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An Exact Model of an Anisotropic Relativistic Sphere

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Abstract

In this paper the field equations of general relativity are solved to obtain an exact solution for a static anisotropic fluid sphere. The solution is free from singularity and satisfies the necessary physical requirements. The physical 3-space of the solution is pseudo-spheroidal. The solution is matched at the boundary with the Schwarzschild exterior solution. Numerical estimates of various physical parameters are briefly discussed.

1. Introduction

The assumption of local isotropy is a common one in astrophysical studies of massive celestial objects. However, the theoretical investigations of Ruderman (1972) and Canuto (1973) on more realistic stellar models indicate that stellar matter may be anisotropic at least in certain density ranges ($\rho < 10^{15}$ g cm^{-3}). According to them the radial pressure may not be equal to the tangential pressure in such anisotropic massive bodies. It is an idealisation to assume that the stellar matter is a perfect fluid. Certainly no astronomical object has a perfect fluid distribution. Therefore it seems worth while to study the behaviour of anisotropic fluid spheres in general relativity.

Anistropy in the pressure could be introduced by the existence of a solid core, by the presence of a type-P superfluid or by other physical phenomena. Our aim is not to study the ways of incorporating anisotropy in stellar matter. Rather, we are interested in constructing models for relativistic anisotropic fluid spheres with physically reasonable behaviour.

Bowers and Linag (1974) have investigated the possible importance of locally anisotropic equations of state for relativistic spheres by generalising the equations of hydrostatic equilibrium to include these effects. Their study indicates that anisotropy—if present in the density range expected for relativistic stars (densities up to at least 10^{15} g cm^{-3})—may have non-negligible effects on such parameters as the maximum equilibrium mass and surface redshift.

Consenza *et al.* (1981), Bayin (1982), Krori *et al.* (1984), Maharaj and Maarten (1989) and Gokhroo and Mehra (1993) have obtained different exact solutions of the Einstein field equations describing the interior gravitational fields of anisotropic fluid spheres. These solution can be used as models of massive compact objects.

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Maharaj and Maaten (1989) discussed a solution for an anisotropic fluid sphere with uniform density. But most astronomical objects have variable density. Therefore, solutions describing the interior fields of anisotropic fluid spheres, with variable density, are physically more realistic. In the present paper we give a physically significant exact solution of the Einstein equations for an anisotropic fluid sphere having a variable density distribution, which is a maximum at the centre and decreasing along the radius.

2. Static Pseudo-spheroidal Space–Time and the Field Equations

A 3-pseudo-spheroid immersed in 4-dimensional Euclidean space with the metric

$$\mathrm{d}\sigma^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + \mathrm{d}w^2 \tag{1}$$

will have the cartesian equation

$$\frac{w^2}{b^2} - \frac{x^2 + y^2 + z^2}{R^2} = 1\,, \tag{2}$$

where b and R are constants. This pseudo-spheroid has the parametric representation

$$x = R\sinh\lambda\sin\theta\cos\phi, \qquad y = R\sinh\lambda\sin\theta\sin\phi,$$

$$z = R\sinh\lambda\cos\theta \qquad w = R\cosh\lambda\,.$$

The metric (1) now becomes

$$\mathrm{d}\sigma^2 = (R^2\cosh^2\lambda + b^2\sinh^2\lambda)\mathrm{d}\lambda^2 + R^2\sinh^2\lambda\,\mathrm{d}\theta^2 + R^2\sinh^2\lambda\sin^2\theta\,\mathrm{d}\phi^2\,.$$

The substitution $r = R\sinh\lambda$ yields a metric in the form

$$\mathrm{d}\sigma^2 = \frac{1 + kr^2/R^2}{1 + r^2/R^2}\,\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,,$$

where $k = 1 + b^2/R^2$. Thus we have $k > 1$. We consider the space–time defined by the line-element

$$\mathrm{d}s^2 = \mathrm{e}^\gamma\,\mathrm{d}t^2 - \frac{1 + kr^2/R^2}{1 + r^2/R^2}\,\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\,, \tag{3}$$

where $\gamma = \gamma(r)$, the physical 3-space of the space–time under consideration, is pseudo-spheroidal. Thomas (1992) has discussed such space–times in detail in connection with Einstein clusters and isotropic fluid spheres. The metric (3) is clearly spherically symmetric. We denote the coordinates $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$ and $x^4 = t$.

