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# Swarms in Periodically Time Dependent Electric Fields*

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*Abstract*

The transport theory of swarms in a time dependent electric field is formulated in terms of the Boltzmann equation expressed as an integral equation in time. It allows for a convenient physical description of the time development of the swarm for the case of a periodically time dependent field. The exact solution of the ideal charge transfer or BGK model, obtained earlier by Robson and Makabe, is expressed in a closed form.

## 1. Introduction

Recently, Robson and Makabe (1994) obtained an exact expression for the distribution function of a swarm with ideal charge transfer interactions in a cold gas described by the BGK model, in the presence of an AC electric field. They also obtained closed formulas for the drift, diffusion and temperature in the long time limit where these quantities become periodic in time with the period and phase simply related to those of the AC field. The distribution function in this limit is also periodic and shows a very interesting peaked structure, of a kind which has been observed in certain experiments.

In this paper an alternative derivation of the Robson and Makabe (1994) results is given. It is shown that the series obtained by them can be summed up in terms of two fairly compact functions and a physical description of the mechanism by which the distribution function comes to be established can also be given.

The method used here has connections with some other papers included in this special *Aust. J. Phys.* issue. Its basic equations are developed in the next section for an arbitrary time dependence of the field and for any type of collision model. This enables us to cover some of the points which arose in discussions at the Workshop.

In subsequent sections specialisation to the BGK model and fields periodic in time is made.

## 2. Basic Equations

We consider the following situation: an initial distribution $f_{\mathrm{in}}(\boldsymbol{r}, \boldsymbol{c})$ is given at time $t_0$, an external source adds a distribution $\mathcal{S}(\boldsymbol{r}, \boldsymbol{c}, t)$, the particles are

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subject to an acceleration $\boldsymbol{a}(t)$ caused by an external field and undergo *direct collisions* due to which a fraction $\nu(\boldsymbol{c})\, f(\boldsymbol{r}, \boldsymbol{c}, t)$ of the particles is lost from the phase space volume element $\mathrm{d}\boldsymbol{r}\, \mathrm{d}\boldsymbol{c}$ at $(\boldsymbol{r},\boldsymbol{c})$, and *inverse collisions* due to which particles of all velocities $\boldsymbol{c}'$ in the volume element $\mathrm{d}\boldsymbol{r}$ at $\boldsymbol{r}$ contribute an amount

$$Kf \equiv \int \mathrm{d}\boldsymbol{c}'\; K(\boldsymbol{c}, \boldsymbol{c}')\, f(\boldsymbol{r}, \boldsymbol{c}', t) \tag{1}$$

to the phase space volume element $\mathrm{d}\boldsymbol{r}\, \mathrm{d}\boldsymbol{c}$ at $(\boldsymbol{r}, \boldsymbol{c})$.

Under the action of the field a particle starting at time $t_0$ with position $\boldsymbol{r} - \boldsymbol{R}(t, t_0)$ and velocity $\boldsymbol{c} - \boldsymbol{C}(t, t_0)$ arrives at the phase space point $(\boldsymbol{r}, \boldsymbol{c})$ at time $t$, where

$$\boldsymbol{C}(t, t_0) = \int_{t_0}^{t} \mathrm{d}t'\; \boldsymbol{a}(t')\,, \tag{2}$$

$$\boldsymbol{R}(t, t_0) = (t - t_0)\boldsymbol{c} + \int_{t_0}^{t} \mathrm{d}t' \int_{t_0}^{t'} \mathrm{d}t''\; \boldsymbol{a}(t'')\,. \tag{3}$$

The probability that a particle completes the trajectory without suffering any *direct collisions* is given by

$$P(\boldsymbol{c}, t|t_0) = \exp\left( - \int_{t_0}^{t} \nu(\boldsymbol{c} - \boldsymbol{C}(t, t'))\; \mathrm{d}t' \right). \tag{4}$$

