{d}t \int \mathrm{d}\boldsymbol{r}\, \exp[-i(\boldsymbol{k}\,.\boldsymbol{r} - \omega t)]\, n(\boldsymbol{r},t)\,, \tag{12}$$

the $\boldsymbol{r}$-integration being over all space and the Laplace transform coordinate ω differing by a factor i from more common representations, following the convention of wave propagation theory (Sessler 1967). Application of this dual transformation to the diffusion equation (10) and using the initial condition (6) yields

$$\bar{n}(\boldsymbol{k},\omega) = \bar{n}_{\mathrm{m}}(\boldsymbol{k},\omega) + \bar{g}(\boldsymbol{k},\omega)\,\boldsymbol{k}\,.\,\bar{\boldsymbol{\Gamma}}(\boldsymbol{k},\omega)\,, \tag{13}$$

where

$$\bar{n}_{\mathrm{m}}(\boldsymbol{k},\omega) = in_0/(\omega - \omega_{\boldsymbol{k}}^{(0)}) \tag{14}$$

is the density transform that would arise if only molecular transport were operative,

$$\bar{g}(\boldsymbol{k},\omega) \equiv (\omega - \omega_{\boldsymbol{k}}^{(0)})^{-1}\,, \tag{15}$$

$$\bar{\boldsymbol{\Gamma}}(\boldsymbol{k},\omega) \equiv \int_0^\infty \mathrm{d}t \int \mathrm{d}\boldsymbol{r}\, \exp[-i(\boldsymbol{k}\,.\,\boldsymbol{r} - \omega t)]\, n(\boldsymbol{r},t)\, \boldsymbol{u}(\boldsymbol{r},t)\,, \tag{16}$$

and

$$\omega_{\boldsymbol{k}}^{(0)} = \boldsymbol{k}\,.\,\boldsymbol{v}_{\mathrm{dr}} - i\mathbf{D} : \boldsymbol{kk} \tag{17}$$

is the hydrodynamic 'dispersion relation'.

The Fourier transform of density is found by inverting the Laplace transform,

$$\tilde{n}(\boldsymbol{k},t) \equiv \int \mathrm{d}\boldsymbol{r}\, \exp(-i\boldsymbol{k}\,.\,\boldsymbol{r})\, n(\boldsymbol{r},t) \tag{18a}$$

$$= (2\pi)^{-1} \int_{\mathrm{C}} \mathrm{d}\omega\, \exp(-i\omega t)\, \bar{n}(\boldsymbol{k},\omega)\,, \tag{18b}$$

where the contour C lies above the singularities of $\bar{n}(\boldsymbol{k},\omega)$ in the complex ω-plane. From (13)–(17) it thus follows that

$$\tilde{n}(\boldsymbol{k},t) = \tilde{n}_{\mathrm{m}}(\boldsymbol{k},t) + \int_0^t \mathrm{d}t_1\, \tilde{g}(\boldsymbol{k},t-t_1)\, \boldsymbol{k}\,.\,\tilde{\boldsymbol{\Gamma}}(\boldsymbol{k},t_1)\,, \tag{19}$$

where

$$\tilde{n}_{\mathrm{m}}(\boldsymbol{k},t) = n_0\, \exp[-(i\boldsymbol{k}\,.\,\boldsymbol{v}_{\mathrm{dr}} + \boldsymbol{kk} : \mathbf{D})t]\,, \tag{20}$$

$$\tilde{g}(\boldsymbol{k},t) = -i\, \exp[-(i\boldsymbol{k}\,.\,\boldsymbol{v}_{\mathrm{dr}} + \boldsymbol{kk} : \mathbf{D})t]\,, \tag{21}$$

$$\tilde{\boldsymbol{\Gamma}}(\boldsymbol{k},t) = \int \mathrm{d}\boldsymbol{r}\, \exp(-i\boldsymbol{k}\,.\,\boldsymbol{r}) n(\boldsymbol{r},t) \boldsymbol{u}(\boldsymbol{r},t) \tag{22a}$$

$$= (2\pi)^{-3} \int \mathrm{d}\boldsymbol{k}_1\, \tilde{n}(\boldsymbol{k}-\boldsymbol{k}_1,t)\, \tilde{\boldsymbol{u}}(\boldsymbol{k}_1,t) \tag{22b}$$

and

$$\tilde{\boldsymbol{u}}(\boldsymbol{k},t) \equiv \int \mathrm{d}\boldsymbol{r}\, \exp(-i\boldsymbol{k}\,.\,\boldsymbol{r})\, \boldsymbol{u}(\boldsymbol{r},t)\,. \tag{23}$$

