

Physical Consequences of the Interpretation of the Skew Part of $g_{\mu\nu}$ in Einstein's Nonsymmetric Unified Field Theory

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Abstract

The electromagnetic interaction in the Einstein–Infeld–Hoffmann (EIH) equations of motion for charged particles in Einstein's unified field theory (EUFT) is found to be *automatically* precluded by the conventional identification of the skew part of the fundamental tensor with the Faraday tensor. It is shown that an alternative identification, suggested by observations of Einstein, Bergmann and Papapetrou, would lead to the expected electromagnetic interaction, were it not for the intervention of an infelicitous (radiation) gauge. Therefore, an EIH analysis of EUFT is *inconclusive* as a test of the physical viability of the theory, and it follows that EUFT cannot be considered necessarily unphysical on the basis of such an analysis. Thus, historically, Einstein's unified field theory was rejected for the wrong reason.

1. Introduction

Recently there has been considerable discussion of the nonsymmetric gravitational theory (NGT) of Moffat (1979, 1991). Damour *et al.* (1992, 1993) have assailed NGT, claiming it to be theoretically inconsistent and possessing unphysical behaviour, and Moffat and his collaborators have defended NGT (Moffat 1993; Cornish and Moffat 1993, 1994; Cornish *et al.* 1993). This debate is still open and, owing to the formal mathematical similarity of NGT to the nonsymmetric unified field theory of Einstein (1950, 1956), this discussion has some bearing on Einstein's theory, and thereby rekindles interest in what many have considered a closed subject. While NGT presumes the skew part of the nonsymmetric fundamental tensor to be a new, unknown field with possible novel couplings to matter, Einstein's theory assumes it to be of electromagnetic origin.

Einstein's unified field theory (EUFT; Einstein 1950, 1956) was considered untenable owing to its apparent failure to produce correct equations of motion for charged particles. This apparent untenability stimulated consideration of various modifications to the theory (Bose 1953; Bonnor 1954; Moffat and Boal 1975; Klotz 1982; Antoci 1989), including the recent work of Damour *et al.* (1992, 1993) and that on NGT (Moffat 1979, 1991, 1993; Cornish and Moffat 1993, 1994; Cornish *et al.* 1993). A comprehensive review and general outline of the early work was given by Goenner (1984).

The ostensible problem in EUFT is that charged particles do not appear to feel the electromagnetic field, a conclusion reached by Infeld (1950) for the earlier, and Callaway (1953) for the later version. Each investigator used a modified form of the Einstein–Infeld–Hoffmann (EIH) approximation scheme (Einstein and

Infeld 1949) which was developed to find the equations of motion of masses in general relativity (GR) from the free-space field equations alone. It turns out that the two versions of EUFT are essentially equivalent insofar as equations of motion are concerned, so we may confine our attention to the earlier, upon which Infeld's analysis is based.

The purpose of this paper is to show that an EIH analysis of EUFT is *inconclusive* as a test of the physical viability of the theory. There is *no* desire to demonstrate that the theory *is viable*—the purpose is merely to show that an EIH analysis is unable to conclude one way or the other.

In the case of the Infeld–Callaway analysis, the conventional interpretation of the $f_{\mu\nu}$ as the Faraday tensor causes the Coulomb interaction to vanish automatically due to 'extra' derivatives inherent in the interpretation. This would seem to suggest that the $f_{\mu\nu}$ would need to be interpreted as potentials rather than as Faraday-tensor-type derivatives of potentials, if there is to be any prospect of a Coulomb interaction in the EIH equations of motion for EUFT.

In fact, such an interpretation of the $f_{\mu\nu}$ is suggested by three separate observations: (i) by the precise way that the EIH equations of motion arise from the field equations of GR, Einstein–Maxwell Theory (EMT) and EUFT; (ii) by the intimation of a gauge-fixing role for equation (6) below by Einstein (1956) and Bergmann (1956); and (iii) by an analysis due to Papapetrou (1948). Under this interpretation of $f_{\mu\nu}$ as potentials, EUFT in its original form would yield the Coulomb interaction in the EIH equations of motion, were it not for the fact that EUFT is put into the radiation gauge by the condition (6).

Since the EIH scheme produces a similarly vanishing Coulomb interaction when applied to EMT in the radiation gauge, the vanishing of the Coulomb term in this gauge for EUFT is not sufficient to conclude that EUFT is therefore necessarily unphysical, any more than the same situation in EMT is sufficient to conclude that EMT is unphysical—one recognises the lack of a Coulomb interaction to result from the choice of *gauge*. Of course, this is very different from asserting that EUFT *is* physical, which is not the intention. The purpose here is merely to show that, since an EIH analysis is inconclusive as a test of the viability of EUFT, historically, EUFT, rejected as it was on the basis of such an analysis, was rejected for the wrong reason.

