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Induced Parity Violation in Odd Dimensions

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Abstract

One of the interesting features about field theories in odd dimensions is the induction of parity-violating terms and well-defined *finite* topological actions via quantum loops if a fermion mass term is originally present and conversely. Aspects of this issue are illustrated for electrodynamics in 2+1 and 4+1 dimensions.

1. Introduction

There are a few curiosities associated with field theories in an odd number of space–time dimensions. The first is that the overall degree of divergence of an integral possessing an odd mass scale cannot be taken at face value, since such an integral behaves like the gamma function at half-integral argument values. This is most easily seen by considering dimensional regularisation of the typical integral as D tends to an odd integer (when r below is an integer),

$$I = -i \int \frac{\Gamma(r)\, \bar{d}^D p}{(p^2 - M^2)^r} = \frac{(-1)^r\, \Gamma(r - D/2)}{(4\pi)^{D/2}\, (M^2)^{r-D/2}}\,. \tag{1}$$

A second noteworthy property is that in odd dimensions, one commonly encounters couplings which are odd powers of mass. This can be understood by considering a free-field theory in D space–time dimensions,

$$\int d^Dx\, [((\partial\phi)^2 - \mu^2\phi^2)/2 + \bar{\psi}(i\gamma.\partial - m)\psi - F^{\mu\nu}F_{\mu\nu}/4]\,, \tag{2}$$

where typically ϕ is a scalar, ψ is a spinor and $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is a Maxwell gauge field. The dimensionlessness of the action (in natural units) specifies the mass dimensions of the fields,

$$[\phi], [A] \sim M^{D/2-1}, \qquad [\psi] \sim M^{(D-1)/2},$$

whereupon interaction Lagrangians like

$$\int d^Dx\, [e\bar{\psi}\gamma.A\psi + f\phi^3 + g\phi\bar{\psi}\psi + \lambda\phi^4 + G(\bar{\psi}\psi)^2 + \ldots] \tag{3}$$

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will have coupling constants with prescribed dimensions,

$$[e],[g] \sim M^{2-D/2}, \quad [f] \sim M^{3-D/2}, \quad [\lambda] \sim M^{4-D}, \quad [G] \sim M^{2-D}. \tag{4}$$

One then observes that in odd dimensions the couplings e, f, g have odd $\sqrt{M}$ scales. That does not matter much for electrodynamics since we meet powers of e^2 in the perturbation expansion but for f, g we can potentially encounter single powers of the coupling. In particular for three dimensions,

$$[f] \sim M^{\frac{3}{2}}, \qquad [e],[g] \sim M^{\frac{1}{2}}, \qquad [\lambda] \sim M, \qquad [G] \sim M^{-1}, \tag{5}$$

telling us that electrodynamics, chromodynamics, $\lambda\phi^4$ and Yukawa interactions become *super*-renormalisable, while Fermi interactions remain unrenormalisable. Combined with the first property, this has the effect of eliminating certain ultraviolet infinities in theories such as $\lambda\phi^4$; thus the tadpole graphs of order λ and higher are perfectly finite and the *only* infinity in that model is the self-mass of ϕ due to the three-ϕ intermediate state. For electrodynamics in 2+1 dimensions, the situation is even better—no infinities at all.

A third peculiar aspect of odd D dimensions stems from the algebra of the Dirac γ-matrices

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu} .$$

When D is even it is well-known that the γ have size $2^{D/2} \times 2^{D/2}$ and there exists a γ_5 matrix which is the product of all the different $D\gamma$ and which anticommutes with each γ_μ; one can always arrange it to have square -1, like all the space-like γ (in our metric +,−,−, ...). It is not so well known that in one higher dimension, the size of the γ remains the same—all that happens is that the 'γ_5' matrix becomes the last element of the D Dirac matrices.

