Aust. J. Phys., 1993, **46**, 503–21

Covariant Formulation of the Ponderomotive Force

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Abstract

A covariant and gauge-invariant theory of the ponderomotive force is developed for both unmagnetised and magnetised collisionless plasmas. The full expression for the force density exerted on the particle background by the waves (or the ponderomotive force in the general sense) is derived for both the unmagnetised and the magnetised cases. In the magnetised case, the 4-dimensional metric is projected into 'parallel' and 'perpendicular' components. The force density on the particle background then has a similar form to that for the unmagnetised case. The usual ponderomotive force is identified in the same way as in the unmagnetised case and has an equivalent form to that in the unmagnetised case when written in terms of the linear response tensor.

1. Introduction

The ponderomotive force is the pressure-like force density imposed on the plasma by high frequency waves with non-uniform amplitudes (Landau and Lifshitz 1960, p. 64; Nicholson 1983, p. 31; Melrose 1986, p. 76). It is a central ingredient in the approach to strong turbulence based on the Zakharov equations (e.g. Zakharov 1972; Goldman 1984). Generalisations of the ponderomotive force involve the inclusion of the effect of an ambient magnetic field (finite Larmor radii) and the inclusion of relativistic effects. Manheimer (1985) derived a covariant expression for the ponderomotive force using the covariant equation of motion in the unmagnetised case. Achterberg (1986) derived the covariant ponderomotive Hamiltonian for a magnetised plasma using a Lie transformation, introduced in this context by Grebogi and Littlejohn (1984). In the covariant formalism the 4-potential $A(x)$ is used to describe the self-consistent electric and magnetic fields. The definition of the ponderomotive force depends on the way the system is separated into background and wave subsystems (Dewar 1977). The covariant ponderomotive force formalism should be gauge-invariant, and the requirement of gauge invariance restricts the choice of separation into subsystems.

In this paper a covariant and gauge-independent theory of the ponderomotive force is presented. The formalism is based on a covariant distribution function

$$F(x,p) = 2m\delta(p^2 - m^2)H(p^0)\, f(\boldsymbol{p}, \boldsymbol{x}, t)\,,$$

0004-9506/93/040503$05.00

where p^0 is the time component of 4-momentum, $H(p^0)$ is the step function and $f(\boldsymbol{p}, \boldsymbol{x}, t)$ is the conventional distribution function defined in the 6-dimensional $\boldsymbol{x}$–$\boldsymbol{p}$ phase space. Natural units (the speed of light $c = 1$ and $\varepsilon_0\mu_0 = 1$) are used throughout the following discussion.

In Section 2, the covariant ponderomotive force for an unmagnetised plasma is presented in the oscillation centre formalism (Dewar 1977; Cary and Kaufman 1981), based on the canonical separation (Dewar 1977). In Section 3, the full expression for the average 4-force density imposed on the particles by the waves, *i.e.* the ponderomotive force based on the physical separation discussed by Dewar (1977), is derived. The covariant guiding-centre theory is outlined in Section 4. The covariant and gauge-independent ponderomotive force for a uniformly magnetised plasma is derived in Section 5. A comparison with 3-tensor notation is given in Section 6. A summary and discussion is given in Section 7.

2. Covariant Ponderomotive Force in an Unmagnetised Plasma

In the oscillation centre (OC) formalism (Dewar 1977; Cary and Kaufman 1981) the plasma system is separated into background and wave subsystems. In this section we specifically consider Dewar's (1977) canonical separation, in which the background subsystem consists of fictitious particles moving along the OC trajectory plus the slowly varying fields (Cary and Kaufman 1981), and the wave subsystem consists of the fast wave fields together with the fast motion of the particles of the plasma (Dewar 1977; Cary and Kaufman 1981). In this case the interaction force density between subsystems takes the simplest form that can be derived from the single-particle approach (by averaging the single-particle formula over the particle distribution function).

Two time scales are introduced to distinguish the fast motion from the slow motion: the characteristic fast time scale is $T_{\mathrm{f}} \approx 1/\omega$, where ω is the characteristic frequency of the high frequency waves, and the slow time scale is $T_{\mathrm{s}} \gg T_{\mathrm{f}}$. On the fast time scale a particle oscillates rapidly in response to the high frequency field, and on the slow time scale the motion of a particle is described in terms of the drift motion of its oscillation centre. The waves are described in terms of a modified plane wave of the form

$$A^{\mu}(x) = A^{\mu}(x,k)\exp[\mathrm{i}\Theta(x)] + \mathrm{c.c.}\,, \tag{1}$$

where $A^{\mu}(x,k)$ is the amplitude, which varies only on the slow time scale, and where $\Theta(x)$ is the wave eikonal and 'c.c.' denotes the complex conjugate. The wave 4-vector is $k^{\mu} = \partial^{\mu}\Theta(x)$, where the notation $\partial^{\mu} \equiv \partial/\partial x_{\mu}$ is used. The Greek indices run over four components, and the metric tensor $g^{\mu\nu}$ is diagonal with components (1, –1, –1, –1). The Einstein summation convention is implied whenever the indices are repeated. Latin indices denote spatial components.

The 4-momentum transfer between the background and wave subsystems is described by the covariant OC motion equation (Dewar 1977)

$$\frac{\mathrm{d}p_{\mathrm{c}}^{\mu}}{\mathrm{d}\tau} + \partial^{\mu}R = 0\,. \tag{2}$$

The covariant canonical momentum $p_{\rm c}^{\mu}$ is defined by

$$p_{\rm c}^{\mu} = -\left(g^{\mu\nu} - u^{\mu}u^{\nu}\right)\frac{\partial R}{\partial u^{\nu}} - u^{\mu}R\,, \tag{3}$$

with R the Lagrangian for the OC orbit, and u^{μ} the 4-velocity of the OC. In equations (2) and (3) and throughout the discussion in the remaining part of this Section, all physical variables referred to are those averaged over the fast time scale, called *OC variables*. Then the covariant OC Lagrangian may be written (see Appendix)

$$R = R^{(0)} + R^{(2)} + \cdots\,, \tag{4a}$$

$$R^{(0)} = -\,m - q u_{\mu}\langle A^{\mu}\rangle\,, \tag{4b}$$

$$R^{(2)} = \frac{q^{2}}{m}a^{\alpha\beta}A_{\alpha}A^{*}_{\beta}\,, \tag{4c}$$

where q and m are the charge and mass of the particles. The tensor $a^{\mu\nu}$ is defined by

$$a^{\mu\nu} = g^{\mu\nu} - \frac{k^{\mu}u^{\nu}}{ku} - \frac{k^{\nu}u^{\mu}}{ku} + k^{2}\frac{u^{\mu}u^{\nu}}{(ku)^{2}}\,, \tag{5}$$

with $ku = k^{\mu}u_{\mu}$ and $k^{2} = k^{\mu}k_{\mu}$. The following identity is implied by the equation of motion (2):

$$0 = \int \mathrm{d}p\, F(x,p)\left[\partial^{\mu}R + u^{\nu}\partial_{\nu}p_{\rm c}^{\mu} + \frac{\mathrm{d}u^{\nu}}{\mathrm{d}\tau}\frac{\partial p_{\rm c}^{\mu}}{\partial u^{\nu}}\right], \tag{6}$$

where $p^{\mu} = mu^{\mu}$ is the 4-momentum and $F(x,p)$ is the covariant OC distribution function, which is a generalisation of the covariant distribution to the OC coordinates (cf. the Appendix). Ignoring the surface terms in the p-integration and using the covariant form of the Vlasov equation (A11), one obtains

$$\partial_{\mu}\int \mathrm{d}p\, F(x,p)\left(Rg^{\mu\nu} + u^{\mu}p_{\rm c}^{\nu}\right) = \int \mathrm{d}p\, R\,\partial^{\nu}F(x,p)\,. \tag{7}$$