Equation (19) constitutes an integral equation for $\tilde{n}(\boldsymbol{k},t)$ and cannot be solved analytically for a general turbulent velocity field $\boldsymbol{u}(\boldsymbol{r},t)$. However, for weak turbulence an iterative solution can be effected. To first order we find, after averaging and making use of (1), (2) and (3),

$$\langle \tilde{n}(\boldsymbol{k},t) \rangle = \tilde{n}_{\mathrm{m}}(\boldsymbol{k},t) + (2\pi)^{-3} \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2\, \tilde{g}(\boldsymbol{k},t-t_1)\, \tilde{g}(\boldsymbol{k}-\boldsymbol{k}_1,t_1-t_2) \times \tilde{n}_{\mathrm{m}}(\boldsymbol{k},t_2)\, \boldsymbol{kk} : \boldsymbol{\Phi}(\boldsymbol{k}_1,t_1-t_2)\,. \tag{24}$$

Note that the density distribution due to molecular transport alone, $\tilde{n}_{\mathrm{m}}(\boldsymbol{r},t)$, is nonrandom, i.e. $\langle n_{\mathrm{m}} \rangle \equiv n_{\mathrm{m}}$. We also observe that the *form* of equation (19) and its solution (24) actually have a generality extending beyond the present discussion, and arise in the full kinetic theory treatment, albeit with different $\tilde{g}(\boldsymbol{k},\omega)$ and $\tilde{n}_{\mathrm{m}}(\boldsymbol{k},\omega)$ (see Section 3). Substitution of (20) and (21) into (24) yields

$$\begin{aligned}\langle \tilde{n}(\boldsymbol{k},t) \rangle &= \tilde{n}_{\mathrm{m}}(\boldsymbol{k},t) \Big(1 - (2\pi)^{-3} \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \int \mathrm{d}\boldsymbol{k}_1 \\ &\qquad \times \exp[(i\boldsymbol{k}\,.\,\boldsymbol{v}_{\mathrm{dr}} - (\boldsymbol{k}_1\boldsymbol{k}_1 - 2\boldsymbol{k}\boldsymbol{k}_1) : \mathbf{D})\,(t_1-t_2)]\, \boldsymbol{kk} : \boldsymbol{\Phi}(\boldsymbol{k}_1,t_1-t_2)\Big) \\ &= \tilde{n}_{\mathrm{m}}(\boldsymbol{k},t) \Big(1 - (2\pi)^{-3} \int_0^t \mathrm{d}\tau\,(t-\tau) \int \mathrm{d}\boldsymbol{k}_1 \\ &\qquad \times \exp[(i\boldsymbol{k}_1\,.\,\boldsymbol{v}_{\mathrm{dr}} - (2\boldsymbol{k}\boldsymbol{k}_1 - \boldsymbol{k}_1\boldsymbol{k}_1) : \mathbf{D})\tau]\, \boldsymbol{kk} : \boldsymbol{\Phi}(\boldsymbol{k}_1,\tau)\Big)\,. \end{aligned} \tag{25}$$

(d) Evaluation of Moments

There are two moments of the density distribution that are of physical interest, namely the position of the centroid,

$$\boldsymbol{r}_{\mathrm{c}} = \frac{1}{n_0} \int \mathrm{d}\boldsymbol{r}\, \boldsymbol{r}\, \langle n(\boldsymbol{r},t) \rangle\,, \tag{26}$$

and the dispersion about the centroid,

$$\boldsymbol{\sigma} = \frac{1}{n_0} \int \mathrm{d}\boldsymbol{r}\, (\boldsymbol{r}-\boldsymbol{r}_{\mathrm{c}})(\boldsymbol{r}-\boldsymbol{r}_{\mathrm{c}})\, \langle n(\boldsymbol{r},t) \rangle\,. \tag{27}$$