For instance, in three dimensions one can take the two-dimensional Pauli $\gamma_0 = \sigma_3$, $\gamma_1 = i\sigma_1$ and simply append $\gamma_2 =$ 'γ_5' $= i\gamma_0\gamma_1 = -i\sigma_2$ to complete the set, without altering the size of the representation. At the same time it should be noticed that one can get a non-zero trace from the product of *three* gamma-matrices, viz. $\mathrm{Tr}[\gamma_\rho\gamma_\sigma\gamma_\tau] = 2i\epsilon_{\rho\sigma\tau}$. Similarly, in five dimensions one can take the usual four-dimensional ones and just append $\gamma_5 \equiv \gamma_0\gamma_1\gamma_2\gamma_3$ as the fifth component; here as well the product of the full five γ gives a non-vanishing trace: $\mathrm{Tr}[\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_\tau] = -4\epsilon_{\mu\nu\rho\sigma\tau}$. The lesson is that when D is odd, one should be careful before discarding traces of odd monomials of gamma-matrices if there are sufficiently many γ, since at least

$$\mathrm{Tr}[\gamma_{\mu_1}\gamma_{\mu_2}\cdots\gamma_{\mu_D}] = (2i)^{(D-1)/2}\epsilon_{\mu_1\mu_2\cdots\mu_D}.$$

Another property worth remembering in odd dimensions is that if one constructs suitably normalised antisymmetric products of r matrices, $\gamma_{[\mu_1\mu_2\cdots\mu_r]}$ (the total set of these from $r = 0$ to $r = D$ generates a complete set into which any $2^{[D/2]} \times 2^{[D/2]}$ matrix can be expanded), then there exists the relation

$$\gamma_{[\mu_1\mu_2\cdots\mu_r]} = i^{(D-1)/2}\epsilon_{\mu_1\mu_2\cdots\mu_D}\gamma^{[\mu_{r+1}\mu_{r+2}\cdots\mu_D]}/(D-r)!$$

This often helps in simplifying products of matrices.

Turning to discrete operations, a charge conjugation operator $\mathcal{C}$ with the transposition property

$$\mathcal{C}\gamma_{[\mu_1\mu_2\cdots\mu_r]}\mathcal{C}^{-1} = (-1)^{[(r+1)/2]}(\gamma_{[\mu_1\mu_2\cdots\mu_r]})^T \tag{6}$$

always exists in even dimensions, but cannot be defined at the odd values $D = 5, 9, 13\ldots$. This is intimately tied with the existence of topological terms in the action for the pure gauge field, as we shall see. As for parity $\mathcal{P}$, it corresponds to an inversion of all the *spatial* coordinates for even D, since that is an improper transformation. However when D is odd, it should be regarded as a reflection of all the space coordinates *except the very last one*, x_{D-1}, in order to ensure that the determinant of the transformation remains negative. It is straightforward to verify that this corresponds to the unitary change

$$\begin{aligned}\mathcal{P}\psi(x_0, x_1, ..., x_{D-2}, x_{D-1})\mathcal{P}^{-1} =& -i\eta\gamma_0\gamma_{D-1}\psi(x_0, -x_1, ..., -x_{D-2}, x_{D-1})\\ =&\eta\gamma_1\cdots\gamma_{D-2}\psi(x_0, -x_1, ..., -x_{D-2}, x_{D-1}),\end{aligned} \tag{7}$$

where η is the intrinsic parity of the fermion field. In that regard, one can just as easily check that a mass term like $m\bar{\psi}\psi$ in the action is *not* invariant under parity for D odd.

This potential to induce other parity-violating interactions in the theory forms the subject of this paper. In Section 2 we shall examine the induction of a Chern–Simons term in 2+1 electrodynamics from a fermion mass; the converse process is considered in an Appendix. In Section 3 we generalise this to 4+1 QED and to higher D, with the converse effect also treated in the Appendix. An explanation of why this is a pure one-loop effect is also provided. Finally we discuss in Section 4 what happens when electrodynamics is purely topological (no free F^2 term in the Lagrangian) as this represents a system quite different from what we are accustomed to.