2. Field Equations

In EUFT, both the fundamental tensor $g_{\mu\nu}$ and the connection $\Gamma_{\mu\nu}^\alpha$ are assumed to be nonsymmetric. The field equations are (Einstein 1950; Infeld 1950)

$$g_{\mu\nu,\lambda} = g_{\sigma\nu}\Gamma_{\mu\lambda}^\sigma + g_{\mu\sigma}\Gamma_{\lambda\nu}^\sigma, \quad (1)$$

$$\Gamma_\mu \equiv \frac{1}{2}(\Gamma_{\mu\tau}^\tau - \Gamma_{\tau\mu}^\tau) = 0, \quad (2)$$

$$R_{(\mu\nu)} = 0, \quad (3)$$

$$R_{[\mu\nu]} = 0; \quad (4)$$

$$\text{where } R_{\mu\nu} = \partial_\sigma \Gamma_{\mu\nu}^\sigma - \frac{1}{2}(\partial_\nu \Gamma_{(\mu\sigma)}^\sigma + \partial_\mu \Gamma_{(\nu\sigma)}^\sigma)$$

$$- \Gamma_{\mu\tau}^\sigma \Gamma_{\sigma\nu}^\tau + \Gamma_{\mu\nu}^\sigma \Gamma_{(\sigma\tau)}^\tau. \quad (5)$$

In this paper, Greek letters denote space-time indices (0,1,2,3) and Latin letters denote spatial indices (1,2,3). Round (square) brackets around indices denote symmetry (skew-symmetry) as usual.

The symmetric part of the nonsymmetric fundamental tensor is identified with the metric tensor of GR, and the skew part $f_{\mu\nu}$ was conventionally identified with the Faraday field tensor of classical electromagnetism. Equation (2) gives rise to the equivalent equation

$$(\sqrt{-g}f_{\mu\sigma})'^{\sigma} = 0, \quad (6)$$

where $g = \det(g_{\mu\nu})$, which thus has the form of Maxwell equations.

It was the failure of EUFT, with this interpretation of $f_{\mu\nu}$, to produce the Coulomb interaction in the EIH equations of motion that led to the widely-held view (Misner *et al.* 1973) that EUFT is unphysical, and prompted numerous modifications of the theory (Goenner 1984).

3. EIH Equations of Motion

The equations of motion for EUFT are obtainable using the EIH approximation scheme as employed by Infeld (1950) [and Callaway (1953)]. Here we simply give an outline of the basic method. Full details may be found elsewhere (Einstein and Infeld 1949; Wallace 1940, 1941; Scheidegger 1953).

The EIH scheme was originally developed to answer the question of whether the free-space field equations of GR were sufficient to produce the equations of motion of particles which were not test particles, i.e. which were also a source of the field (the geodesic principle applies to test particles only). Einstein suspected the non-linearity of the field equations of GR might give rise to constraints on the motions of particles, and thereby yield the equations of motion.

The field equations are split, with the time component distinguished. The basic assumption is of 'slow' motion with respect to the speed of light, which may be easily formalised in terms of an expansion parameter based on the 'speed of motion' rather than on 'strength of field'. Particles are represented as singularities in the field. Since the field equations do not therefore hold at the positions of the particles, they are each surrounded by a closed surface upon which the field equations do hold (and each surface encloses only one particle). The field equations are integrated, and it turns out that the values of the surface integrals are independent of the shapes of the surfaces—they depend only on the co-ordinates of the singularities and their time derivatives. One finds that in order for the whole system of equations to remain consistent at each successive instant of time, the surface integrals must take certain values. Since the value of the surface integrals depends only upon the motion of the enclosed singularities, the singularities are thereby effectively constrained to move in certain ways. In other words, the surface integrals imply integrability conditions and these conditions are the equations of motion of the particles.

The Newtonian equations of motion arise in the first iteration of the EIH approximation, at the fourth order in the expansion parameter. When there is an electromagnetic energy-momentum tensor in the field equations, the correct charged-particle equations of motion emerge.