Based on the canonical separation (Dewar 1977), the energy momentum tensor for the background subsystem (the particle motion with its fast oscillation removed plus slowly varying fields) is identified as

$$T_{\rm b}^{\mu\nu}(x) = \int \mathrm{d}p\, u^{\mu}p_{\rm c}^{\nu}\, F(x,p)\,. \tag{8}$$

Using (8), equation (6) can be further cast into the form

$$\partial_{\mu}T_{\rm b}^{\mu\nu}(x) = -\int \mathrm{d}p\, F\,\partial^{\nu}R\,. \tag{9}$$

The right-hand side of equation (9) is denoted by

$$f_{\rm b}^{\nu}(x) = -\int {\rm d}p\, F\, \partial^{\nu} R\,. \tag{10}$$

We assume that $\langle A\rangle$ is small in the sense that $O(\langle A\rangle) \sim O(\langle\partial_{\mu}|A|^2\rangle)$. Then, keeping terms up to first order in the spatial–time derivative, one has

$$f_{\rm b}^{\mu}(x) = -\frac{q^2}{m}\int {\rm d}p\, F(x,p)\, \partial^{\mu}(a^{\alpha\beta}A_{\alpha}A_{\beta}^{*})\,. \tag{11}$$

Equation (11) is the covariant form of the ponderomotive force for an unmagnetised plasma (cf. Manheimer 1985). Equation (11) is the usual ponderomotive force in the sense that, apart from the integration over the particle distribution, it can be derived from the usual single-particle approach (see e.g. Manheimer 1985). The formalism (11), or more specifically, the canonical separation of the background and wave subsystems, is covariant and gauge-independent.

The generalisation of the treatment discussed in this Section to magnetised plasmas is obscured by the difficulty in deriving the magnetised counterpart of the OC Lagrangian (4a)–(4c). This is due to the fact that two fast time scales are involved: one is associated with gyration and the other with the waves. In the remaining sections, to derive the ponderomotive force for a magnetised plasma, we proceed in a different way by considering the average 4-momentum transfer between the particle background and wave subsystems under the physical separation (Dewar 1977).

3. Average 4-Force in an Unmagnetised Plasma

The average 4-force density on the background plasma is required for a description of the transfer of 4-momentum between the particle and wave subsystems. This 4-force density turns out not to be identical to the covariant ponderomotive force (11), and the relation between the two can be understood in terms of the different separation into background and wave subsystems. The average 4-force density imposed on the particles by the waves can be obtained from the exact equation of motion, which is the same as equation (2) except that R is the exact Lagrangian defined by (A4) in the Appendix, and the canonical 4-momentum p_c corresponds to that derived from the Lagrangian (A4). By analogy with the discussion in the previous section, using equation (6) with R the exact Lagrangian, one has the identity

$$\left\langle \partial_{\nu}\int {\rm d}p\, F(p,x) m u^{\mu}u^{\nu}\right\rangle = \langle F^{\mu\nu}(x)\, J_{\nu}(x)\rangle\,, \tag{12}$$

where the angle brackets represent the local average (Dewar 1977) and the Maxwell tensor $F^{\mu\nu}$ is defined by

$$F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}\,. \tag{13}$$

In equation (12), the expansion

$$J = J^{(0)} + J^{(1)} + J^{(2)} + \cdots \tag{14}$$

is inserted, so that $J^{(1)}(x)$ is the induced current to first order in $A(x)$. The 4-velocity can be written as an average part $\bar{u}^\mu$ plus the quiver velocity u^μ_{Q} (Bezzerides *et al.* 1977), i.e.

$$u^\mu = \bar{u}^\mu + u^\mu_{\mathrm{Q}}\,, \tag{15}$$

$$u^\mu_{\mathrm{Q}} = (g^{\mu\nu} - \bar{u}^\mu\bar{u}^\nu)\dot{\xi}_\nu + \cdots\,, \tag{16}$$

where the remaining terms are of higher order than $O(A)$, and $\dot{\xi} = \mathrm{d}\xi/\mathrm{d}\bar{\tau}$, with ξ the displacement due to the fast oscillation and $\bar{\tau}$ the averaged proper time (cf. Appendix). Since the local averaging and differentiation commute asymptotically (Dewar 1977), the left-hand side of (12) implies

$$\begin{aligned}\partial_\mu\Big\langle\int \mathrm{d}p\,F(p,x)mu^\mu u^\nu\Big\rangle &= \partial_\mu\int \mathrm{d}\bar{p}\,F(\bar{p},\bar{x})m\bar{u}^\mu\bar{u}^\nu + \partial_\mu\Big\langle\int \mathrm{d}\bar{p}\,F(\bar{p},\bar{x})m\Big\{\dot{\xi}^\mu\dot{\xi}^\nu \\ &\quad + \bar{u}^\mu\dot{\xi}^\nu\bar{u}\dot{\xi} + \bar{u}^\nu\dot{\xi}^\mu\bar{u}\dot{\xi} - \tfrac{1}{2}\bar{u}^\mu\bar{u}^\nu\big[\dot{\xi}^2 - 3(\bar{u}\dot{\xi})^2\big]\Big\}\Big\rangle\,. \end{aligned} \tag{17}$$

Denoting the first term by f^ν_{B} and using equations (12) and (17), one then has

$$f^\nu_{\mathrm{B}} \equiv \partial_\mu\int \mathrm{d}\bar{p}\,F(\bar{p},\bar{x})m\bar{u}^\mu\bar{u}^\nu = \partial_\mu T^{\mu\nu}_{\mathrm{Q}} + f^\nu_{\mathrm{L}}(\bar{x})\,, \tag{18}$$

with

$$\begin{aligned} T^{\mu\nu}_{\mathrm{Q}} = -\Big\langle\int \mathrm{d}p\,F(p,x)m\Big[\dot{\xi}^\mu\dot{\xi}^\nu + u^\mu\dot{\xi}^\nu u\dot{\xi} \\ + u^\nu\dot{\xi}^\mu u\dot{\xi} - \tfrac{1}{2}\big(\dot{\xi}^2 - 3(u\dot{\xi})^2\big)u^\mu u^\nu\Big]\Big\rangle\,, \end{aligned} \tag{19}$$

$$f^\mu_{\mathrm{L}}(x) = \langle F^{\mu\nu}J^{(1)}_\nu\rangle\,. \tag{20}$$

In (19) and (20), and in the following discussion, the bar on the variables x and u is omitted. The average stress tensor (19) is due to the particle quiver motion and f^μ_{L} is the average Lorentz force density. Equation (18) can be interpreted as the average force density exerted on the particles by the waves. By solving the covariant equation of motion for a particle in a perturbed field, one has