These can be evaluated directly from the Fourier transform (25), using the following identities,

$$\int \mathrm{d}\boldsymbol{r}\, \boldsymbol{r}\, \langle n(\boldsymbol{r},t) \rangle = i \frac{\partial \langle \tilde{n} \rangle}{\partial \boldsymbol{k}}\bigg|_{\boldsymbol{k}=0}\,, \tag{28a}$$

$$\int \mathrm{d}\boldsymbol{r}\, \boldsymbol{r}\boldsymbol{r}\, \langle n(\boldsymbol{r},t) \rangle = i \frac{\partial^2 \langle \tilde{n} \rangle}{\partial \boldsymbol{k}\, \partial \boldsymbol{k}}\bigg|_{\boldsymbol{k}=0}\,, \tag{28b}$$

which follow from differentiation of (18a) with respect to $\boldsymbol{k}$. Evidently we need evaluate $\tilde{n}(\boldsymbol{k},t)$ only up to second order in $\boldsymbol{k}$ to obtain these moments.

Since the integral term on the right-hand side of (25) is $O(k^2)$, it makes no contribution to the centroid position and it therefore follows from (26), (28a) and (20) that

$$\boldsymbol{r}_{\mathrm{c}} = \frac{i}{n_0}\frac{\partial \tilde{n}_{\mathrm{m}}}{\partial \boldsymbol{k}}\bigg|_{k=0} = \boldsymbol{v}_{\mathrm{dr}}t\,. \tag{29}$$

Thus the position of the swarm centroid is unaffected by turbulent fluctuations in the background gas. On the other hand, the second-order moment (28b), and hence the dispersion (27), is obviously affected by the turbulent term in (25), and we find

$$\begin{aligned}\boldsymbol{\sigma} = & -\frac{1}{n_0}\frac{\partial^2 \tilde{n}_{\mathrm{m}}}{\partial \boldsymbol{k}\,\partial \boldsymbol{k}}\bigg|_{k=0} - \boldsymbol{v}_{\mathrm{dr}}\boldsymbol{v}_{\mathrm{dr}}t^2 \\ & + 2(2\pi)^{-3}\int_0^t \mathrm{d}\tau\,(t-\tau)\int \mathrm{d}\boldsymbol{k}_1\,\exp[(i\boldsymbol{k}_1\,.\,\boldsymbol{v}_{\mathrm{dr}} - \boldsymbol{k}_1\boldsymbol{k}_1 : \mathbf{D})\tau]\,\boldsymbol{\Phi}(\boldsymbol{k}_1,\tau)\,.\end{aligned}$$

From (20) it can be shown that

$$-\frac{1}{n_0}\frac{\partial^2 \tilde{n}_{\mathrm{m}}}{\partial \boldsymbol{k}\,\partial \boldsymbol{k}}\bigg|_{k=0} = (2\mathbf{D} - \boldsymbol{v}_{\mathrm{dr}}\boldsymbol{v}_{\mathrm{dr}}t)t$$

and hence

$$\boldsymbol{\sigma} = 2\mathbf{D}t + 2(2\pi)^{-3}\int_0^t \mathrm{d}\tau\,(t-\tau)\int \mathrm{d}\boldsymbol{k}_1\,\exp[(i\boldsymbol{k}_1\,.\,\boldsymbol{v}_{\mathrm{dr}} - \boldsymbol{k}_1\boldsymbol{k}_1 : \mathbf{D})\tau]\,\boldsymbol{\Phi}(\boldsymbol{k}_1,\tau)\,. \tag{30}$$

The first term on the right-hand side corresponds to molecular diffusion, while the integral includes both a turbulent diffusion term and cross effects. Equation (30) is actually the complete specification of dispersion in a turbulent gas, for given molecular transport coefficients $\boldsymbol{v}_{\mathrm{dr}}$ and $\mathbf{D}$ and specified turbulent autocorrelation function $\boldsymbol{\Phi}(\boldsymbol{k},\tau)$. We now consider explicit, but approximate, evaluation of the integral.