The distribution function $f(\boldsymbol{r}, \boldsymbol{c}, t)$ which gives the number of particles in the phase space volume element $\mathrm{d}\boldsymbol{r}\, \mathrm{d}\boldsymbol{c}$ at $(\boldsymbol{r}, \boldsymbol{c})$ at time $t$ may now be obtained as the sum of contributions from the initial distribution at time $t_0$, from the source and from other parts of the distribution, and is given by

$$\begin{aligned} f(\boldsymbol{r}, \boldsymbol{c}, t) = & \int_{t_0}^{t} \mathrm{d}t'\; P(\boldsymbol{c}, t|t') \int \mathrm{d}\boldsymbol{c}''\; K(\boldsymbol{c} - \boldsymbol{C}(t, t');\; \boldsymbol{c}'')\, f(\boldsymbol{r} - \boldsymbol{R}(t, t');\; \boldsymbol{c}'';\; t') \\ & + \int_{t_0}^{t} \mathrm{d}t'\; P(\boldsymbol{c}, t|t')\, S(\boldsymbol{r} - \boldsymbol{R}(t, t');\; \boldsymbol{c} - \boldsymbol{C}(t, t');\; t') \\ & + h(\boldsymbol{r}, \boldsymbol{c}, t|t_0)\,. \end{aligned} \tag{5}$$

The first term on the right-hand side arises because at any time $t'$, with $t_0 \le t' \le t$, inverse collisions transfer particles to the phase space volume element at $(\boldsymbol{r}', \boldsymbol{c}')$ according to (1), and these then travel under the influence of the field along certain trajectories. The contribution to the phase space volume element at $(\boldsymbol{r}, \boldsymbol{c})$ at time $t$ is obtained by multiplying $Kf$ with the probability of survival along the trajectory and integrating over $t'$ with such values of $(\boldsymbol{r}', \boldsymbol{c}')$ that the trajectory ends at $(\boldsymbol{r}, \boldsymbol{c})$ at time $t$. The second term is similar, with the source putting particles into the volume element at $(\boldsymbol{r}', \boldsymbol{c}')$ at $t'$. The last term is the contribution from the initial distribution given by

$$h(\boldsymbol{r}, \boldsymbol{c}, t|t_0) = P(\boldsymbol{c}, t|t_0)\, f_{\mathrm{in}}(\boldsymbol{r} - \boldsymbol{R}(t, t_0);\; \boldsymbol{c} - \boldsymbol{C}(t, t_0))\,. \tag{6}$$

If the initial distribution is thought of as a source proportional to $\delta(t-t_0)$ it may be included in the second term.

It can be verified (see the Appendix) by direct differentiation of (5) that $f$ satisfies the Boltzmann equation

$$(\partial_t - \boldsymbol{a}\,.\,\partial_c + \boldsymbol{c}\,.\,\partial_r + \nu - K) f(\boldsymbol{r}, \boldsymbol{c}, t) = \mathcal{S}(\boldsymbol{r}, \boldsymbol{c}, t)\,, \tag{7}$$

and that at $t = t_0$

$$f(\boldsymbol{r}, \boldsymbol{c}, t_0) = h(\boldsymbol{r}, \boldsymbol{c}, t_0 | t_0) = f_{\text{in}}(\boldsymbol{r};\ \boldsymbol{c})\,. \tag{8}$$

Time $t_0$ is arbitrary and there is no significance to $t \le t_0$ in these considerations.

The (integro-) differential equation (7) is usually written without a source term. Without a proper specification of the initial condition it does not have a unique solution, since given any solution of (7) one can always obtain another by adding an arbitrary solution of the equation

$$(\partial_t + \boldsymbol{a}\,.\,\partial_c + \boldsymbol{c}\,.\,\partial_r + \nu) h(\boldsymbol{r}, \boldsymbol{c}, t) = 0\,. \tag{9}$$

With these provisos equations (7) and (8) may in their turn be converted to (5), but the argument above leading to (5) has the advantage of displaying the physical requirements which lead to a unique solution. However, starting from (5), especially in numerical and approximate considerations, does not confer automatic immunity from errors that may be committed by insufficient attention to this aspect of the problem.

The following spatial moments of the distribution function are specially important in the calculation of transport coefficients

$$F^{(0)}(\boldsymbol{c}, t) = \int \mathrm{d}\boldsymbol{r}\ f(\boldsymbol{r}, \boldsymbol{c}, t)\,, \tag{10a}$$

$$\boldsymbol{F}^{(1)}(\boldsymbol{c}, t) = \int \mathrm{d}\boldsymbol{r}\ \boldsymbol{r} f(\boldsymbol{r}, \boldsymbol{c}, t)\,, \tag{10b}$$

$$\boldsymbol{F}^{(2)}(\boldsymbol{c}, t) = \tfrac{1}{2} \int \mathrm{d}\boldsymbol{r}\ \boldsymbol{r}\boldsymbol{r} f(\boldsymbol{r}, \boldsymbol{c}, t)\,, \tag{10c}$$

since their velocity moments are related to the total number of particles, the position of the centroid and the spread of the swarm as follows:

$$N(t) = \int \mathrm{d}\boldsymbol{c}\ \boldsymbol{F}^{(0)}(\boldsymbol{c}, t)\,, \tag{11a}$$

$$\langle \boldsymbol{r} \rangle_t = \int \mathrm{d}\boldsymbol{c}\ \boldsymbol{F}^{(1)}(\boldsymbol{c}, t)\,, \tag{11b}$$

$$\langle \boldsymbol{r}\boldsymbol{r} \rangle_t = 2 \int \mathrm{d}\boldsymbol{c}\ \boldsymbol{F}^{(2)}(\boldsymbol{c}, t)\,. \tag{11c}$$

Equations for the determination of $\boldsymbol{F}^{(n)}(\boldsymbol{c},t)$ can be obtained from (5). They form a hierarchy in which the first member is

$$F^{(0)}(\boldsymbol{c},t) = \int_{t_0}^{t} \mathrm{d}t'\ P(\boldsymbol{c},t|t') \int \mathrm{d}\boldsymbol{c}''\ \boldsymbol{K}(\boldsymbol{c}-\boldsymbol{C}(t,t');\ \boldsymbol{c}'')\, F^{(0)}(\boldsymbol{c}'',t')$$
$$+ \int_{t_0}^{t} \mathrm{d}t'\ P(\boldsymbol{c},t|t')\, \mathcal{S}^{(0)}(\boldsymbol{c}-\boldsymbol{C}(t,t');\ t') + h^{(0)}(\boldsymbol{c},t|t_0)\,, \qquad (12)$$

where the superscripted symbols are the spatial moments of the corresponding quantities in (5). The equations for higher moments are similar except that they involve the lower moments in the inhomogeneous term. A hierarchy of differential equations for the same functions was derived in Kumar (1981). The relationship between that hierarchy and the present one is similar to the relationship between equations (7) and (5).

In either form the hierarchy is not equivalent to the original single equation it is derived from, because of the intervening assumption that the moments exist. The physical content of this assumption is very similar to that of assuming that a gradient expansion is a good representation of the spatial dependence of the distribution function. In the literature the experience with this hierarchy is usually limited to the case of a time independent field with the number of particles conserved so that only $F^{(0)}$ and $\boldsymbol{F}^{(1)}$ need be considered in the calculation of drift and diffusion. [See, however, Robson and Ness (1986) and Ness and Robson (1986) where the latter restriction is removed.]

In experiments, other than the steady state ones, the source acts only for a finite time. It is then possible to choose $t_0$ large enough so that the source term vanishes. If $\max_c \nu(\boldsymbol{c}) = \nu_{\mathrm{m}} > 0$, then

$$P(\boldsymbol{c},t|t_0) \le \exp(-\nu_{\mathrm{m}}(t-t_0))$$

and the memory of the initial distribution is lost in the long time limit. On the basis of this the inhomogeneous term in (12) is dropped, the distribution function is taken to be independent of time and the time integration taken to extend from 0 to $\infty$. In this way a homogeneous time independent equation is obtained for the space integrated distribution function $F^{(0)}$. It is valid if $F^{(0)}$ exists and is independent of time in the long time limit. Skullerud and Kuhn (1983) have given reasons why this may be the case in many physically interesting situations and pointed out a case where it is not so. In favourable circumstances it can be shown that in the long time limit both $(\langle \boldsymbol{rr}\rangle_t - \langle \boldsymbol{r}\rangle_t\, \langle \boldsymbol{r}\rangle_t)$ and $\langle \boldsymbol{r}\rangle_t$ are proportional to $t$ and that $F^{(0)}$ and $\boldsymbol{F}^{(1)}$ suffice for the determination of the drift velocity and diffusion coefficient. In such cases $\boldsymbol{F}^{(1)}$ may be expressed in terms of $F^{(0)}$ but careful handling of the relationship between these functions is necessary to ensure that the diffusion coefficient is correctly obtained. Skullerud and Kuhn (1983) pointed out a case in connection with the calculation of the diffusion coefficient where errors were committed at this point. Note that if the number of particles is not conserved one needs both $F^{(0)}$ and $\boldsymbol{F}^{(1)}$ in the calculation of drift and $\boldsymbol{F}^{(1)}$ and $\boldsymbol{F}^{(2)}$ in the calculation of diffusion.

Equations of this kind arising from (12) are used in the path integral methods. These equations may be converted into equations for $KF^{(0)}$ as the unknown function, which could then be thought of as the basis of another independent 'method'. The remarks above are equally relevant to both these methods and may help in clarifying their relationship to other approaches.