(3a) *EIH for GR and EMT*

In the EIH method, the character of the entire approximation scheme depends upon the choice of solution of the lowest-order equations. No use is made of the more familiar exact solutions of the field equations, such as the Schwarzschild in GR or Reissner–Nordström in EMT—they have no role whatsoever in the EIH scheme. The EIH approximation scheme is independent of the results of the (full non-linear) field equations, such as these exact solutions. The point of contact between the field equations and the EIH scheme lies in the assumed character of the field functions.

Thus in GR, where the gravitational functions (i.e. the symmetric $g_{\mu\nu}$) are interpreted as potentials, the lowest-order (i.e. second-order) EIH functions were chosen to be Newtonian potentials φ , with φ being the total sum of each particle's own m/r -type Newtonian gravitational potential.

In EMT, the electromagnetic functions A_μ are interpreted as electromagnetic potentials. To the lowest order (second), one obtains simply the Coulomb potential Φ , with Φ being the total sum of each particle's own q/r -type Coulomb potential.

In the 'Newtonian' approximation, the EIH equations of motion for EMT arise (Wallace 1941) from a surface integral whose integrand is formed from the fourth-order part of

$$P_{ij} + \frac{1}{2}\delta_{ij}\eta^{\alpha\beta}P_{\alpha\beta} + 2(T_{ij} + \frac{1}{2}\delta_{ij}\eta^{\alpha\beta}T_{\alpha\beta}) = 0, \quad (7)$$

where $P_{\mu\nu}$ is the usual Ricci tensor of GR, $T_{\mu\nu}$ is the usual electromagnetic energy-momentum tensor, and $\eta_{\mu\nu}$ is the flat space metric, $\text{diag}(1, -1, -1, -1)$. The integrand contains derivatives and/or products of the second-order Newtonian and Coulomb potentials. The GR equations of motion are obtained by putting $T_{\mu\nu} = 0$.

For later comparison, we show the result for the simplest case—two particles—and give only the equations of motion for particle '1'. Those of particle '2' are analogous. The equations of motion of particle '1' are found to be (in 3-vector form)

$$m_1\ddot{\mathbf{x}} + 2m_1\nabla[m_2/r] = 0, \quad (8)$$

where r is the distance between the two masses, $\mathbf{x}$ denotes the position 3-vector of particle '1', dots denote time derivatives, and m_1 and m_2 denote the masses of particles '1' and '2' respectively. We see that this is the Newtonian equation of motion for particle '1' moving in the gravitational field of particle '2'.

The modification to the gravitational equations of motion (8), brought about by the electromagnetic field in (7), is found from the surface integral of the non-gravitational terms. These are (Wallace 1940, 1941)

$$-4\Phi_{,i}\Phi_{,j} + 2\delta_{ij}\Phi_{,s}\Phi_{,s}, \quad (9)$$

and yield a term of the form

$$q_1\nabla[q_2/r] \quad (10)$$

which is clearly the Coulomb force on particle ‘1’ (with charge q_1) due to the field of particle ‘2’ (with charge q_2).

(3b) *EIH for EUFT*

In EUFT, the modification to the fourth-order gravitational equations of motion due to the electromagnetic field is found in a way exactly analogous to that used for EMT. The $f_{\mu\nu}$ were conventionally interpreted as Faraday tensor derivatives of the electromagnetic 4-potential. The EIH integrand is formed from the fourth-order part of (3) by way of

$$R_{(ij)} + \frac{1}{2}\delta_{ij}\eta^{\alpha\beta}R_{(\alpha\beta)} = 0, \quad (11)$$

and contains the gravitational part of (7) (the P) which yields the same equations of motion (8) as in GR, together with non-gravitational terms containing products of the $f_{\mu\nu}$ and their derivatives.

The EUFT modification to the gravitational equations of motion is found by evaluating the surface integral of these non-gravitational terms, as in EMT, although in EUFT these terms appear in an integrand more complicated than that for EMT.

This was the basis of Infeld’s (1950) approach. However, Infeld did not fully simplify his non-gravitational integrand and, by use of the so-called Lemma of the EIH procedure (a formal algebraic result), he was able to correctly conclude, *without evaluating any of the terms*, that the integral vanished. This use of the Lemma, however, obscures what turns out to be an important fact.