$$\dot{\xi}^\mu = -\frac{q}{m}G^{\mu\nu}A_\nu(x,k)\exp(-\mathrm{i}kx) + \mathrm{c.c.}\,, \tag{21}$$

with

$$G^{\mu\nu} = g^{\mu\nu} - \frac{k^\mu u^\nu}{ku}\,. \tag{22}$$

Equation (21) can also be derived by another method set out in the Appendix [see equation (A21)]. One can simplify (19) using the identity

$$u_\mu G^{\mu\nu} = 0\,.$$

Using equation (21) and taking the local average of (19), one finds

$$T_{\mathrm{Q}}^{\nu\mu} = -\alpha^{\mu\alpha\nu\beta} A_\alpha^* A_\beta\,, \tag{23}$$

with

$$\alpha^{\mu\alpha\nu\beta} = \frac{q^2}{m} \int \mathrm{d}p\, F(p) \Big[G^{\mu\alpha} G^{\nu\beta} + G^{\nu\alpha} G^{\mu\beta} - G^{\rho\alpha} G_\rho{}^{\beta} u^\mu u^\nu \Big] . \tag{24}$$

In deriving equation (24), the zeroth-order distribution function is approximated by $F(p)$. [We assume that the x-dependence of the zeroth-order distribution is due to the contribution of terms of second order in $A(x,k)$.] In an unmagnetised plasma, the Fourier transform (over the fast time scale), $J^{(1)}(x,k)$, of the linear current is identified using the linearised covariant Vlasov equation:

$$J_\nu^{(1)}(x,k) = q \int \mathrm{d}p\, u_\nu\, F^{(1)}(x,p,k)\,, \tag{25}$$

$$F^{(1)}(x,p,k) = \sum_{n=0}^{\infty} \Big[-\frac{\mathrm{i}}{ku} u\partial \Big]^n \Big[qG^{\mu\nu} A_\nu(x,k) - \frac{\mathrm{i}q}{ku} F^{\mu\nu}(x,k) u_\nu \Big] \frac{\partial F}{\partial p^\mu} . \tag{26}$$

The following condition needs to be satisfied (Cary and Kaufman 1981):

$$|u\partial\Phi(x,k)| \ll |ku\Phi(x,k)|\,, \tag{27}$$

where $\Phi(x,k)$ represents either $A^\mu(x,k)$ or $\partial^\mu A^\nu(x,k)$. Substituting the linear response current (25) into (20) and using (1), one obtains the average Lorentz force density:

$$f_{\mathrm{L}}^\mu(x) = \partial^\mu(\alpha^{\alpha\beta} A_\alpha^* A_\beta) - \partial^\nu \Big(k^\mu \frac{\partial \alpha^{\alpha\beta}}{\partial k^\nu} A_\alpha^* A_\beta \Big) - \Big[\partial_\alpha(\alpha^{\alpha\beta} A^{*\mu} A_\beta) + \mathrm{c.c.} \Big] . \tag{28}$$

Substituting (23) and (28) into (18), the resulting expression for the average force density exerted on the particles by the waves is

$$\begin{aligned} f_{\mathrm{B}}^\mu(x) \;&=\; \partial^\mu(\alpha^{\alpha\beta} A_\alpha^* A_\beta) - \partial^\nu \Big(k^\mu \frac{\partial \alpha^{\alpha\beta}}{\partial k^\nu} A_\alpha^* A_\beta \Big) \\ &\quad - \Big[\partial_\alpha(\alpha^{\alpha\beta} A^{*\mu} A_\beta) + \partial_\alpha(\alpha^{\alpha\beta} A^\mu A_\beta^*) + \partial_\nu \alpha^{\nu\alpha\mu\beta} A_\alpha^* A_\beta \Big] , \end{aligned} \tag{29}$$

where $\alpha^{\alpha\beta}$ is the linear response tensor, identified by writing (on neglecting x-derivative terms)

$$J_\nu^{(1)}(k) = \alpha_{\nu\rho}(k)\, A^\rho(x,k)\,, \tag{30}$$

and given by

$$\alpha^{\mu\nu}(k) = -\frac{q^2}{m} \int \mathrm{d}p\, F(p) \Big[g^{\mu\nu} - \frac{k^\mu u^\nu + k^\nu u^\mu}{ku} + k^2 \frac{u^\mu u^\nu}{(ku)^2} \Big] . \tag{31}$$

In deriving equation (29), and throughout the following calculation, only terms up to $O(A^2)$ are kept.

Note that because (20) represents the Lorentz force, equation (28) must be gauge-invariant. A gauge transformation is defined by

$$A^{\mu}(x,k) \to A^{\mu}(x,k) - \mathrm{i}k^{\mu}\psi(x,k) + \partial^{\mu}\psi(x,k)\,, \tag{32}$$

where $\psi(x,k)$ is any differentiable function of x. The gauge independence of the tensor $T_{\mathrm{Q}}^{\mu\nu}$ is guaranteed by the identity

$$k_{\nu}G^{\mu\nu} = 0.$$

Therefore, the resulting expression (29) is explicitly gauge-independent.

In comparison with the formalism derived in the previous section [cf. equation (11) of Section 2], the first term on the right-hand side of (29) is the usual covariant gauge-independent ponderomotive force (Manheimer 1985). The expression (29) can also be identified as the (generalised) ponderomotive force provided that the background subsystem is defined to consist of the average motion of the particles, and the remaining part of the system is regarded as the wave subsystem. Such a separation is the physical split-up (Dewar 1977), since the right-hand side of (29) can be rewritten as the 4-divergence of the stress tensor that reproduces equation (229) of Dewar (1977) in the Lorentz gauge. The terms involving the time-derivative components in equation (29) represent time-dependent ponderomotive effects (Klíma and Petržílka 1978; Kentwell 1985). Other formalisms for the ponderomotive force in 3-tensor notation, such as those derived by Hora (1985), Barash and Karpman (1983) and Kentwell (1985), can be reproduced from equation (29).

4. Covariant Guiding Centre Theory

In order to derive the ponderomotive force for a magnetised plasma, a covariant guiding centre theory is presented here. The static magnetic field is defined in terms of the Lorentz invariant

$$B := \left[\tfrac{1}{2}F_{0}^{\mu\nu}F_{0\mu\nu}\right]^{\frac{1}{2}}. \tag{33}$$

This definition reproduces the static magnetic field B_0 in any frame with electric field $E_0 = 0$. Let such a frame, with its z-axis along $\boldsymbol{B}_0$, be denoted by $\mathcal{K}_0$.

When the plasma is magnetised, particles drift as a result of the familiar drift motions of the centre of gyration, as well as through the effects of the waves. Thus covariant guiding centre theory is required to take into account both the effect of the waves and the effect of a finite Larmor radius.