It is worth noting that if the source function and the initial distribution in (5) are independent of $\boldsymbol{r}$, then the distribution function will also be independent of $\boldsymbol{r}$ and (5) will then appear to be the same as (12). This could be the source of some confusion which occasionally appears. The resolution is simple: a space independent distribution function has no spatial moments, there is no hierarchy and no equation (12). Put differently, in such a case the swarm may have an average velocity but it does not have a centroid. Consequently there is no sensible way of identifying a drift velocity or diffusion which is related to the spatial spread of the swarm about the centroid. When these quantities are expressed in terms of $F^{(0)}$ the use of a hierarchy of equations is essential.

Equation (5) is convenient for studying the effects of a time dependent field. It seems that only the sinusoidally varying AC fields are important for experiments. These are a special case of periodic fields where in general one may expect that the distribution function and the transport coefficients become periodic in time after a large number of periods. Considerations similar to those described above for the time independent field may then be applied, taking into account the expected asymptotic behaviour.

## 3. Ideal Charge Transfer Model

In the ideal charge transfer model, singly charged ions move in the parent gas of the same species. Upon collision the charges on the colliding particles are interchanged. The process is described by the so called BGK collision integral, where the collision frequency is independent of velocity, $\nu(\boldsymbol{c}) = \nu$, and the operator in equation (1) takes the form

$$Kf \equiv \nu w(\boldsymbol{c}) \int \mathrm{d}\boldsymbol{c}\, f(\boldsymbol{r}, \boldsymbol{c}, t)\,, \tag{13}$$

where $w(\boldsymbol{c})$ is the Maxwellian distribution with the usual meaning of the symbols

$$w(\boldsymbol{c}) = (\alpha^2/2\pi)^{\frac{3}{2}} \exp(-\tfrac{1}{2}\alpha^2 c^2)\,, \qquad \alpha^2 = m/kT_0\,.$$

In the cold-gas limit the particles of the host gas are stationary and $w(\boldsymbol{c})$ is replaced by a delta-function. In physical terms this means that given a collision frequency $\nu$, at any time $t$ a fraction $\nu dt$ of the particles is brought to rest due to inverse collisions and is reaccelerated in the AC field starting from this zero velocity.

### *(3a) Spatially Uniform Distribution in a Cold Gas*

We now take advantage of the simplicity of this model to analyse the distribution function in a different way. Consider an initial distribution $f_{\mathrm{in}}(\boldsymbol{c})$ of ions introduced in such a gas at time $t = 0$. If there were no collisions this will become a periodic distribution $f_{\mathrm{in}}(\boldsymbol{c} + (\boldsymbol{a}/\omega)\sin\omega t)$ under the action of the

AC field $\boldsymbol{a}\cos\omega t$. As a result of direct collisions particles are lost from this distribution and at time $t$ only a fraction $e^{-\nu t} f_{\rm in}$ remain. We may regard the lost particles as forming a new distribution whose form is determined by the cycle in progress. Let this distribution, due to the uncompleted cycle, be called $f_{\rm u}(\boldsymbol{c},\tau)$, where $\tau$ is the time from the beginning of the cycle. At the end of the cycle the distribution so generated becomes the input to the next cycle. Each completed cycle therefore gives rise to a contribution $f_{\rm c}(\boldsymbol{c},\tau)$ multiplied by the probability of its survival which depends on the time it came into being. After $N$ completed cycles at time $t = (2\pi/\omega)N + \tau/\omega$, $0 \le \tau \le 2\pi$, the total distribution function thus has the form

$$f(\boldsymbol{c},t) = f_{\rm u}(\boldsymbol{c},\tau) + (e^{-2\pi\nu N/\omega} + e^{-2\pi\nu(N-1)/\omega} + \dots e^{-2\pi\nu/\omega}) f_{\rm c}(\boldsymbol{c},\tau)$$
$$+ e^{-\nu t} f_{\rm in}(\boldsymbol{c} + (\boldsymbol{a}/\omega)\sin\omega\tau)\,, \tag{14a}$$

$$\lim_{N\to\infty} f(\boldsymbol{c},t) = f_{\rm u}(\boldsymbol{c},\tau) + (e^{2\pi\nu/\omega} - 1)^{-1} f_{\rm c}(\boldsymbol{c},\tau)\,. \tag{14b}$$