(e) Dispersion Tensor

We now evaluate the integral (30) for times short on the turbulent scale but long on the molecular scale, assuming such a separation to exist. Firstly, we assume that the argument of the exponential is small for all times τ over which $\boldsymbol{\Phi}(\boldsymbol{k},\tau)$ is appreciable, so that

$$\begin{aligned}\boldsymbol{\sigma} = 2\mathbf{D}t + 2(2\pi)^{-3}\bigg(&\int_0^t \mathrm{d}\tau\,(t-\tau)\int \mathrm{d}\boldsymbol{k}_1\,\boldsymbol{\Phi}(\boldsymbol{k}_1,\tau) \\ & \times [1 + (i\boldsymbol{k}_1\,.\,\boldsymbol{v}_{\mathrm{dr}} - \boldsymbol{k}_1\boldsymbol{k}_1 : \mathbf{D})\tau + O(\tau^2)]\bigg)\,.\end{aligned} \tag{31}$$

For isotropic turbulence, with $\boldsymbol{\Phi}(\boldsymbol{k},\tau)$ given by (5), all integrals of odd functions of $\boldsymbol{k}$ with the autocorrelation function must vanish identically and, in particular,

$$\int \mathrm{d}\boldsymbol{k}_1\, \boldsymbol{k}_1 \boldsymbol{.} \boldsymbol{v}_{\mathrm{dr}}\, \boldsymbol{\Phi}(\boldsymbol{k}_1,\tau) \equiv 0\,. \tag{32}$$

Moreover, as can readily be shown from the definitions (2) and (3), for isotropic turbulence

$$(2\pi)^{-3} \int \mathrm{d}\boldsymbol{k}_1\, \boldsymbol{\Phi}(\boldsymbol{k}_1,\tau) \equiv \tfrac{1}{3}\mathbf{1}\langle\, \boldsymbol{u}(\boldsymbol{r},0)\boldsymbol{.}\boldsymbol{u}(\boldsymbol{r},\tau)\,\rangle\,. \tag{33}$$

Thus (31) simplifies to

$$\boldsymbol{\sigma} = 2\mathbf{D}t + \tfrac{2}{3}\mathbf{1} \int_0^t \mathrm{d}\tau\,(t-\tau)\,\langle\, \boldsymbol{u}(\boldsymbol{r},0)\boldsymbol{.}\boldsymbol{u}(\boldsymbol{r},\tau)\,\rangle - \boldsymbol{\sigma}^*\,, \tag{34}$$

where the second term is the familiar turbulent dispersion integral (Taylor 1921),

$$\boldsymbol{\sigma}^* \equiv 2 \int_0^t \mathrm{d}\tau\,(t-\tau)\tau\,\boldsymbol{\Delta}(\tau) \tag{35}$$

accounts for cross effects, and

$$\boldsymbol{\Delta}(\tau) \equiv (2\pi)^{-3} \int \mathrm{d}\boldsymbol{k}_1\,(\boldsymbol{k}_1\boldsymbol{k}_1 : \mathbf{D})\,\boldsymbol{\Phi}(\boldsymbol{k}_1,\tau) \tag{36}$$

At short times $[t < \zeta^{-1}$, where ζ is defined in (40) below$]$ the cross term (35) is

$$\begin{aligned}\boldsymbol{\sigma}^*(t) &= 2 \int_0^t \mathrm{d}\tau\,(t-\tau)\tau\,[\boldsymbol{\Delta}(0) + \tau\boldsymbol{\Delta}'(0) + \ldots] \\ &= \tfrac{1}{3}t^3\,\boldsymbol{\Delta}(0) + O(t^4)\,.\end{aligned} \tag{37}$$