2. 3D Electrodynamics

Turning to QED in 2+1 dimensions, we are blessed with a coupling with positive mass dimension $[e^2] \sim M$, so we anticipate a finite number of ultraviolet singularities. But in fact *none exists* thanks to gauge invariance. The standard diagrams for photon polarisation, electron self-energy and vertex corrections are all perfectly finite—barring infrared problems, which actually correct themselves non-perturbatively through the dressing of the photon line, as first demonstrated by Jackiw and Templeton (1981), although some doubts about the loop expansion have been expressed by Pisarski and Rao (1985).

Straightforward evaluation of the graphs produces the one-loop results

$$\Pi_{\mu\nu}(k) = (k_\mu k_\nu - k^2\eta_{\mu\nu})\frac{e^2}{2\pi}\int\frac{\alpha(1-\alpha)d\alpha}{\sqrt{m^2 - k^2\alpha(1-\alpha)}}$$

$$+ im\epsilon_{\lambda\mu\nu}k^\lambda\frac{e^2}{4\pi}\int\frac{d\alpha}{\sqrt{m^2-k^2\alpha(1-\alpha)}}, \tag{8}$$

$$S(p) = \langle T(\bar\psi(p)\psi(0))\rangle = \frac{e^2}{16\pi}\int\frac{dw}{w(\gamma.p - w)}\left[\frac{\xi}{w} - \frac{4m}{(w-m)^2}\right], \tag{9}$$

where ξ is a parameter that fixes a Lorentz-covariant gauge ($\xi = 0$ is the Landau gauge). The vertex corrections are obviously finite too, because '$Z_1 = Z_2$'. Higher powers of e^2 only serve to make the diagrams more convergent than they already are in lowest order, since the series expansion of a physical amplitude will take the form

$$T(k) = T_0(k)\left[1 + c_1\frac{e^2}{m} + c_2\frac{e^4}{m^2} + \ldots\right]; \quad c_i = f(k/m),$$

where k signifies the external momentum variables. Broadhurst *et al.* (1993) have calculated some of these coefficients for any D.

An important aspect of (8) is the induced parity-violating Chern–Simons interaction, $\epsilon_{\lambda\mu\nu}A^\lambda F^{\mu\nu}$. It is not surprising that it should have sprouted, since we started with a fermion mass $m\bar\psi\psi$ term which is intrinsically $\mathcal{P}$-violating in odd dimensions; but the value of that induced photon term is finite and disappears as $m \to 0$. Note, however, that in the infrared limit it reduces to $ie^2\epsilon_{\lambda\mu\nu}k^\lambda/4\pi$ *provided that* $m \neq 0$.

There has been some argument in the literature that this Chern–Simons term may produce anomalies in some processes because it contains an ϵ tensor specific to three dimensions (in much the same way that the axial anomaly is connected with the ϵ tensor in four dimensions). This cannot be true because vacuum polarisation to order e^2 is perfectly sensible and *finite* when evaluated by any reasonable regularisation. Anomalies only arise when a *divergence* multiplies an apparent *zero*, as in the Pauli–Villars method, when a mass regulator contribution M^2 multiplies an integral of order $1/M^2$; or in a dimensional context, when a pole $1/(D-3)$ multiplies an evanescent zero $(D-3)$. But, as we have already seen, the photon self-energy diagram contains no such singularity (Delbourgo and Waites 1993). Thus an anomaly is indeed absent.

3. Electrodynamics in Higher Dimensions

A subject of debate has been whether the induced Chern–Simons term suffers from higher-order loop corrections. There is a simple proof, presented later, which says this cannot be, but before describing it, let us exhibit the induced term for any odd D dimension, as it is a 'clean' result. We begin as before with massive fermion QED. For arbitrary D the induced topological term takes the form of an n-point function,

$$C\epsilon_{\mu_1\mu_2\cdots\mu_D}A^{\mu_1}F^{\mu_2\mu_3}\ldots F^{\mu_{D-1}\mu_D}; \qquad n=(1+D)/2. \tag{10}$$

Notice that this conforms perfectly with charge conjugation: when $D=3$ and $\mathcal{C}$ is conserved, the topological term involves an even number $n=2$ of photons; when $D=5$ and $[e^2]\sim M^{-1}$, we encounter three photon lines but then $\mathcal{C}$ is no longer valid; when $D=7$, $\mathcal{C}$-invariance becomes operative again and the number of photon lines is $n=4$; and so on.