Since the number of particles is conserved in this model $n_0 = \int f(\boldsymbol{c},t)\,\mathrm{d}\boldsymbol{c}$ is independent of time. Because of the initial condition chosen here equation (5) may be taken independent $\boldsymbol{r}$. The field only affects the component $v$ of velocity in the direction of the field so that the transverse degrees of freedom can be eliminated. Making the change of variables $\omega t \to t$, $\nu/\omega \to \nu$, $\boldsymbol{c}\cdot\boldsymbol{a}/\omega a^2 \to v$ and setting $n_0 = 1$ we have, from (5), the equation for the relevant one dimensional distribution

$$f(v,t) = e^{-\nu t} f_{\rm in} + \int_0^t \mathrm{d}t'\, e^{-\nu(t-t')} \nu\delta(v - \sin t + \sin t')\,. \tag{15}$$

Splitting the time integration in cycles and changing the variables of integration in each cycle appropriately, we obtain the one dimensional analogues of the functions introduced in (14a):

$$f_{\rm c}(v,\tau) = \int_0^{2\pi} e^{-\nu(\tau-\tau')} \nu\delta(v - \sin\tau + \sin\tau')\,\mathrm{d}\tau'\,, \tag{16}$$

$$f_{\rm u}(v,\tau) = \int_0^{\tau} e^{-\nu(\tau-\tau')} \nu\delta(v - \sin\tau + \sin\tau')\,\mathrm{d}\tau'\,. \tag{17}$$

These can be expressed in a closed form in terms of the variables

$$\mu = |\sin\tau - v|; \quad \sigma = \mathrm{sgn}(\sin\tau - v); \quad \theta = \sin^{-1}\mu\,,$$

and the function

$$F(v,\tau) = \left(\cosh\frac{\nu\pi}{2} - \sigma\sinh\frac{\nu\pi}{2}\right) e^{-\nu(\tau-\pi)}/\sqrt{1-\mu^2}\,, \tag{18}$$

as

$$f_c(v,\tau) = F(v,\tau)\,2\cosh(\nu(\theta - \pi/2))\,, \tag{19}$$

$$f_u(v,\tau) = F(v,\tau)\left[e^{\nu(\theta-\pi/2)}\Theta\left(\tau - \theta - \frac{1-\sigma}{2}\pi\right) + e^{-\nu(\theta-\pi/2)}\Theta\left(\tau + \theta - \frac{1-\sigma}{2}\pi\right)\right], \tag{20}$$

with $\Theta(x) = 1$ for $x > 0$ and $\Theta(x) = 0$ for $x < 0$. These expressions apply for $\mu < 1$ and represent a summation of the series given by Robson and Makabe (1994). Both functions vanish for $\mu > 1$.

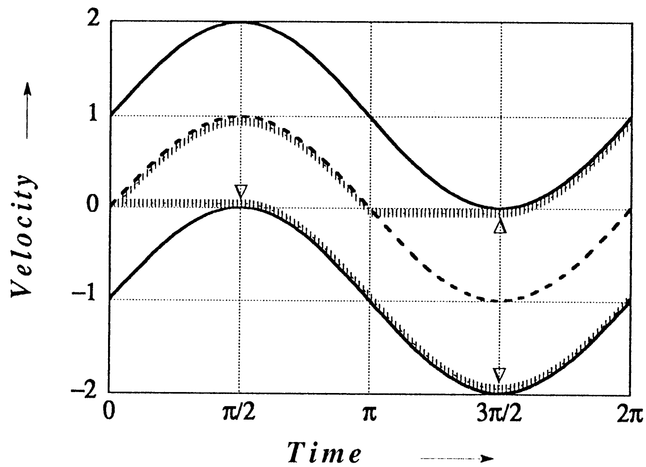


**Fig. 1.** The distribution function vanishes outside the area enclosed by the continuous curves. The function $f_u$ vanishes outside the area enclosed by the hatched curves. See the text for further explanation.