Let the particle orbits be described in terms of guiding-centre coordinates plus gyration components. Explicitly, a particle orbit is written as

$$x^{\mu} = x_{\mathrm{g}}^{\mu} + \zeta^{\mu}\,,$$

where $x_{\rm g}^{\mu}$ and ζ^{μ} are the guiding-centre coordinates and gyration, respectively. The time scale of the gyration satisfies the condition $T_{\rm g} \ll T_{\rm s}$. Then $A^{\mu}(x,k)$ and $\Theta(x)$ in (1) can be expanded about the guiding centre coordinates $x_{\rm g}$:

$$A^{\mu}(x_{\rm g}+\zeta,k) = A^{(0)\mu} + A^{(1)\mu} + \cdots , \tag{34a}$$

$$\Theta(x_{\rm g}+\zeta) = \Theta^{(0)} + \Theta^{(1)} + \cdots , \tag{34b}$$

with $A^{(0)\mu} = A^{\mu}(x_{\rm g})$ and $\Theta^{(0)} = \Theta(x_{\rm g})$; $A^{(1)\mu}$ and $\Theta^{(1)}$ are of first order in ζ. Since $A(x,k)$ is a slowly varying function, it suffices to consider the leading term in the expansion (34a) and up to the first-order terms in expansion (34b). In our discussion we are concerned with the functionals which depend only on $A(x)$ and its derivatives, as well as the 4-velocity. The gyro-average is usually done by putting all gyrophase dependence in the exponential and integrating over the gyrophase. Since the first-order term is retained only in the expansion of $\Theta(x)$ in ζ, the gyrophase dependence of $A(x)$ and its derivatives automatically appears in the exponential. The gyrophase dependence of the 4-velocity may be put into the exponential by writing it in terms of Bessel functions (cf. the discussion in the next section). For this purpose one first needs to separate the 4-velocity into 'perpendicular' and 'parallel' parts, where the 'perpendicular' part is associated with gyration. More generally, the metric tensor is separated into 'perpendicular' and 'parallel' parts (Melrose 1982)

$$g^{\mu\nu} = g_{\perp}^{\mu\nu} + g_{\parallel}^{\mu\nu} , \tag{35}$$

with respect to the definition of the static magnetic field (33). The matrix representation of $g^{\mu\nu}$ in the frame $\mathcal{K}_0$ is

$$g_{\perp}^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \tag{36}$$

$$g_{\parallel}^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{37}$$

Once defined in the frame $\mathcal{K}_0$, $g_{\parallel}^{\mu\nu}$ and $g_{\perp}^{\mu\nu}$ may be written down in any frame $\mathcal{K}$ by making the appropriate Lorentz transformation. In any frame, one has

$$g_{\parallel\alpha\beta}\, g_{\perp}^{\beta\sigma} = 0 .$$

Using (35)–(37) any 4-vector a^{μ} may be decomposed into $a^{\mu} = a_{\parallel}^{\mu} + a_{\perp}^{\mu}$, where the 4-vector $a_{\perp}^{\mu}$ is space-like. Writing $a_{\perp}^{2} = -a_{\perp}^{\mu} a_{\perp\mu} = -g_{\perp}^{\mu\nu} a^{\mu} a^{\nu} > 0$, the invariant $a_{\perp} > 0$ may be used to rewrite the perpendicular components in the form

$$a^{\mu} = (a_0, a_{\perp}\cos\psi, a_{\perp}\sin\psi, a_z)$$

in $\mathcal{K}_0$, where ψ is an appropriate angle.

In summary, the average over the gyration may be performed as follows. First choose a particular frame, say $\mathcal{K}_0$, in which the 4-velocity is

$$u^\mu = (u_0, u_\perp \cos\psi, u_\perp \sin\psi, u_z)\,.$$

In this case the angle ψ is the gyrophase angle: $\psi = \psi_0 + \eta\Omega t$, where $\eta = q/|q|$ is the sign of the particle charge and the invariant $\Omega = |q|B/m$ reduces to the nonrelativistic gyrofrequency in the frame $\mathcal{K}_0$. If functions like $u^\mu A^\nu(x)$ or $u^\mu \partial^\nu A^\alpha(x)$ are to be averaged, one first writes them in terms of Bessel functions to make all gyrophase angles appear only in the exponential. After averaging over the gyrophase angle ψ, the variables remaining in the theory are $u_\parallel^\mu$, $u_\perp$ and x_g^μ.

5. Magnetised Plasma

Using the covariant guiding centre theory developed in the previous section, one may generalise the unmagnetised formalism $f_{\rm B}^\mu$ to the magnetised case. The usual covariant gauge-independent ponderomotive force $f_{\rm p}^\mu$ may be identified in a similar way to the unmagnetised case. In the following discussion, it is assumed that the unperturbed distribution function is spatially homogeneous, and the plane-wave form (1) is assumed with $\Theta(x) = kx$. To obtain the average Lorentz force density, one considers the integral solution for the linearised covariant Vlasov equation for a magnetised plasma:

$$F^{(1)}(x,p) = -\int_{-\infty}^{\tau} d\tau' \exp(-ikx')\Big\{iku'G^{\mu\nu}(k,u')\,A_\mu(x',k) + qF^{\mu\nu}(x',k)u_\nu{}'\Big\}\frac{\partial F(p')}{\partial p'^\mu} \tag{38}$$

The tensor $G^{\mu\nu}(k,u)$ is defined by equation (22). The zeroth-order distribution function $F(p)$ corresponds to the unperturbed distribution. The integration path is along an unperturbed orbit, which is chosen in such way that it passes through the point (x,p). Therefore the coordinates and velocity at any point on the unperturbed orbit can be obtained from a given point (x,u) by the following equation:

$$\begin{aligned} x'^\mu - x^\mu &= [t^{\mu\nu}(0) - t^{\mu\nu}(\tau-\tau')]u_\nu\,, \\ u'^\mu &= \dot{t}^{\mu\nu}(\tau-\tau')u_\nu\,, \end{aligned} \tag{39}$$

with $x' = x'(x,p,\tau-\tau')|_{\tau'=\tau} = x$, $u' = u'(x,p,\tau-\tau')|_{\tau'=\tau} = u$ and $\dot{t}^{\mu\nu}(\tau) := dt^{\mu\nu}(\tau)/d\tau$. The matrix representation of $t^{\mu\nu}$ in $\mathcal{K}_0$ is

$$t^{\mu\nu}(\tau-\tau') = \frac{1}{\Omega}\begin{pmatrix} \Omega(\tau-\tau') & 0 & 0 & 0 \\ 0 & -\sin\Omega(\tau-\tau') & -\eta\cos\Omega(\tau-\tau') & 0 \\ 0 & \eta\cos\Omega(\tau-\tau') & -\sin\Omega(\tau-\tau') & 0 \\ 0 & 0 & 0 & -\Omega(\tau-\tau') \end{pmatrix} \tag{40}$$

It is convenient to define

$$R^{\mu\nu}(\sigma) = t^{\mu\nu}(0) - t^{\mu\nu}(\sigma)\,, \tag{41}$$

with $\sigma = \tau - \tau'$. Then the average Lorentz force density can be written

$$\begin{aligned} f_{\mathrm{L}}^{\mu} =& q^2 \int \mathrm{d}p\, F(p) \left(g_{\|\alpha\beta} \frac{\partial}{\partial p^{\beta}} + \frac{p_{\|\alpha}}{p_{\perp}} \frac{\partial}{\partial p_{\perp}} \right) \int_0^{\infty} \mathrm{d}\sigma\, \exp(-\mathrm{i}kRu) u A^*(x,k) \\ &\times \Bigg\{ (-\mathrm{i}k^{\mu}) \left[\mathrm{i}ku' G^{\alpha\nu}(k,u') A_{\nu}(x',k) + F^{\alpha\nu}(x',k) u'_{\nu} \right] \\ &+ \mathrm{i}ku' G^{\alpha\nu}(k,u') \frac{\partial A_{\nu}(x',k)}{\partial x_{\mu}} \Bigg\}\,. \end{aligned} \tag{42}$$