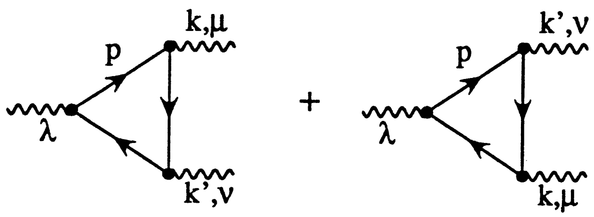

Fig. 1. One-loop induction of a Chern–Simons amplitude in five dimensions.

The result of the one-loop contribution to the topological term has already been quoted in equation (8). Looking at the next odd dimension, $D=5$, the relevant one-loop graphs are shown in Fig. 1, leading to the induced vertex

$$\Gamma_{\lambda\mu\nu}(k,k')=-2ie^3\int \bar{d}^5p\,\frac{\mathrm{Tr}[\gamma_\nu(\gamma.p+m)\,\gamma_\mu(\gamma.(p+k)+m)\,\gamma_\lambda(\gamma.(p-k')+m)]}{(p^2-m^2)\,[(p+k)^2+m^2]\,[(p-k')^2-m^2]}.$$

Introducing Feynman parameters in the usual way to combine denominators and picking out the term with five gamma-matrices in the trace, we end up with

$$\begin{aligned}\Gamma_{\lambda\mu\nu}(k,k')&=-\,16ie^3m\int \bar{d}^5p\,\frac{d\alpha\,d\beta\,d\gamma\,\delta(1-\alpha-\beta-\gamma)\epsilon_{\lambda\mu\nu\rho\sigma}\,k^\rho\,k'^\sigma}{[p^2-m^2+k^2\alpha\beta+k'^2\gamma\alpha+(k+k')^2\beta\gamma]^3}\\&=-\,\frac{e^3m\epsilon_{\lambda\mu\nu\rho\sigma}\,k^\rho\,k'^\sigma}{8\pi^2}\int_0^1\frac{d\alpha\,d\beta\,d\gamma\,\delta(1-\alpha-\beta-\gamma)}{\sqrt{m^2-k^2\alpha\beta-k'^2\gamma\alpha-(k+k')^2\beta\gamma}}.\end{aligned} \tag{11}$$

One can regard this amplitude as the five-dimensional description of the process $\pi^0\to 2\gamma$, because one of the indices (4) of the Levi–Civita tensor just corresponds to the standard pseudoscalar and the residual four indices (0 to 3) are the normal 4-vector ones. Just as with 2+1 QED, we see that the induced term in 4+1 QED vanishes with the fermion mass m. [Contrariwise, one can check that if $m=0$, but a term (10) is present from the word go, then a fermion mass term, amongst other parity-violating ones, will arise. See the Appendix.]

We are now in a position to quote the topological vertex induced for arbitrary odd D by the fermion mass term. Introduce $n=(D+1)/2$ and Feynman parameters α_i, $i=1\ldots n$, for each internal line (Fig. 2). Call the momentum flowing across

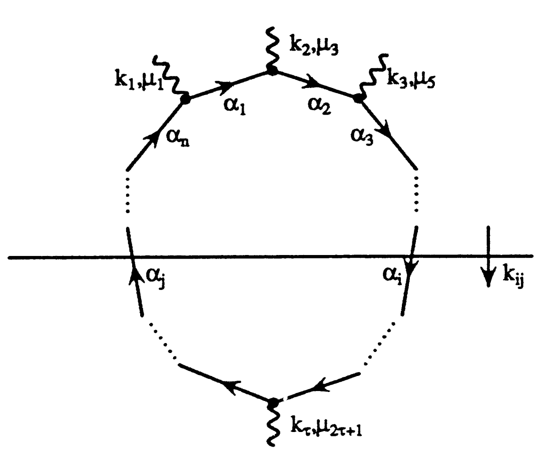

Fig. 2. One-loop induction of a Chern–Simons term in D dimensions.