Using (5) and (8), it can be shown that

$$\boldsymbol{\Delta}(0) = \Delta_{\perp}\mathbf{1} + (\Delta_{||} - \Delta_{\perp})\,\hat{\boldsymbol{E}}\hat{\boldsymbol{E}}\,, \tag{38}$$

where

$$\Delta_{||} = \tfrac{1}{15}(4D_{\perp} + D_{||})\zeta^2\,, \tag{39a}$$

$$\Delta_{\perp} = \tfrac{1}{15}(3D_{\perp} + 2D_{||})\zeta^2\,, \tag{39b}$$

and ζ is the vorticity of the turbulent motion,

$$\zeta^2 \equiv \langle\, |\,\mathrm{curl}\,\boldsymbol{u}(\boldsymbol{r},0)\,|^2\,\rangle\,. \tag{40}$$

Thus (34), (37) and (39) together yield

$$\boldsymbol{\sigma}^* = \sigma_{\perp}\mathbf{1} + (\sigma_{||} - \sigma_{\perp})\,\hat{\boldsymbol{E}}\hat{\boldsymbol{E}}\,, \tag{41}$$

where

$$\sigma_\perp = 2D_\perp t + \tfrac{2}{3}\int_0^t \mathrm{d}\tau\,(t-\tau)\,\langle\,\boldsymbol{u}(\boldsymbol{r},0)\,.\,\boldsymbol{u}(\boldsymbol{r},\tau)\,\rangle - \tfrac{1}{45}(3D_\perp + 2D_{||})\zeta^2 t^3 + O(t^4)\,, \tag{42a}$$

$$\sigma_{||} = 2D_{||}t + \tfrac{2}{3}\int_0^t \mathrm{d}\tau\,(t-\tau)\,\langle\,\boldsymbol{u}(\boldsymbol{r},0)\,.\,\boldsymbol{u}(\boldsymbol{r},\tau)\,\rangle - \tfrac{1}{45}(4D_\perp + D_{||})\zeta^2 t^3 + O(t^4)\,. \tag{42b}$$

For zero field, or when diffusion is otherwise isotropic, $D_\perp = D_{||} \equiv D$ and $\sigma_\perp = \sigma_{||} \equiv \sigma$, where

$$\sigma(t) = 2Dt + \tfrac{2}{3}\int_0^t \mathrm{d}\tau\,(t-\tau)\,\langle\,\boldsymbol{u}(\boldsymbol{r},0)\,.\,\boldsymbol{u}(\boldsymbol{r},\tau)\,\rangle - \tfrac{1}{9}D\zeta^2 t^3 + O(t^4)\,. \tag{43}$$

A similar expression was first obtained by Saffman (1960), who corrected an error in both the sign and the magnitude of the cross term derived by Townsend (1954). Note that this term acts to *reduce* the overall dispersion of the swarm. Equations (42) are the more general results allowing for anisotropic diffusion. Moffatt (1983) has given a lucid physical explanation of the origin of the cross effect in the case of strong turbulence, but the same qualitative considerations could be expected to apply to the present case. Note that the *difference*,

$$\sigma_{||} - \sigma_\perp = 2(D_{||} - D_\perp)t\,(1 + \zeta^2 t^2/90)\,, \tag{44}$$

between the dispersions along and transverse to the field is *enhanced* by turbulent fluctuations in the medium.

3. Further Discussion: A Possible Kinetic Theory Treatment

Non-hydrodynamic effects may be important at short times after emission from the source, close to boundaries or wherever the density $n(\boldsymbol{r},t)$ varies rapidly over distances of the order of a mean free path and/or times of the order of the mean free time between collisions (Kumar *et al.* 1980). We have already pointed out the strict inconsistency of using the diffusion equation (10) in the event that the swarm is maintained permanently in the 'kinetic' (non-hydrodynamic) regime by rapidly varying turbulent fluctuations in the gas. We outline below how one might proceed with a more general analysis of molecular–turbulent diffusion cross effects, based upon solution of the kinetic Boltzmann equation and avoiding the density gradient expansion implicit in (10).

In keeping with the spirit of 'passive additive' transport theory, all properties of the gas are assumed to be prescribed. The phase space distribution function f_0 of the gas molecules with velocity $\boldsymbol{c}_0$ is therefore assumed to be given, and for our purposes it can be taken to have the Maxwellian form

$$f_0(\boldsymbol{r},\boldsymbol{c}_0,t) = N\left(\frac{m_0}{2\pi k_\mathrm{B}T_0}\right)^{\frac{3}{2}} \exp\left\{-\,\frac{\frac{1}{2}m_0[\boldsymbol{c}_0 - \boldsymbol{u}(\boldsymbol{r},t)]^2}{k_\mathrm{B}T_0}\right\}, \tag{45}$$

where T_0 is the gas temperature, m_0 the mass of a gas molecule and k_B Boltzmann's constant.