In equation (42), the assumed boundary conditions are $F(p_\|, p_\perp) \to 0$ for $p_\|^{\mu} \to \pm\infty$ or $p_\perp \to +\infty$. The local average is implied in (42). To simplify (42), we choose the particular frame $\mathcal{K}_0$ defined in Section 4, with the wavevector $\boldsymbol{k}$ in the xy plane. Using

$$u'^{\mu} = (u_0, u_\perp \cos(\Omega\sigma + \eta\psi), \eta u_\perp \sin(\Omega\sigma + \eta\psi), u_z)\,,$$

one has

$$u'^{\mu} \exp\left[-\mathrm{i}k_\perp \rho \sin(\Omega\sigma + \eta\psi) \right] = \sum_l \exp\left[-\mathrm{i}l(\Omega\sigma + \eta\psi) \right] \tilde{u}^{\mu}(l,k)\,,$$

where $\tilde{u}^{\mu}(l,k)$ in this special frame has the representation

$$\tilde{u}^{\mu} = [u^0 J_l, (lu_\perp / k_\perp \rho) J_l, -\mathrm{i}\eta u_\perp J'_l, u_z J_l], \qquad \rho = u_\perp/\Omega, \qquad J_l = J_l(k_\perp \rho)\,,$$

where $J_l(k_\perp\rho)$ is a Bessel function. The reality condition is

$$\tilde{u}^{*\alpha}(-l,-k) = \tilde{u}^{\alpha}(l,k)\,,$$

where $*$ denotes complex conjugation. Let $\Phi(x,k)$ represent either $A(x,k)$ or $\partial^{\mu} A^{\nu}(x,k)$. Then, by definition, $\Phi(x + Ru, k)$ is a slowly varying function of $x + Ru$, where u is the 4-velocity, k is the wave 4-vector associated with the high frequency waves, and $R^{\mu\nu} = t^{\mu\nu}(0) - t^{\mu\nu}(\sigma)$ is defined above. The function $\Phi(x + Ru, k)$ may be written as a slowly varying function of x by integrating over the σ-variable in Ru. However, $\Phi(x + Ru, k)$ as a function of $x + Ru$ is generally unknown. We proceed as follows: after repeated integration by parts, the leading-order approximation is

$$\begin{aligned} &\int \mathrm{d}\sigma\, \exp[\mathrm{i}(k_\| u_\| - l\Omega)\sigma]\, \Phi(x + Ru, k) \\ &\qquad = \frac{\mathrm{i}}{k_\| u_\| - l\Omega} \sum_{m=0}^{\infty} \left(\frac{-\mathrm{i}}{k_\| u_\| - l\Omega} u_\| \frac{\partial}{\partial x} \right)^m \Phi(x,k) + O\left(\frac{T_{\mathrm{g}}}{T_{\mathrm{s}}} \right), \end{aligned} \tag{43}$$

provided

$$\left| \rho \frac{\partial}{\partial x} \Phi(x,k) \right| \ll 1, \qquad \left| u_\| \frac{\partial}{\partial x} \Phi(x,k) \right| \ll \left| (k_\| u_\| - l\Omega) \Phi(x,k) \right|. \tag{44}$$

In the right-hand side of equation (43), and in the conditions (44), the variable x is referred to as the guiding-centre coordinates, where the subscript g is now omitted. The conditions (44) were first proposed by Cary and Kaufman (1981) in deriving the ponderomotive Hamiltonian for a magnetised plasma. The physical interpretation of the conditions is that the amplitude of the wave must vary slowly along an unperturbed orbit in order for the description to be valid. The conditions (44) also imply that the time scale associated with gyration must be small compared with the slow time scale.

By applying (43) to equation (42) it can be shown that terms like $u'_\perp \partial A(x',k)/\partial x'_\perp$ correspond to next higher order of the expansion; such terms are neglected in the following calculations. The gyrophase angle appears only in the exponential and in the velocity u. Upon integration over the gyrophase, one finally arrives at an expression for $f^\mu_{\rm L}$ similar to equation (28), except that the second term is now replaced by

$$-\partial^\nu\left(k^\mu \frac{\partial \alpha^{\alpha\beta}}{\partial k^\nu_\parallel} A^*_\alpha A_\beta\right),$$

and the response tensor corresponds to the gyrophase-averaged linear response tensor for the magnetised plasma:

$$\alpha^{\alpha\beta} = \frac{q^2}{m}\int \mathrm{d}p_\parallel\, p_\perp \mathrm{d}p_\perp\, F(p_\parallel, p_\perp)\bigg\{ g^{\alpha\beta}_\parallel - \sum_l \left(k_\parallel \frac{\partial}{\partial u_\parallel} + \frac{k_\parallel u_\parallel}{u_\perp}\frac{\partial}{\partial u_\perp}\right) \frac{\tilde{u}^{*\alpha}(l,k)\tilde{u}^\beta(l,k)}{k_\parallel u_\parallel - l\Omega}\bigg\}, \tag{45}$$

which, after carrying out the derivatives, becomes

$$\alpha^{\alpha\beta} = \frac{q^2}{m}\sum_l \int \mathrm{d}p_\parallel\, p_\perp \mathrm{d}p_\perp\, F(p_\parallel, p_\perp)\bigg\{ G^{\nu\alpha}(l,k,u_\parallel,u_\perp) t_{\nu\rho}(l,k,u_\parallel,u_\perp) \times G^{*\rho\beta}(l,k,u_\parallel,u_\perp)\bigg\}, \tag{46}$$

with

$$G^{\mu\nu}(l,k,u_\parallel,u_\perp) := g^{\mu\nu} J_l(k_\perp \rho) - \frac{k^\mu \tilde{u}^\nu(l,k)}{k_\parallel u_\parallel - l\Omega}. \tag{47}$$

In (45)–(47), one has

$$\tilde{u}^\mu(l,k) = [u_0 J_l, (l u_\perp / k_\perp \rho) J_l, -\mathrm{i}\eta u_\perp J'_l, u_z J_l]$$

in the $\mathcal{K}_0$ frame, and $t^{\mu\nu}(l,k,u_\parallel,u_\perp)\equiv t^{\mu\nu}(\omega_l/\Omega)$ is defined by (Melrose 1987)

$$t^{\mu\nu}(\omega_l)=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \dfrac{\omega_l^2}{\omega_l^2-\Omega^2} & \dfrac{i\eta\omega_l\Omega}{\omega_l^2-\Omega^2} & 0\\ 0 & -\dfrac{i\eta\omega_l\Omega}{\omega_l^2-\Omega^2} & \dfrac{\omega_l^2}{\omega_l^2-\Omega^2} & 0\\ 0 & 0 & 0 & -1\end{pmatrix}, \tag{48}$$

with

$$\omega_l=k_\parallel u_\parallel-l\Omega\,. \tag{49}$$

The absence of the perpendicular k^μ-derivative of the response tensor in the magnetised counterpart of the second term in $f^\mu_{\rm L}$ is due to the fact that only the leading-order terms are retained in the gyrophase expansion (34a). The condition for the validity of this approximation is that the gyro-radius be small compared with the slowly varying spatial scale.