each possible cutting of two lines k_{ij} if those lines have parameters α_i, α_j. The calculation then produces the result

$$\Gamma_{\mu_1\mu_2\cdots\mu_n}(k) = -\frac{me^n i^{n-1}}{2(2\pi)^{n-1}}\epsilon_{\mu_1\mu_2\cdots\mu_D}\, k_1^{\mu_2}\, k_2^{\mu_4}\ldots k_n^{\mu_D}$$

$$\times \int_0^1 \prod_{k=1}^{n} d\alpha_k \frac{\delta\left(1-\sum_k \alpha_k\right)}{\sqrt{m^2 - \sum_{i<j=1}^{n} k_{ij}^2\, \alpha_i\, \alpha_j}}\,. \tag{12}$$

One may readily check that this collapses to the results (8) and (11) for $D = 3$ and $D = 5$ respectively. It corresponds to the Chern–Simons term (10) where $C = e^n/2n!(4\pi)^{n-1}$ if one goes to the soft photon limit, always assuming $m \neq 0$.

To finish off this section let us explain why this one-loop answer (12) is all there is. In three dimensions, the Lagrangian $\epsilon_{\lambda\mu\nu}A^\lambda F^{\mu\nu}$ will change by a pure divergence under the gauge transformation, $\delta A \to \partial\chi$, so the action remains invariant for all normal field configurations that vanish at ∞. However, a fourth-order interaction like

$$\epsilon_{\lambda\mu\nu}A^\lambda F^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}$$

will *not* be invariant under the gauge change; thus it is not permitted. More generally, in odd D dimensions the interaction

$$\epsilon_{\mu_1\mu_2\cdots\mu_D}A^{\mu_1}F^{\mu_2\mu_3}\ldots F^{\mu_{D-1}\mu_D}(F_{\rho\sigma}F^{\rho\sigma})^N; \qquad N \geq 1\,, \tag{13}$$

and ones like it, are forbidden by gauge invariance and thus cannot be produced. On the other hand a two-loop contribution to the fundamental topological term can be regarded as an integration of (13), with $N = 1$, over one of the photon momenta. Since we have just concluded that (13) must be absent, we deduce that the induced topological term (10) cannot receive any two-loop (or higher-loop) quantum corrections. In this respect, it is a pristine result similar to the Adler–Bardeen theorem for the axial anomaly; nevertheless it is only of academic interest in as much as QED becomes unrenormalisable (cf. the dimensions of e^2) when $D \geq 5$, unless the space–time is compact, e.g. in some Kaluza–Klein geometries.

4. Topological QED

So far we have considered models where the initial Lagrangian contains the normal free gauge kinetic energy $F_{\mu\nu}F^{\mu\nu}$ term, and seen what transpires as a result of parity violation primarily through the fermion field. Now we shall consider what happens when the initial Lagrangian has no gauge field kinetic energy but starts off life instead with a Chern–Simons piece such as (10). In 2+1 dimensions, this still means a bilinear term in the gauge field capable of launching a propagator,

$$D_{\mu\nu} = \left[\frac{i\epsilon_{\mu\nu\lambda}k^\lambda}{\mu k^2} - \xi\frac{k_\mu k_\nu}{k^4}\right], \tag{14}$$

where we have taken account of gauge-fixing, with parameter ξ. (The very same expression can be obtained by adding a conventional kinetic term $-ZF_{\mu\nu}F^{\mu\nu}/4$ and taking the limit $Z \to 0$.) Evaluating the fermion self-energy now yields

$$\Sigma(p) = e^2 \int d^3k \, \frac{\gamma_\mu(\gamma.p - \gamma.k)\gamma_\nu}{(p-k)^2} D^{\mu\nu}(k) = -e^2\left[\frac{\xi\gamma.p}{16\sqrt{-p^2}} + \frac{\sqrt{-p^2}}{8\mu}\right], \tag{15}$$

