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Quasi-particle Interference Effect in the Coulomb Blockade Problem

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Abstract

The Coulomb energy in a tunnel junction of very small capacitance, in which there are two or more kinds of quasi-particles, will involve coupling between pairs of quasi-particle species. The variation in the number of one of them will affect the motion of the others. This should be detectable in a current-biased experimental setup. Taking Josephson and single-electron tunnelling as an example, we calculate the Fourier coefficients of the voltage across the junction, which is a periodic function of time. We show that the interference effect is significant in that the frequency dependence of the Fourier coefficients is completely different from that associated with tunnelling of a single species.

1. Introduction

Tunnelling of charged quasi-particles through ultra-small junctions at low temperatures has received much attention lately (Averin and Likharev 1986; Ben-Jacob *et al.* 1988; Likharev 1988; Mullen *et al.* 1988). If there were more than one species of quasi-particle tunnelling through the ultra-small junction, they would be coupled through the Coulomb term $(Q^2/2C)$ in the Hamiltonian of the system, Q being the sum of the quasi-particle charges, and C being the capacitance of the junction. The object of this note is to present an analysis of the interference effects arising out of the above coupling in a system consisting of two species of quasi-particles. A typical example of such a system would be a Josephson junction with a 'weak' link (Likharev 1979), in which single electrons and Cooper pairs can tunnel through. The analysis has obvious relevance to situations that involve the tunnelling of quasi-particles representing highly correlated states, such as those exhibiting fractional charges.

2. The Model Hamiltonian

We consider the situation in which there are two kinds of quasi-particles in the tunnel junction, carrying charges $Q_\alpha = \alpha e$ and $Q_\beta = \beta e$ and with tunnelling coefficients T_α and T_β respectively. The total charge in the junction is $Q = Q_\alpha + Q_\beta$. The Coulomb energy of the junction is equal to $(Q_\alpha + Q_\beta)^2/2C$.

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The canonical coordinates (Ben-Jacob and Gefen 1985) conjugate to the number operators of the quasi-particles α and β are ϕ_α and ϕ_β respectively. Thus,

$$Q_\alpha = -\mathrm{i}\alpha e\,\frac{\partial}{\partial\phi_\alpha}\,, \tag{1}$$

$$Q_\beta = -\mathrm{i}\beta e\,\frac{\partial}{\partial\phi_\beta}\,. \tag{2}$$

The Hamiltonian has the form

$$H = -\frac{e^2}{2C}\left(\alpha\,\frac{\partial}{\partial\phi_\alpha} + \beta\,\frac{\partial}{\partial\phi_\beta}\right)^2 + T_\alpha\cos\phi_\alpha + T_\beta\cos\phi_\beta\,. \tag{3}$$

To solve the problem we make the variable changes

$$\phi = \tfrac{1}{2}\left(\frac{\phi_\alpha}{\alpha} + \frac{\phi_\beta}{\beta}\right), \tag{4}$$

$$\bar{\phi} = \frac{\phi_\alpha}{\alpha} - \frac{\phi_\beta}{\beta}\,, \tag{5}$$

so that

$$H = -\frac{e^2}{2C}\,\frac{\partial^2}{\partial\phi^2} + T_\alpha\cos[\alpha(\phi+\bar{\phi}/2)] + T_\beta\cos[\beta(\phi-\bar{\phi}/2)]\,. \tag{6}$$

Clearly, $\bar{\phi}$ is a constant of motion. It is now not difficult to solve for the eigenvalues.

We consider a simple case where the quasi-particle of type α provides the larger tunnelling current. The term in the Hamiltonian (6) that involves tunnelling of the quasi-particle of type β can then be treated by perturbation theory. The unperturbed Schrödinger equation is a Mathieu equation whose eigenvalues form an energy band E_k, and the corresponding eigenfunctions are Bloch functions,

$$\psi_k(\phi) = \exp(\mathrm{i}k\phi)\sum_n c_{k,n}\exp(\mathrm{i}n\alpha\phi)\,. \tag{7}$$