The average stress tensor $T^{\mu\nu}_{\rm Q}$ due to the particle quiver motion can be calculated in a similar manner to the unmagnetised case. From the perturbed equation of motion for a particle in a uniform magnetic field, one has (Melrose 1987)

$$\dot\xi^\mu=\int_{-\infty}^{\tau}{\rm d}\tau'\,\dot t^{\mu\nu}(\tau-\tau')S_\nu(\tau'), \tag{50}$$

where

$$S^\mu(\tau')=\frac{{\rm i}q}{m}\exp[-{\rm i}kx'(\tau')]ku'(\tau')\,G^{\mu\nu}[k,u'(\tau')]\,A_\nu(x,k)+{\rm c.c.}\,, \tag{51}$$

with $x'(\tau')$, the unperturbed orbit, given by equations (39). In view of equations (40) and (48), the explicit evaluation of (50) gives

$$\dot\xi^\mu=-\frac{q}{m}\sum_l t^{\mu\alpha}(\omega_l)\,G_\alpha{}^\beta(l,k,u_\parallel,u_\perp)\,A_\beta\exp(-{\rm i}kx+{\rm i}k_\perp\rho\sin\eta\psi-{\rm i}\eta l\psi)+{\rm c.c}\,.$$

Therefore one has [cf. equations (23) and (24)]

$$\alpha^{\mu\alpha\nu\beta}=\frac{q^2}{m}\sum_l\int{\rm d}p_\parallel\,p_\perp{\rm d}p_\perp\,F(p_\parallel,p_\perp)\Big\{t^{*\mu\lambda}(\omega_l)\,G^{*\alpha}_\lambda(l)\,t^{\nu\sigma}(\omega_{l'})\,G_\sigma{}^\beta(l')$$
$$-\tfrac12\sum_{s,s'}t^{*\rho\lambda}(\omega_l)\,t_{\rho\sigma}(\omega_{l'})\,G_\lambda{}^{*\alpha}(l)\,G^{\sigma\beta}(l')\,\tilde u^\mu(s,k)\,\tilde u^{*\nu}(s',k)+{\rm c.c.}\Big\}, \tag{52}$$

with $l'=l-s+s'$ and $G^{\alpha\beta}(l)\equiv G^{\alpha\beta}(l,k,u_\parallel,u_\perp)$ given by (47). It follows that, on neglecting the second-order current, the force density exerted by the waves for a magnetised plasma has an equivalent form to (29), with $\alpha^{\mu\nu}$ and $\alpha^{\mu\alpha\nu\beta}$ now replaced by (45) and (52) respectively.

Similar to the unmagnetised case, as discussed in Section 3, the usual ponderomotive force for a magnetised plasma is identified as

$$f_{\rm p}^{\mu}(x) = \frac{q^2}{m}\int \mathrm{d}p_{\parallel}\, p_{\perp}\mathrm{d}p_{\perp}\, F(p_{\parallel}, u_{\perp})\left\{|A|^2 - \sum_l \left(k_{\parallel}\frac{\partial}{\partial u_{\parallel}} + ql\frac{\partial}{\partial \mu}\right)\frac{|H(l,k)|^2}{k_{\parallel}u_{\parallel} - l\Omega}\right\}, \quad (53)$$

with $\mu = mu_{\perp}^2/2B$ and where $H(l,k) = \tilde{u}(l,k)A$ is a Lorentz invariant. In the frame $\mathcal{K}_0$, one has

$$\tilde{u}(l,k)A = A_0 u_0 J_l - \frac{\Omega}{k_{\perp}}A_x l J_l - \mathrm{i}\eta u_{\perp} A_y J_l' - u_z A_z J_l\,. \quad (54)$$

Equation (53) can be considered as the covariant version of the ponderomotive force derived by Grebogi and Littlejohn (1984) for a uniformly magnetised plasma.

6. Ponderomotive Force in 3-Tensor Notation

In order to compare the covariant formalism derived in the earlier sections with the results given by other authors, we reformulate the covariant formalism in terms of 3-tensor notation. In 4-tensor notation the spatial components of the linear response tensor are obtained from

$$J^i(k) = \alpha^{ij}(k)\, A_j(k)\,,$$

where J^i is a contravariant 3-vector, which is numerically equal to the ith component of the 3-vector $\boldsymbol{J}$, and A_j is a covariant 3-vector, which is numerically equal to the jth component of the 3-vector $-\boldsymbol{A}$, where i and j run over x, y, z. In the 3-tensor notation all indices are written as subscripts, so that the linear response tensor is identified by writing

$$J_i(\mathbf{k},\omega) = \alpha_{ij}(\boldsymbol{k},\omega)A_j(\boldsymbol{k},\omega)\,. \quad (55)$$

We distinguish the 3-vector components from the contravariant 4-vector components by showing the variables $\mathbf{k}$ and ω as explicit arguments in the 3-tensor form. Comparing the two notations we have the following translations:

$$J^i(k) \rightarrow J_i(\boldsymbol{k},\omega)\,, \quad (56)$$

$$A_i(k) \rightarrow -A_i(\boldsymbol{k},\omega)\,, \quad (57)$$

$$\alpha^{ij}(k) \rightarrow -\alpha_{ij}(\boldsymbol{k},\omega)\,. \quad (58)$$

In 3-tensor notation, the equivalent dielectric tensor is defined as

$$K_{ij} = \delta_{ij} + \frac{1}{\omega^2\varepsilon_0}\alpha_{ij}\,. \quad (59)$$

As an example, for transverse waves in an isotropic plasma one has

$$K_{ij}^{\mathrm{T}} = \left(1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2}\right)\delta_{ij}\,, \tag{60}$$

where ω_{p} is the plasma frequency. In the nonrelativistic limit, using the equivalent dielectric tensor (60), the force density (29) for an unmagnetised plasma becomes

$$\boldsymbol{f}_{\mathrm{B}}(\boldsymbol{k},\omega) = -\varepsilon_0 \frac{\omega_{\mathrm{p}}^2}{\omega^2}\nabla_x |\boldsymbol{E}|^2\,, \tag{61}$$

where $\nabla_x = \partial/\partial \boldsymbol{x}$ and the temporal gauge is chosen. Alternatively, equation (61) can be derived from a single-particle approach (Melrose 1986); it is just the 3-vector component of the usual ponderomotive force defined by the contravariant 4-vector f_{p}^{μ}. Note that in (61) the time-dependent ponderomotive terms are absent for transverse waves (Kentwell 1985).