The physical process occurring in each cycle may be understood by reference to Fig. 1. Consider the process of formation of $f_u$ first. At each instant $\tau$ during the cycle, due to inverse collisions occurring over the whole distribution of particles, a fraction $\nu d\tau$ of the particles is brought to rest. We may picture this as being introduced along the line $v = 0$ at the point $\tau$. This infinitesimal group then travels along the sine curve passing through this point in the diagram and looses particles due to direct collisions at a rate determined by $\nu$. When the curve crosses the line $v = 0$ again, a new group is added to it causing a sudden jump in the number in the group. Therefore the function can be nonvanishing only over the region enclosed by the hatched curve in the figure. At the points indicated by the triangles the acceleration vanishes while the velocity is going through an extremum. A bunching of particles occurs which are then propagated along the continuous lines causing characteristic peaks in the distribution as the velocity approaches the values on the continuous lines. When $\tau = 2\pi$ there is a discontinuity at $v = 0$ because particles with $v < 0$ have had longer flight times and therefore more of them have been lost. This distribution at $\tau = 2\pi$ becomes the input distribution for the next cycle which is simply transported along sinusoidal curves. It retains its shape but is scaled down due to a uniform loss of particles from direct collisions and constitutes the function $f_c$.

One can also picture the development for a thermal gas in this model. The only difference is that at each instant of time the fraction $\nu d\tau$ of particles is introduced into the diagram as a thermal distribution in $v$ and each part of that is transported along relevant sinusoidal lines. In analytical terms the distribution $f_{\mathrm{Th}}$ for the case of a *finite temperature BGK model* is obtained from the cold-gas distribution by a convolution

$$f_{\mathrm{Th}}(v,\tau) = \int_{-\infty}^{\infty} w(v')\, f(v-v',\tau)\, \mathrm{d}v' \,. \tag{21}$$

One can allow for the process of *attachment and ionisation* by slightly altering equation (15) so that we have

$$f(v,t) = e^{-\nu' t} f_{\mathrm{in}} + e^{-(\nu'-\nu)t} \int_0^t \mathrm{d}t'\; e^{-\nu(t-t')} \nu\delta(v - \sin t + \sin t') \,. \tag{22}$$

The number of particles now varies as $n(t) = n_0 \exp(-(\nu'-\nu)t)$ and the distribution can still be expressed in terms of the functions $f_c$ and $f_u$ given above.

The discussion above was phrased in terms of a spatially independent distribution function. It is equally applicable to the space integrated distribution function denoted by $F^{(0)}$ in the previous section. When used in the calculation of drift velocity that is the proper interpretation.

Robson and Makabe (1994) have given plots of the distribution function over the $(v,\tau)$-plane calculated from their series solution. The singularities and discontinuities evident in the discussion above appear to be smoothed out in their plots, but the general features of the distributions are in agreement with what is found here. In particular the dependence on the parameter $\nu/\omega$ is clearly seen. Actually their distributions look very much like what one may expect from a thermal BGK model!

### *(3b) Distribution with Space Dependence in a Cold Gas*

For this model, equation (5) without the source term takes the form

$$f(\boldsymbol{r},\boldsymbol{c},t) = h(\boldsymbol{r},\boldsymbol{c},t|0) + \nu \int_0^t \mathrm{d}t'\; e^{-\nu(t-t') - \boldsymbol{r}(t,t')\cdot\nabla} \delta(\boldsymbol{c}(t,t'))\, n(\boldsymbol{r},t') \,, \tag{23}$$

where

$$n(\boldsymbol{r},t) = \int \mathrm{d}\boldsymbol{c}\; f(\boldsymbol{r},\boldsymbol{c},t) \,, \tag{24}$$

$$\boldsymbol{r}(t,t') = (\boldsymbol{a}/\omega^2)(\cos\omega t' - \cos\omega t - \omega(t-t')\sin\omega t) \,, \tag{25}$$

$$\boldsymbol{c}(t,t') = \boldsymbol{c} - (\boldsymbol{a}/\omega)(\sin\omega t - \sin\omega t') \,. \tag{26}$$

The term $h$, due to initial conditions decays as $\exp(-\nu t)$ and thus makes a negligible contribution at long times. Omitting this contribution in (27b) and (27c) we have the following equations for the quantities defined in (11):

$$N(t) = e^{-\nu t} N_0 + \nu \int_0^t \mathrm{d}t' \, \exp(-\nu(t-t'))\, N(t')\,, \tag{27a}$$

$$\langle \boldsymbol{r} \rangle_t = \nu \int_0^t \mathrm{d}t' \, \exp(-\nu(t-t'))[\langle \boldsymbol{r} \rangle_{t'} + \boldsymbol{r}(t,t')]\,, \tag{27b}$$

$$\langle \boldsymbol{r}\boldsymbol{r} \rangle_t = \nu \int_0^t \mathrm{d}t' \, \exp(-\nu(t-t'))[\langle \boldsymbol{r}\boldsymbol{r} \rangle_{t'} + \boldsymbol{r}(t,t')\langle \boldsymbol{r} \rangle_{t'} + \langle \boldsymbol{r} \rangle_{t'}\, \boldsymbol{r}(t,t') + \boldsymbol{r}(t,t')\, \boldsymbol{r}(t,t')]\,. \tag{27c}$$