The first Brillouin zone has the range $-\alpha/2 < k < \alpha/2$, because the 'potential', i.e. the tunnelling term has a period of $2\pi/\alpha$. The tunnelling of the type β quasi-particle can be taken into account through perturbation theory. The term $T_\beta\cos[\beta(\phi-\bar{\phi}/2)]$ in (6) couples two Bloch states. Let the coupling matrix element be W_{lk}. Then

$$\begin{aligned} W_{lk} &\equiv \int \psi_l^*(\phi)\,T_\beta\cos[\beta(\phi-\bar{\phi}/2)]\,\psi_k(\phi)\,d\phi \\ &= \frac{T_\beta}{2}\sum_{m,n} c_{l,m}^*\,c_{k,n}\big[\exp(-\mathrm{i}\beta\bar{\phi})\delta_{k-l+(n-m)\alpha+\beta} + \exp(\mathrm{i}\beta\bar{\phi})\delta_{k-l+(n-m)\alpha-\beta}\big]\,, \end{aligned} \tag{8}$$

where (7) has been used. Equation (8) is valid only when α and β are rational numbers. Hence α and β can be written in nonreducible form as n_1/n_2 and n_3/n_4 respectively with n_1, n_2, n_3 and n_4 being integers. The resulting energy band has a period μ/ν, where μ is the highest common factor of n_1 and n_3, and ν is the lowest common multiple of n_2 and n_4. The perturbed energy band can be found by diagonalising the matrix W, and the problem is thus solved. If, for simplicity, only one band is considered, then one has to solve the following secular equation:

$$\begin{vmatrix} E_k & W_{kl} \\ W_{lk} & E_l \end{vmatrix} = 0\,, \tag{9}$$

where k and l satisfy the relation given by (8). We shall give an example.

In a current-biased junction, the interaction Hamiltonian is

$$H_{\mathrm{I}} = I\Phi/c\,, \tag{10}$$

where Φ is the flux. Since

$$\Phi_\gamma = \frac{\hbar c}{\gamma e}\,\phi_\gamma \tag{11}$$

for γ equal to either α or β, we have

$$\Phi = \Phi_\alpha + \Phi_\beta = \frac{2\hbar c}{e}\,\phi\,. \tag{12}$$

Hence

$$H_{\mathrm{I}} = \frac{2\hbar I}{e}\,\phi\,, \tag{13}$$

which is completely analogous to the model Hamiltonian for describing the dynamics of a band electron in a uniform electric field (Shockley 1950). It is well known that in the latter case

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{F}{\hbar}\,, \tag{14}$$

where s is the crystal wavevector and F is the applied force. Therefore, in the Coulomb blockade problem,

$$\frac{\mathrm{d}k}{\mathrm{d}t} = \frac{2I}{e}\,. \tag{15}$$

As pointed out by Widom *et al.* (1982), the voltage V across the junction corresponds to the group velocity of a band electron in a crystal in a uniform field. It thus has the periodicity of the energy band in k-space. We expand it in a Fourier series,

$$V_k = \sum_n V_n \sin(2n\pi k/\xi)\,, \tag{16}$$

where ξ is the size of the Brillouin zone, as indicated after equation (7). From (14) we find that the frequency of V_k is $4\pi I/e\xi$, whereas for tunnelling of type α quasi-particles only, the frequency is $2\pi I/e\alpha\xi_\alpha$. Note that $\xi_\alpha = 1$ is the size of the Brillouin zone if the tunnelling of type β quasi-particles is ignored in (3). Therefore the interference effect leading to a change of the periodicity of V should be observable at suitably low temperatures, where the noise level will not mask the effect (i.e. for $k_B T < e^2/2C$).

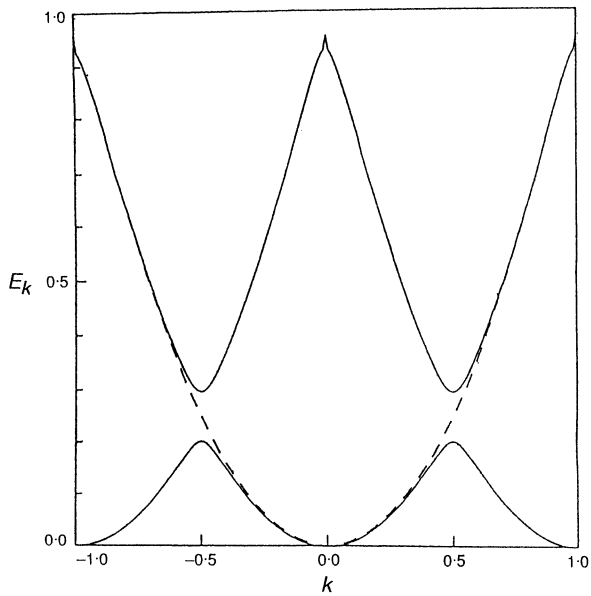

Fig. 1. Eigenvalues of the Hamiltonian (6) as a function of k. The dashed curve was evaluated under the conditions $\bar{\phi} = 0$, $e^2/2C : T_2 : T_1 = 1{\cdot}0 : 0{\cdot}3 : 0{\cdot}0$. The lower and upper solid curves show E_k^- and E_k^+ as a function of k. They were calculated under the conditions $\bar{\phi} = 0$, $e^2/2C : T_2 : T_1 = 1{\cdot}0 : 0{\cdot}3 : 0{\cdot}1$.