A second example is the case of longitudinal waves, with equivalent dielectric tensor then given by

$$K_{ij}^{\mathrm{L}} = K^{\mathrm{L}}\frac{k_i k_j}{|\boldsymbol{k}|^2}\,.$$

In the nonrelativistic limit K^{L} is given by

$$K^{\mathrm{L}} = 1 - \frac{q^2}{m}\int \mathrm{d}\boldsymbol{p}\,\frac{f(\boldsymbol{p})}{(\omega - \boldsymbol{k}\cdot\boldsymbol{v})}\,,$$

which can be obtained from (31). Then the force density (29) gives

$$\begin{aligned}\boldsymbol{f}_{\mathrm{B}} &= \varepsilon_0(K^{\mathrm{L}}-1)\nabla_x|\boldsymbol{E}|^2 + \boldsymbol{k}\frac{\partial[\varepsilon_0(K^{\mathrm{L}}-1)]}{\partial\omega}\frac{\partial}{\partial t}|\boldsymbol{E}|^2 \\ &\quad - \boldsymbol{k}\frac{\partial(\varepsilon_0 K^{\mathrm{L}})}{\partial \boldsymbol{k}}\cdot\nabla_x|\boldsymbol{E}|^2\,,\end{aligned} \tag{62}$$

where we again choose the temporal gauge. The first term in (62) is the usual ponderomotive force, and the second term is the time-dependent ponderomotive effect (Barash and Karpman 1983; Klíma and Petržílka 1978; Akama and Nambu 1981). The third term is the thermal correction (Barash and Karpman 1983; Kentwell 1985).

For a uniformly magnetised plasma with high frequency electromagnetic waves travelling in it, the 3-vector component of the usual ponderomotive force part may be derived as follows. Choose the x-, y- and z-axes along $\hat{\boldsymbol{k}}_\perp$, $\hat{\boldsymbol{b}}\times\hat{\boldsymbol{k}}_\perp$ and $\hat{\boldsymbol{b}} = \boldsymbol{B}/B$, respectively. In the temporal gauge, $H(l,k)$ in equation (53) becomes

$$H(l,\boldsymbol{k},\omega) = -\boldsymbol{A}\cdot\left[J_l u_z\hat{\boldsymbol{b}} + \frac{\Omega}{k_\perp}\left(lJ_l\hat{\boldsymbol{k}}_\perp + 2\mathrm{i}\eta\mu\frac{\partial J_l}{\partial\mu}\hat{\boldsymbol{b}}\times\hat{\boldsymbol{k}}_\perp\right)\right]\,. \tag{63}$$

The usual ponderomotive force for a magnetised plasma is then

$$f_{\mathrm{P}}^{i}=\frac{q^{2}}{m}\partial^{i}\int\frac{\mathrm{d}k}{(2\pi)^{4}}\int\mathrm{d}p_{\|}\,p_{\perp}\mathrm{d}p_{\perp}\,F(p_{\|},p_{\perp})$$
$$\times\left\{-|\boldsymbol{A}|^{2}-\sum_{l}\left(k_{z}\frac{\partial}{\partial u_{z}}+ql\frac{\partial}{\partial\mu}\right)\frac{|H(l,\boldsymbol{k},\omega)|^{2}/\gamma}{\omega-k_{z}v_{z}-l\Omega_{0}}\right.$$
$$\left.+\sum_{l}\frac{\omega^{2}}{(\omega-k_{z}v_{z}-l\Omega_{0})^{2}}|H(l,\boldsymbol{k},\omega)|^{2}/\gamma^{2}\right\},\tag{64}$$

with $\Omega_0 = |q|B/\gamma m$ being the relativistic gyrofrequency. For $\omega \to \infty$ the last term reduces to $|u_z A_z|^2 + \frac{1}{2}u_\perp^2(|A_x|^2 + |A_y|^2)$, which is consistent with Grebogi and Littlejohn's (1984) result.

7. Conclusions

In this paper we present a covariant, gauge independent ponderomotive force for both magnetised and unmagnetised Vlasov plasmas. In Section 2, the covariant ponderomotive force was derived in terms of the OC Lagrangian, based on the canonical separation with the stress tensor for the background subsystem defined by equation (8). The ponderomotive force derived under such a separation is consistent with the $\mathcal{K}$–χ theorem proposed by Cary and Kaufman (1977), i.e. the ponderomotive force is related to the linear response tensor. The ponderomotive force (11) can be obtained by integration of the single-particle formula (Manheimer 1985) over the particle distribution.

The force density f_{B}^{ν}, equation (18), for both the unmagnetised and the magnetised cases can be regarded as the generalised ponderomotive force, provided that the system is separated in such a way that the background subsystem consists of the average motion of particles and the remaining part of the system is regarded as the wave subsystem, i.e. according to the physical separation of Dewar (1977). With such a separation of the system, the ponderomotive force defined by f_{B}^{μ}, in the unmagnetised case, can reproduce the extended ponderomotive force in 3-tensor notation given by other authors (Klíma and Petržílka 1978; Barash and Karpman 1983; Hora 1985).

In the derivation of the covariant ponderomotive force, the assumptions [cf. equations (27) and (44)] made are similar to those made by Cary and Kaufman (1981) in the Lie transformation formalism of the ponderomotive Hamiltonian. In the formalism developed here the finite Larmor radius effect is retained, and the theory is manifestly covariant and gauge-independent.

Acknowledgments

We thank Dr P. A. Robinson and an anonymous referee for helpful comments.

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Appendix: Covariant Lagrangian for OC Orbits

In the case of an unmagnetised plasma the total Lagrangian for the system is

$$\mathcal{L}(x) = \mathcal{L}_{\rm p}(x) + \mathcal{L}_{\rm em}(x), \tag{A1}$$

where $\mathcal{L}_{\rm p}(x)$ and $\mathcal{L}_{\rm em}(x)$ are the Lagrangians for the particles and the electromagnetic fields, respectively. In the notation used here one has

$$\mathcal{L}_{\rm p}(x) = \int {\rm d}p\, F[A(x),p]\, R\big[A(x),u\big]\,, \tag{A2}$$

$$\mathcal{L}_{\rm em}(x) = -\frac{1}{\mu_0}\left[\partial^\mu A^\nu(x) - \partial_\nu A^\mu(x)\right]\left[\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\right], \tag{A3}$$

where $R(A(x),u)$ is the covariant Lagrangian for a single particle. For an unmagnetised plasma one has

$$R[A(x),u] = -m - quA(x), \tag{A4}$$

where $A(x)$ is the 4-potential for the electromagnetic field in the waves. There is no explicit dependence on x in the Lagrangian, which is a functional of A, ∂A, and u. The action of the system is

$$I = \int {\rm d}x\, \mathcal{L}(x)\,. \tag{A5}$$

We introduce the oscillation centre coordinates $(\bar{x},\bar{p})$ which are defined by $x=\bar{x}+\xi$ with $\bar{x}=\langle x\rangle$, where $\langle\cdots\rangle$ represents the average over the fast time scale. One may also define the proper time, $\bar{\tau}$, on the OC orbit by (Dewar 1977)

$$\frac{\mathrm{d}\bar{\tau}}{\mathrm{d}\tau} = \left[1 + 2\bar{u}\frac{\mathrm{d}\xi}{\mathrm{d}\bar{\tau}} + \left(\frac{\mathrm{d}\xi}{\mathrm{d}\bar{\tau}}\right)\left(\frac{\mathrm{d}\xi}{\mathrm{d}\bar{\tau}}\right)\right]^{-1/2}, \tag{A6}$$

where

$$\bar{u}^{\mu} = \frac{\mathrm{d}\bar{x}^{\mu}}{\mathrm{d}\bar{\tau}} \tag{A7}$$

is the mean 4-velocity and

$$u^{\mu} = \frac{\mathrm{d}\bar{\tau}}{\mathrm{d}\tau}\left(\bar{u}^{\mu} + \frac{\mathrm{d}\xi^{\mu}}{\mathrm{d}\bar{\tau}}\right) \tag{A8}$$

is the exact 4-velocity, with $\mathrm{d}/\mathrm{d}\bar{\tau}$ defined by

$$\frac{\mathrm{d}}{\mathrm{d}\bar{\tau}} \equiv \bar{u}\frac{\partial}{\partial\bar{x}} + \frac{\mathrm{d}\bar{u}}{\mathrm{d}\bar{\tau}}\frac{\partial}{\partial\bar{u}^{\mu}}. \tag{A9}$$