Simplifying Infeld’s integrand (Infeld 1950), we find it is

$$\Phi_{,is}\Phi_{,js} - \Phi_{,s}\Phi_{,ijs}, \quad (12)$$

which is to be compared with (9). The gravitational equations of motion in EUFT are thereby modified (Voros 1994) by a term of the form

$$q_1\nabla[\nabla^2(q_2/r)]. \quad (13)$$

The ‘extra’ derivatives in (12) compared to (9) result in the appearance of the Laplacian operator ∇^2 in the EUFT analogue (13) of the EMT Coulomb force term (10) of the equations of motion. Since q/r is a harmonic function, this term will vanish identically *even if the Coulomb potential does not*, and there is therefore no electromagnetic modification to the gravitational equations of motion, whence Infeld’s (and thus Callaway’s) inference that charged particles do not feel the electromagnetic field. Infeld’s use of the Lemma meant he did not observe this. However, with the details of the calculation laid bare in this way we see that any possible Coulomb contribution to the EIH equations of motion is *automatically* precluded, entirely on account of these ‘extra’ derivatives, which are engendered by the conventional identification of $f_{\mu\nu}$ with *derivatives* of electromagnetic potentials.

These ‘extra’ derivatives would be avoided by an identification of the $f_{\mu\nu}$ with potentials, which is consonant with the following observations of Einstein, Bergmann and Papapetrou.

Einstein (1956) noted that the curvature tensor, formed as usual from the $\Gamma_{\mu\nu}^\alpha$, is invariant under the substitution (' λ -transformation')

$$\Gamma_{\mu\nu}^\alpha \rightarrow \Gamma_{\mu\nu}^\alpha + \delta_\mu^\alpha \lambda_{,\nu}, \quad (14)$$

where λ is an arbitrary function of the coordinates. This $U(1)$ invariance he termed ' λ -invariance'. The non-symmetric Ricci tensor, formed as usual from the curvature tensor, is also invariant under a λ -transformation. Einstein then noted that postulating the equations (2) involves a normalisation of the Γ -field, which removes the λ -invariance of the system of equations—the non-symmetric Ricci tensor reduces to the $R_{\mu\nu}$ defined in equation (5) if equation (2) is assumed.

Bergmann (1956) noted that λ -transformations are related to electromagnetic $U(1)$ gauge transformations. Thus (2) implies, by way of (6), that the $f_{\mu\nu}$ are potentials, since gauge conditions are generally of the nature of single-derivative constraints.

The possibility of interpreting the $f_{\mu\nu}$ as potentials was explicitly noted earlier by Papapetrou (1948), who analysed the field equations of EUFT in order to compare them with those of EMT. He concluded that the simplest interpretation of $f_{\mu\nu}$ was not as the field itself, but as the potential of the field. The apparent concern, that the skew tensor $f_{\mu\nu}$ is of a different character from the vector potential A_μ of Maxwell's electromagnetic theory, and thus that it might not be able to give rise to the correct number of degrees of freedom for a photon field, when examined in detail, turns out to be unfounded (see Section 4 below).

As we noted earlier, in GR, where the gravitational functions are interpreted as potentials, the lowest-order EIH functions were chosen to be Newtonian potentials. In EMT, the electromagnetic functions are interpreted as electromagnetic potentials, and this gives rise, at the lowest order, to the Coulomb potential. Thus also in EUFT, informed by the interpretation of $f_{\mu\nu}$ as potentials, the lowest-order electromagnetic functions in the $f_{\mu\nu}$ must correspond to Coulomb potentials. This must be done in a way which follows the identification made for the metric in GR and EMT, i.e. that the lowest-order $f_{\mu\nu}$ be treated as a set of Coulomb potentials in accordance with the manner in which the lowest-order functions in the metric are treated as a set of Newtonian potentials in the EIH approach to GR and EMT.

When this is done, we find (Voros 1994) that in EUFT the gravitational equations of motion are modified by a term of the form

$$q_1 \nabla(q_2/r). \quad (15)$$

This term is to be compared with equation (10). It is clearly the Coulomb force on particle 1 moving in the field of the other particle.

It follows from (15), therefore, that a Coulomb term *may* exist in the EIH equations of motion of EUFT in the interpretation of $f_{\mu\nu}$ as potentials, in contrast to the situation under the conventional interpretation as Faraday-tensor-type derivatives of potentials.

However, equation (6), which has been identified, following the observations of Einstein and Bergmann, as a gauge-fixing equation, can be shown to take the

form $\Phi = 0$, $\nabla \cdot \mathbf{A} = 0$ (to third order in the EIH scheme), which specifies the radiation gauge.

In EMT, imposing the radiation gauge on the EIH scheme causes all the particle charges to be constrained to vanish. Since all the electromagnetic EIH functions depend on the charges, it follows that if the radiation gauge is imposed, then all the EIH functions vanish to all orders, the scheme ‘collapses’, and is thus inapplicable even in the case of EMT. There is nothing mysterious about this—the EIH scheme was developed to describe the interaction of *particles*; it is ill-suited to describing *radiation*.