which contains a mass-like term where none previously existed. In the same vein, we may compute the vacuum polarisation correction to (14) and arrive at

$$\Pi_{\mu\nu}(k) = ie^2\mathrm{Tr}\int d^3p \, \frac{\gamma_\mu\gamma.p\gamma_\nu\gamma.(p-k)}{p^2(p-k)^2} = (-k^2\eta_{\mu\nu} + k_\mu k_\nu)\frac{e^2}{8\sqrt{-k^2}}, \tag{16}$$

which has the effect of leaving $D(k) \sim 1/k$. In higher orders of perturbation theory we may expect that

$$\Sigma(p) = \gamma.p \, f\left(\frac{e^2}{\sqrt{-p^2}}, \frac{e^2}{\mu}\right) + \sqrt{-p^2}\, g\left(\frac{e^2}{\sqrt{-p^2}}, \frac{e^2}{\mu}\right),$$

$$\Pi_{\mu\nu} = (-k^2\eta_{\mu\nu} + k_\mu k_\nu)\,\pi\left(\frac{e^2}{\sqrt{-k^2}}, \frac{e^2}{\mu}\right),$$

where f, g and π are scalar functions of their arguments. It is fascinating to speculate on the full form of those functions by applying some non-perturbative method of solution.

The situation is radically different in 4+1 dimensions since the Chern–Simons term is *trilinear* in the gauge field and alone cannot engender a propagator. Rather, one must resort to quantum corrections to get something of that ilk;

the vacuum polarisation graph (initially from massless fermions) produces a hard quantum loop contribution:

$$D_{\mu\nu} = \left(-\frac{\eta_{\mu\nu}}{k^2} + \frac{k_\mu k_\nu}{k^4} \right) \frac{512\pi}{3e^2\sqrt{-k^2}} - \xi \frac{k_\mu k_\nu}{k^4}. \tag{17}$$

Taken with the trilinear gauge interaction, this can produce a vacuum polarisation effect from the gauge field itself, namely

$$\Pi_{\mu\nu} = i\left(\frac{512\pi C}{3e^2}\right)^2 \int \frac{\bar{d}^5 k'}{[k'^2(k-k')^2]^{\frac{3}{2}}} \epsilon_{\mu\rho\sigma\alpha\beta}\, k^\alpha\, k'^\beta\, \epsilon_\nu{}^{\rho\sigma\gamma\delta}\, k_\gamma\, k'_\delta$$

or

$$\Pi_{\mu\nu} = (\eta_{\mu\nu}\, k^2 - k_\mu k_\nu) \left(\frac{512C}{3e^2}\right)^2 \frac{\sqrt{-k^2}}{12}.$$

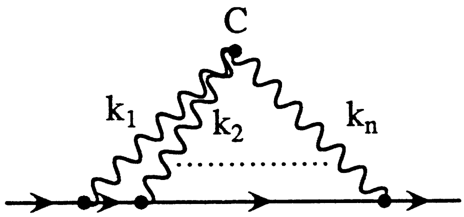

Fig. 3. Induction of a fermion mass term through a topological interaction.

Interestingly, (17) does not give birth to a mass-like fermion self-energy at the one-loop level—five gamma-matrices are needed to obtain that. This means we have to consider two-loop effects, either to order e^4 or to first order in the Chern–Simons coupling C, as sketched in Fig. 3. Quite generally in 4+1 dimensions we may anticipate that

$$\Sigma(p) = \gamma{\cdot}p\; f(e^2\sqrt{-p^2}, C/e^3) + \sqrt{-p^2}\; g(e^2\sqrt{-p^2}, C/e^3),$$

$$\Pi_{\mu\nu} = (-k^2\eta_{\mu\nu} + k_\mu k_\nu)\; \pi(e^2\sqrt{-k^2}, C/e^3).$$

However, we must be on guard that higher-order contributions in e^2 and C are very likely unrenormalisable now and possibly of academic interest only. Still, our discussion does indicate the nature of such parity-violating contributions in these models and how they spring from just one source.