The OC distribution function $F(\bar{x},\bar{p})$ can be formally defined by (Dewar 1977)

$$\mathrm{d}x\,\mathrm{d}p\,F(x,p) = \mathrm{d}\bar{x}\,\mathrm{d}\bar{p}\,F(\bar{x},\bar{p})\frac{\mathrm{d}\tau}{\mathrm{d}\bar{\tau}}, \tag{A10}$$

and it satisfies the Vlasov equation

$$\bar{u}\frac{\partial}{\partial\bar{x}}F(\bar{x},\bar{p}) + \frac{\partial}{\partial\bar{u}}\left[\frac{\mathrm{d}\bar{u}}{\mathrm{d}\bar{\tau}}F(\bar{x},\bar{p})\right] = 0, \tag{A11}$$

where $\mathrm{d}\bar{u}^{\mu}/\mathrm{d}\bar{\tau} = \mathcal{F}^{\mu}(\bar{x},\bar{p})$ is the 4-force that determines the OC orbits.

The action (A5) in the OC coordinates may expanded about ξ:

$$\begin{aligned} I = &\int \mathrm{d}\bar{x}\,\mathrm{d}\bar{p}\,F(\bar{x},\bar{p})\left[-m\frac{\mathrm{d}\tau}{\mathrm{d}\bar{\tau}} - q\left(\bar{u}^{\alpha} + \frac{\mathrm{d}\xi^{\alpha}}{\mathrm{d}\bar{\tau}}\right)A_{\alpha}(\bar{x}+\xi)\right] \\ &+ \int \mathrm{d}\bar{x}\left[-\frac{1}{4\mu_0}\bar{\partial}^{[\mu,}A^{\nu]}(\bar{x}+\xi)\bar{\partial}_{[\mu,}A_{\nu]}(\bar{x}+\xi)\right], \end{aligned} \tag{A12}$$

where the abbreviation $\bar{\partial}^{[\mu,}A^{\nu]} := \partial A^{\nu}(\bar{x})/\partial\bar{x}_{\mu} - \partial A^{\mu}(\bar{x})/\partial\bar{x}_{\nu}$ is used. In equation (A12), the rapidly varying component of the exact distribution has been incorporated within the large square brackets and later its average is included in the averaged Lagrangian. From here on the bars on x, u, τ are omitted. Up to the second order of $A(x)$, the Lagrangian can be identified as

$$\begin{aligned} \mathcal{L}(x) = \int \mathrm{d}p\,F(x,p)\Big[&-m - mu\frac{\mathrm{d}\xi}{\mathrm{d}\tau} - quA(x) - \tfrac{1}{2}m(g^{\alpha\beta} - u^{\alpha}u^{\beta})\frac{\mathrm{d}\xi_{\alpha}}{\mathrm{d}\tau}\frac{\mathrm{d}\xi_{\beta}}{\mathrm{d}\tau} \\ &-q\frac{\mathrm{d}\xi}{\mathrm{d}\tau}A(x) - qu^{\alpha}\xi^{\beta}\partial_{\beta}A_{\alpha}(x)\Big] - \frac{1}{4\mu_0}\partial^{[\mu},A^{\nu]}(x)\partial_{[\mu},A_{\nu]}(x). \end{aligned} \tag{A13}$$

A Fourier transformation gives

$$\xi(x) = \int \frac{dk}{(2\pi)^4}\, e^{-ikx}\xi(k)\,, \tag{A14}$$

$$A(x) = \int \frac{dk}{(2\pi)^4}\, e^{-ikx}A(k)\,, \tag{A15}$$

where $\xi(k)$ and $A(k)$ are slowly varying functions of x. Generally ξ depends on u as well as x. Since $du/d\tau \propto |A|^2$, provided higher order terms are neglected, and ξ is of at least first order in A, we have $d\xi/d\tau \approx u\partial\xi/\partial x$. Therefore we have the Fourier transform

$$\frac{d\xi}{d\tau} = \int \frac{dk}{(2\pi)^4}\, e^{-ikx}\left(- iku\xi(k) + \frac{d\xi(k)}{d\tau}\right), \tag{A16}$$

$$\partial^{[\mu}A^{\nu]}(x) = \int \frac{dk}{(2\pi)^4} e^{-ikx}\Big\{ - ik^{[\mu}A^{\nu]} + \partial^{[\mu,}A^{\nu]}\Big\}. \tag{A17}$$

The average Lagrangian for the quasi-particles is denoted by

$$\langle \mathcal{L}_{\rm p}(x)\rangle = \int dp\, F(x,p)\, R(x,u)\,, \tag{A18}$$

with

$$\begin{aligned} R = -m + \int \frac{dk}{(2\pi)^4}\Big\{ &- \tfrac{1}{2}m\big(g^{\alpha\beta} - u^\alpha u^\beta\big)\left(- iku\xi + \frac{d\xi}{d\tau}\right)_\alpha \left(iku\xi^* + \frac{d\xi^*}{d\tau}\right)_\beta \\ &-\tfrac{1}{2}\Big[- iqk^\beta u^\alpha \xi^*_\beta A_\alpha + qu^\alpha \xi^*_\beta \partial^\beta A_\alpha + \text{c.c.}\Big] \\ &-\tfrac{1}{2}\Big[iqku\xi^*_\alpha A^\alpha + q\frac{d\xi^*_\alpha}{d\tau}A^\alpha + \text{c.c.}\Big]\Big\}, \end{aligned} \tag{A19}$$

and with ξ determined by the Euler equation

$$\frac{d}{d\tau}\left(\frac{\partial R}{\partial \dot{\xi}^*}\right) - \frac{\partial R}{\partial \xi^*} = 0\,. \tag{A20}$$

We have

$$\frac{d\xi^\mu}{d\tau} = -\frac{iq}{mku}G^{\mu\sigma}u\partial A_\sigma\,, \tag{A21}$$

$$\xi^\mu = -\frac{iq}{mku}G^{\mu\sigma}A_\sigma - \frac{q}{m(ku)^2}\partial^{[\mu,}A^{\sigma]}u_\sigma - \frac{2q}{m(ku)^2}G^{\mu\sigma}u\partial A_\sigma\,, \tag{A22}$$

where $G^{\mu\nu}$ is defined by equation (22). We use the ordering $\epsilon \sim \mathrm{d}/\mathrm{d}\tau$. To $O(1)$ we have

$$\xi^{\mu} \approx -\frac{\mathrm{i}q}{mku} G^{\mu\nu}(k,u)\, A_{\nu}(k)\,. \tag{A23}$$

Substituting (A23) into (A19), the Lagrangian for the OC orbit is identified as (4a)–(4c).

Manuscript received 10 February 1992, accepted 14 April 1993