It therefore follows that this gauge constraint in EUFT, similarly constraining all charges to be zero, precludes a charged-particle interaction in the EIH equations of motion for EUFT. There is thus no Coulomb interaction in the equations of motion, despite the provision for one in (13), because it is proscribed by the gauge entailed by equation (6).

4. Comments

Some comments on the interpretation of $f_{\mu\nu}$ as potentials outside the context of the EIH approximation scheme are in order.

At first glance it may seem unlikely that the skew tensor $f_{\mu\nu}$ could contain the degrees of freedom of the two helicity states of a massless spin-1 particle; one might be tempted to think that because the theory has a 2-form playing the role of gauge potentials, it might describe instead an axion field or something similar. However, such a conclusion is premature. The actual behaviour of the $f_{\mu\nu}$, at least in the linearised case, is very suggestive, as we now report.

A spin-projection analysis of the implied particle spectrum performed recently by Moffat (1993) on the linearised skew field equations of NGT (which has similar field equations to the later Einstein theory, and identical linearised skew free-space field equations to those of the later EUFT) reveals that the six functional degrees of freedom in the skew $f_{\mu\nu}$ actually behave like two spin-1 fields (i.e. each having three functional degrees of freedom), both massless. It turns out that one of these does not propagate, so the six $f_{\mu\nu}$ effectively possess the same behaviour as a single propagating massless spin-1 field.

However, this spin-1 field is further constrained to have only a scalar degree of freedom, due to the presence of the constraint equation (6) in the theory. [It can be shown that in the classical linearised skew field equations, the apparently six degrees of freedom in the $f_{\mu\nu}$ are reduced, by the linearised form of the constraint (6), to a single degree of freedom.]

Moffat interprets this one remaining scalar degree of freedom as indicating the skew field to be a scalar field; however, we can see from the details of Moffat’s result that the linearised skew free-space field equations of EUFT for the $f_{\mu\nu}$ may also be interpreted as a constrained propagating massless spin-1 field. Indeed, by adding mass to the skew field $f_{\mu\nu}$ of their theory, Damour *et al.* (1993) were able to explicitly extract all three spin-1 functional degrees of freedom, whence their interpretation of $f_{\mu\nu}$ as a massive fifth-force-type vector field.

What role the constraint equation (6) has in EUFT remains an interesting question—within the limited context of the EIH approximation, it behaves like a gauge-fixing equation. It would have a more complex role outside this approximation.

5. Conclusion

Equation (6) reduces to Maxwell's equations given the *a priori* identification of $f_{\mu\nu}$ with the electromagnetic (Faraday) field tensor. We have seen that this identification frustrates the inference of a Coulomb interaction via the EIH procedure owing to the presence of 'extra' derivatives on the Coulomb potential introduced by this identification. The (EIH-independent) gauge-fixing character of equation (6), to which Einstein and Bergmann allude, however, suggests an alternative identification of $f_{\mu\nu}$ with electromagnetic potentials, which was independently noted earlier by Papapetrou to be the simplest interpretation of the $f_{\mu\nu}$.

Given this identification of $f_{\mu\nu}$ as potentials, equation (6) imposes, within the context of the EIH scheme, a gauge which precludes the existence of a charged-particle interaction in the equations of motion—the radiation gauge. The absence of a Coulomb interaction in EMT in the radiation gauge is not usually taken to mean that EMT is therefore physically unviable. It is clear that the absence of such an interaction in this circumstance for Einstein's theory also does not allow the standard inference that EUFT is therefore necessarily unphysical owing to the lack of such an interaction, which latter was the contention of Infeld (1950) and Callaway (1953).

Thus, an EIH analysis of EUFT is inconclusive as a test of the physical viability of the theory. This is not to say, of course, that the theory is viable—we have merely found that an EIH analysis, which, historically, was the basis of rejecting EUFT, is unable to determine the physical viability of EUFT one way or the other.

Given that the skew field $f_{\mu\nu}$ has been found to behave like a constrained massless propagating spin-1 field in the linearised approximation, one is led to wonder what might emerge from Einstein's unified field theory if the constraint equation (6) on the $f_{\mu\nu}$ were to be relaxed.

Acknowledgments

The author would like to thank Dr H. S. Perlman for his inestimable support, for a critical reading of the manuscript and for many helpful discussions and suggestions, and to the referee for pertinent observations and helpful suggestions. This work was supported in part by an Australian Postgraduate Research Award.

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