If we could trust some non-perturbative method for resumming the Feynman diagrams then we might be able to estimate the quantum effects associated with the dimensional couplings e^2 and C. The same of course applies with even greater force to pure Chern–Simons theory in higher odd dimensions.

Acknowledgments

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Appendix

Here we shall examine the converse of Sections 2 and 3, in as much as we deal with massless electrodynamics (A and ψ) but introduce the parity violation through a primary Chern–Simons term, not a fermion mass. Our treatment is to be contrasted with that in Section 4, where a kinetic term for the photon was absent. In the present circumstances,

$$\mathcal{L} = \bar{\psi}\gamma.(i\partial - eA)\psi - F_{\mu\nu}F^{\mu\nu}/4 + C\epsilon_{\mu_1\mu_2\cdots\mu_D}A^{\mu_1}F^{\mu_2\mu_3}\cdots F^{\mu_{D-1}\mu_D},$$

we can be certain the gauge field will propagate at the bare level in any dimension D. It so happens that when $D = 3$ the Chern–Simons piece is also bilinear and can be incorporated with the standard F^2 term to give the initial two-point function,

$$D_{\mu\nu} = \frac{-\eta_{\mu\nu} + k_\mu k_\nu/k^2}{k^2 - \mu^2} + i\frac{\mu\epsilon_{\mu\nu\lambda}k^\lambda}{k^2(k^2 - \mu^2)} - \xi\frac{k_\mu k_\nu}{k^4}.$$

Parity-violating terms then arise through quantum corrections in other Green functions.

Probably the most significant of these is in the fermion self-energy,

$$\Sigma(p) = \frac{e^2\gamma.p}{16\pi p^2}\left[\frac{p^4 - \mu^4}{\mu^2 p}\ln\left(\frac{\mu + p}{\mu - p}\right) - \frac{2(p^2 - \mu^2)}{\mu} + \frac{\pi p^2\sqrt{-p^2}}{\mu^2} - \tfrac{3}{2}\pi\xi\sqrt{-p^2}\right]$$
$$+ \frac{e^2}{8\mu\pi}\left[\frac{p^2 - \mu^2}{p}\ln\left(\frac{\mu + p}{\mu - p}\right) - 2\mu + \pi\sqrt{-p^2}\right].$$

We should notice that in the limit of small μ, this expression reduces to

$$\Sigma(p) = \frac{e^2\gamma.p}{4\pi p^2}\left[\frac{\mu}{2} - \frac{3\pi\xi\sqrt{-p^2}}{8}\right] + \frac{e^2\mu}{8\sqrt{-p^2}}.$$

and could have been directly evaluated by regarding ϵAF as an interaction, rather than combining it with the bare photon propagator as above. (It will disappear, of course, when $\mu \to 0$ in the Landau gauge.)

Indeed this is the only sensible treatment for higher dimensions D since the Chern–Simons term is no longer bilinear. For $D = 5$ to first order in $C\epsilon AFF$, one engenders the mass term (and no kinetic term)

$$\Sigma(p) = -ie^3 C \int \bar{d}^D k\, \bar{d}^D k'\, \frac{\epsilon_{\mu\nu\lambda\alpha\beta}\, k^\alpha\, k'^\beta\, \gamma^\mu\, \gamma.(p-k-k')\gamma^\nu\, \gamma.(p-k)\gamma^\lambda}{k^2\, k'^2\, (k+k')^2\, (p-k-k')^2\, (p-k)^2}$$

$$= \frac{3\Gamma(3-D)\, p^4 e^3 C}{(16)^3 \pi^4};$$

unfortunately this is divergent as $D \to 5$, which is not too surprising. There is likewise a two-loop contribution of the same type to the photon self-energy, but this cannot add a parity-violating part to Π because such a term would violate gauge invariance for $D = 5$, as we have already explained in Section 3.

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