3. Discussion

We have applied the above ideas to the case of a Josephson junction, in which the two kinds of quasi-particles are the Cooper pairs with charge $2e$ and the unpaired electrons with charge e. In most experimental situations Josephson tunnelling produces a much larger current than that due to single-electron tunnelling. Thus we have $\alpha = 2$ and $\beta = 1$. In view of (8), the states $|k\rangle$ $(k > 0)$ and $|k-1\rangle$ are coupled. From (9) we get

$$E_k^{\pm} = \frac{E_k + E_{k-1}}{2} \pm \sqrt{\frac{(E_k - E_{k-1})^2}{4} + |W_{k,k-1}|^2}\,. \tag{17}$$

After the coupling, the size of the first Brillouin zone is reduced to be in the range $0 < k < 1$. To show explicitly what has been derived, we have plotted in Fig. 1 the energy bands before and after coupling. The dashed curve is the lowest band without single-electron tunnelling. The lower and upper solid curves show E_k^- and E_k^+ as functions of k. These were calculated under the conditions that $\bar{\phi} = 0$ and $e^2/2C : T_2 : T_1 = 1 : 0{\cdot}3 : 0{\cdot}1$.

Table 1. Fourier coefficients of junction voltage [equation (16)]
All V_j are normalised, with V_1 equal to unity

$(e^2/2C)$: T_2 : T_1	V_2	V_3	V_4
$1\cdot0:0\cdot3:0\cdot00^{A}$	$-0\cdot481$	$0\cdot305$	$-0\cdot216$
$1\cdot0:0\cdot3:0\cdot05^{B}$	$-0\cdot474$	$0\cdot291$	$-0\cdot203$
$1\cdot0:0\cdot3:0\cdot10^{B}$	$-0\cdot436$	$0\cdot229$	$-0\cdot149$
$1\cdot0:0\cdot3:0\cdot15^{B}$	$-0\cdot402$	$0\cdot170$	$-0\cdot113$

[A] No single-electron tunnelling, V_j is the Fourier coefficient of the voltage at the frequency $j\pi I/|e|$.
[B] Single-electron tunnelling included, V_j (from E_k^-) is the Fourier coefficient at the frequency $4j\pi I/|e|$.

In Table 1 we have listed the ratios of the Fourier coefficients of the junction voltage at different values of T_1. According to the discussion above, if only Josephson tunnelling occurs, then the fundamental frequency of voltage oscillations is $\omega = \pi I/|e|$, and the coefficient V_j is associated with the oscillation of frequency $\omega_j = j\pi I/|e|$. If there is also single electron tunnelling, then $\omega = 4\pi I/|e|$, and the coefficient V_j is associated with the oscillation $\omega_j = 4j\pi I/|e|$. This distinction should be identifiable in experiments. The temperature should be low enough to quench the noise effectively, and, at the same time, should be in a range that would enable variation of the relative populations of the two quasi-particle species, i.e. close to T_c. Thus the preferred material for making a junction that would show this interference effect would be one that has a low T_c.

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References

Averin, D. V., and Likharev, K. K. (1986). *J. Low Temp. Phys.* **62**, 345.
Ben-Jacob, E., and Gefen, Y. (1985). *Phys. Lett.* A **108**, 289.
Ben-Jacob, E., Gefen, Y., Mullen, K., and Schuss, Z. (1988). *Phys. Rev.* B **37**, 7400.
Likharev, K. K. (1979). *Rev. Mod. Phys.* **51**, 101.
Likharev, K. K. (1988). *IBM J. Res. Dev.* **32**, 144.
Muller, K., Ben-Jacob, E., Jaklevic, R. C., and Schuss, Z. (1988). *Phys. Rev.* B **37**, 98.
Shockley, W. (1950). 'Electrons and Holes in Semiconductors', Chs 7, 14 and 15 (Van Nostrand: New York).
Widom, A., Megaloudis, G., Clark, T. D., Prance, H., and Prance, R. J. (1982). *J. Phys.* A **15**, 3887.

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