It can be verified that the total number is conserved. The drift velocity is given by

$$\boldsymbol{W} = \frac{\mathrm{d}}{\mathrm{d}t} \langle \boldsymbol{r} \rangle_t = \frac{\boldsymbol{a}}{\sqrt{\nu^2+\omega^2}} \cos(\omega t - \phi) + \boldsymbol{a}\, e^{-\nu t}\left(\frac{\nu}{\nu^2+\omega^2} + \frac{\sin \omega t}{\omega}\right), \tag{28a}$$

$$\tan\phi = \omega/\nu\,. \tag{28b}$$

With $t = (2\pi/\omega)N + \tau$ in the limit $N \to \infty$, the second term disappears and the result is in agreement with that of Robson and Makabe. Note that the second term here is not due to the initial conditions. Further, if we take the limit $\omega \to 0$ the stationary field result $\boldsymbol{W} = \boldsymbol{a}/\nu$ is obtained as may be expected. Similar remarks apply also to the expressions obtained for the diffusion coefficient from

$$\boldsymbol{D}(t) = \tfrac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t}(\langle \boldsymbol{r}\boldsymbol{r} \rangle_t - \langle \boldsymbol{r} \rangle_t \langle \boldsymbol{r} \rangle_t)\,, \tag{29}$$

and for the temperature tensor from

$$k\boldsymbol{T} = m\,\langle \langle \boldsymbol{c}\boldsymbol{c} \rangle - \langle \boldsymbol{c} \rangle \langle \boldsymbol{c} \rangle \rangle\,. \tag{30}$$

The expressions for drift, diffusion and temperature obtained in this way agree with those given by Robson and Makabe.

*(3c) Remarks*

The singularities and discontinuities of the distribution function obtained here are very much a function of the particular time dependence, $\boldsymbol{a}\cos\omega t$, of the field. The physical reason for this was pointed out in Section 3$a$ above. Mathematically it is contained in equations (18)–(20) where in particular it is associated with the factor $\sqrt{1-\mu^2}$ in (18). Other types of field may not lead to such structure. In particular, sectionally linear time dependence of the field, the so-called saw tooth type of periodicity, does not lead to singularities.

The averages calculated in Section 3$b$ have a fairly transparent meaning but their actual determination in swarm experiments may require new types of arrangements. Theoretical investigation of the continuity equation with time dependent transport coefficients will also be relevant.

The corresponding quantities for the thermal BGK model are obtained by first repeating the above considerations with a moving cold gas, i.e. using $\delta(\boldsymbol{c}-\boldsymbol{c}_0)$ instead of $\delta(\boldsymbol{c})$ and then using a convolution similar to (21). The drift velocity does not change but temperature corrections to other quantities are obtained.

## 4. Collision Models of a General Kind

The equations for the model of Section 3 could be solved completely because the collision frequency $\nu$ is independent of velocity $\boldsymbol{c}$ and, more importantly, because the distribution $Kf$ generated by indirect collisions is the product of a known function, $\delta(\boldsymbol{c})$ or $w(\boldsymbol{c})$, of velocity and the density $n(\boldsymbol{r},t)$. This greatly simplified the integral equations.

In the general case the kernel of the collision operator,

$$J(\boldsymbol{c},\boldsymbol{c}') \equiv \nu(\boldsymbol{c})\,\delta(\boldsymbol{c}-\boldsymbol{c}') - K(\boldsymbol{c},\boldsymbol{c}')\,, \tag{31}$$

admits a spectral decomposition of the type

$$J(\boldsymbol{c},\boldsymbol{c}') = w(\boldsymbol{c}) \sum_{\alpha} \phi_{\alpha}^{*}(\boldsymbol{c})\,\nu_{\alpha}\,\phi_{\alpha}(\boldsymbol{c}')\,. \tag{32}$$

The BGK model corresponds to eigenvalues

$$\nu_{\alpha} = \nu(1-\delta_{\alpha 0})\,, \tag{33}$$

with $\phi_0(\boldsymbol{c}) = 1$. Hence by comparison, for a more general type of collision the complication is in carrying additional velocity moments corresponding to $\phi_{\alpha}$.