On the other hand, the phase space distribution function f for the swarm particles must be found as the solution of Boltzmann's equation,

$$\left(\frac{\partial}{\partial t} + \boldsymbol{c}.\nabla + \frac{e\boldsymbol{E}}{m}.\frac{\partial}{\partial \boldsymbol{c}}\right) f = -J(f, f_0)\,, \tag{46}$$

where J is the well known linear swarm particle–neutral molecule collision operator (Kumar *et al.* 1980). Here e, m and $\boldsymbol{c}$ denote the charge, mass and velocity, respectively, of a swarm particle. Velocity moments of f yield quantities of physical interest, e.g. the number density,

$$n(\boldsymbol{r}, t) = \int \mathrm{d}\boldsymbol{c}\, f(\boldsymbol{r}, \boldsymbol{c}, t)\,. \tag{47}$$

The task then is to solve (46), find $n(\boldsymbol{r}, t)$ from (47) and hence evaluate the averages over density and turbulent fluctuations given in Section 2. All this must be done without any approximations involving assumptions about the smallness of terms on the left-hand side of (46): no density gradient expansion (Kumar *et al.* 1980) and certainly no Chapman–Enskog approximation (Chapman and Cowling 1970) can be made if one wishes to go beyond the level of the hydrodynamic theory presented in Section 2. This is indeed a formidable task if realistic representation of the collision term $J(f, f_0)$ is required, and there seems to be no alternative to numerical analysis from the outset in that case. However, if one is prepared to accept an approximate, qualitative analysis that sheds light on the physical nature of the cross effects operating in the kinetic regime, there is an alternative, which is currently under investigation.

We propose in future work to follow an earlier analysis (Robson 1975) of the full space–time evolution of a swarm in quiescent gas using the so-called Krook model collision term,

$$J(f, f_0) = \nu(f - f_{\mathrm{eq}})\,, \tag{48}$$

where

$$f_{\mathrm{eq}} = n(\boldsymbol{r}, t)\left(\frac{m}{2\pi k_B T_0}\right)^{\frac{3}{2}} \exp\left\{-\frac{\frac{1}{2}m[\boldsymbol{c} - \boldsymbol{u}(\boldsymbol{r}, t)]^2}{k_B T_0}\right\} \tag{49}$$

is the distribution function the swarm would have if it were in equilibrium with the gas molecules, and ν is a collision frequency. Laplace transformation in time and Fourier transformation in both configuration and velocity space will be employed. Viewed from the perspective of this model, a hydrodynamic description is possible, using equation (10), if the velocity field $\boldsymbol{u}$ varies only slowly during one collision time ν^{-1}. In that case the results of Section 2 are valid. Otherwise a kinetic theory analysis must be carried out. The details are lengthy and will be left to a subsequent paper. We merely observe here that an expression similar in form to (24) arises for the density, in which, however, the functions $\tilde{n}_{\mathrm{m}}(\boldsymbol{k},t)$ and $\tilde{g}(\boldsymbol{k},t)$ are by no means as simple as in (20) and (21) respectively.

In summary then, we have:

(i) obtained expressions for the dispersion of a charged particle swarm in a turbulent gas in directions parallel and transverse to an applied electrostatic field;
(ii) examined cross effects between molecular and turbulent diffusion and have shown that the result of Saffman (1960) is recovered in the limit of isotropic molecular diffusion; and
(iii) pointed out possible deficiencies in the hydrodynamic treatment of the swarm, and outlined a more rigorous alternative procedure involving solution of the kinetic Boltzmann equation.

Finally the reader is directed to another analysis of non-traditional swarm phenomena in an inhomogeneous time-varying gaseous medium subject to a sound wave (Robson and Paranjape 1992). As in that paper, we conclude that the experimental implications of the theory need further examination.

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