It appears that the decomposition of the time development of the distribution function into completed and uncompleted cycles is well based both in the physical and mathematical sense and can still be used, but the distribution functions for the completed cycles can no longer be represented in terms of a single function $f_{\mathrm{c}}$ as in the previous case. There is a sort of memory effect involved. To exhibit this we introduce the notation

$$f_n(\tau) \equiv f(\boldsymbol{r},\boldsymbol{c},t) \quad \text{for} \quad t=(2\pi n+\tau)/\omega\,, \quad 0 \le \tau \le 2\pi\,. \tag{34}$$

During the first cycle (5) is an equation for $f_0(\tau)$ of the form

$$f_0(\tau) = h_0 + \int_0^{\tau} \mathrm{d}\tau'\, L_0(\tau,\tau')\, f_0(\tau')\,, \tag{35}$$

where for sake of brevity we have not shown the $\boldsymbol{r}$ and $\boldsymbol{c}$ dependence explicitly. The expressions for the function $h_0$ and the operator $L_0$ can be inferred from (5). During the next cycle (5) is an equation for $f_1(\tau)$ which acquires an additional contribution to the inhomogeneous term due to the first completed cycle,

$$f_1(\tau) = h_1 + \int_0^{2\pi} \mathrm{d}\tau'\, L_1(\tau,\tau')\, f_0(\tau') + \int_0^{\tau} \mathrm{d}\tau'\, L_0(\tau,\tau')\, f_1(\tau')\,. \tag{36}$$

Proceeding in this way equations satisfied by any $f_n(\tau)$ may be obtained. For a given $n$ the inhomogeneous term contains contributions depending on all previous cycles and involves operators $L_n$ to $L_1$.

It is to be expected that for most interactions the inhomogeneous term and therefore $f_n(\tau)$ has a limit as $n \to \infty$, but it is not possible to give a simple characterisation of it at this stage. The hierarchy of equations developed here provides a systematic way of approaching the problem, although it appears to be a difficult prospect in as much as this hierarchy occurs on top of other hierarchies associated with $\boldsymbol{r}$ and $\boldsymbol{c}$ dependence that occur in swarm theory.

## 5. Conclusion

As with other formulations of transport theory, a formulation in terms of the time dependent integral equation (5) has its advantages. This comes particularly to the fore in a discussion of the problem of a periodically time dependent field, where the resolution of the distribution function according to different time cycles is found to be convenient in the physical description of the development of the swarm. This also leads to a convenient summation of the series for the exact solution of the BGK model problem found earlier by Robson and Makabe.

In Section 2 a discussion of the problem of the time independent field was also given to show that physical assumptions very similar to the gradient expansion used in other methods have to be used to arrive at a proper identification of transport coefficients.

## Acknowledgment

I wish to thank the referee for suggesting the opportunity to add the further explanation in the Appendix.

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## Appendix: Notes on Equations (4)–(9)

To verify that equation (5) leads to (7), let the $h$ term be absorbed in $\mathcal{S}$ as indicated below (6). Then the differentiation w.r.t. the upper limit of integrations gives $Kf + \mathcal{S}$, the differentiation w.r.t. the $t$ in $\boldsymbol{C}(t,t')$ gives $-\boldsymbol{a}\,.\,\partial_c f$ and that w.r.t. the $t$ in $\boldsymbol{R}(t,t')$ gives $-\boldsymbol{c}\,.\,\partial_r f$. Rearranging the terms we get (7). It is assumed that the functions involved are well enough behaved to permit such differentiation. The difference between equation (9) and the homogeneous Boltzmann equation, i.e. equation (7) with $\mathcal{S} = 0$, should be noted.

The function $P(\boldsymbol{c}, t|t_0)$ of equation (4) satisfies

$$(\partial_t + \boldsymbol{a}\,.\,\partial_c + \nu)P(\boldsymbol{c}, t|t_0) = 0\,,$$

with the initial condition $P(\boldsymbol{c}, t_0|t_0) = 1$. This shows that the rate of change of $P$ in time is equal to the sum of losses due to streaming in the velocity space, $\boldsymbol{a}\,.\,\partial_c P$, and due to direct collisions, $\nu(\boldsymbol{c})P$. Hence we may interpret $P(\boldsymbol{c}, t|t_0)$ as the probability that a particle starting from $t_0$ survives such processes and completes the trajectory to acquire the velocity $\boldsymbol{c}$ at time $t$. At time $t_0$ the velocity would have been $\boldsymbol{c} - \boldsymbol{C}(t, t')$. This interpretation also holds if a discretised approximation for the right-hand side of (4) is considered.

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