We shall develop the Einstein field equations for a static anisotropic fluid sphere with the metric (3) as the metric associated with the distribution. Einstein equations for non-empty space–times are

$$R_{ik} - \tfrac{1}{2} R g_{ik} = -8\pi T_{ik}\,. \tag{4}$$

The energy–momentum tensor is taken here in the form

$$T_{ik} = \rho v_i\, v_k + p h_{ik} + t_{ik}\,, \tag{5}$$

where $v_i = \mathrm{e}^{\nu/2}\delta_i^t$ represents the four-velocity, ρ is the energy density, p is the isotropic fluid pressure and $h_{ik} = v_i\, v_k - g_{ik}$ is the projection tensor. Here t_{ik} is the anisotropic pressure (stress) tensor given by

$$t_{ik} = \sqrt{3}\, s(r)\, (c_i\, c_k - \tfrac{1}{3}\, h_{ik})\,, \tag{6}$$

where $c_i = e^{\lambda/2}\delta_i^r$ is a unit radial vector and $s(r)$ is the magnitude of the stress tensor.

Now, the surviving components of T_k^i given by (5) can be written as

$$\begin{aligned} T_4^4 &= \rho, \qquad T_1^1 = -(p + 2s/\sqrt{3}), \\ &T_2^2 = T_3^3 = -(p - s/\sqrt{3})\,. \end{aligned} \tag{7}$$

The radial pressure p_{r} and the tangential pressure $p_\perp$ are given by

$$p_{\mathrm{r}} = p + 2s/\sqrt{3}, \quad p_\perp = p - s/\sqrt{3}, \quad s = (p_{\mathrm{r}} - p_\perp)/\sqrt{3}\,. \tag{8}$$

The Einstein field equations (5) along with (6) and (7) for the metric (3) reduce to the following system of three equations:

$$8\pi p_{\mathrm{r}} = \left[\left(1 + \frac{r^2}{R^2}\right)\frac{\nu'}{r} - \frac{k-1}{R^2}\right]\left(1 + \frac{kr^2}{R^2}\right)^{-1}, \tag{9}$$

$$8\pi\rho = \frac{3(k-1)}{R^2}\left(1 + \frac{kr^2}{3R^2}\right)\left(1 + \frac{kr^2}{R^2}\right)^{-2}, \tag{10}$$

$$\begin{aligned} 32\pi\sqrt{3}\, s r^2 &= \frac{4(1-k)r^2/R^2}{1 + kr^2/R^2} - \frac{4(1-k)r^2/R^2}{(1 + kr^2/R^2)^2} \\ &+ \frac{2r\nu'}{(1 + kr^2/R^2)^2}\left(1 + \frac{2kr}{R^2} + \frac{kr^2}{R^4}\right) - \frac{r^2(1 + r^2/R^2)(2\nu'' + \nu'^2)}{1 + kr^2/R^2}\,. \end{aligned} \tag{11}$$

This is a system of three equations for one metric function ν and three physical variables ρ, p_{r} and s. So we have to put one additional restriction on the behaviour of these variables. In the next section we shall specify the anisotropy function S and obtain a solution of the system of equations (9)–(11).

Differentiating (10) with respect to r, we get

$$8\pi\rho' = -\frac{2k(k-1)r}{R^4}\left(5+\frac{kr^2}{R^2}\right)\left(1+\frac{kr^2}{R^2}\right)^{-3}. \tag{12}$$

From (10) it is easy to see that ρ is positive. The result (12) indicates that ρ' is negative. Hence as r increases the density decreases from the maximum value ρ_0 at the centre.

3. Solution of the Field Equations

We now specify the anisotropy function S for the explicit solution of the system (9)–(11). We assume that

$$8\pi\sqrt{3}\,S = -\frac{\beta r^2}{R^4(1+kr^2/R^2)^2}, \tag{13}$$

where β is an arbitrary constant. Using (13) in equation (11) and introducing new variables u and F defined by

$$u^2 = \frac{k}{k-1}(1+r^2/R^2), \qquad F = \mathrm{e}^{\nu/2}, \tag{14}$$

we get the differential equation

$$(1-u^2)\mathrm{d}^2F/\mathrm{d}u^2 + u\,\mathrm{d}F/\mathrm{d}u + (1-k+\beta/k)F = 0. \tag{15}$$

Equation (15) is integrable when $k = 1+\sqrt{1+\beta}$. Its general solutions, in this case, can be expressed in the form

$$\mathrm{e}^{\nu/2} = F = Au + B[u\log(u+\sqrt{u^2-1}) - \sqrt{u^2-1}], \tag{16}$$

where A and B are constants of integration. The density ρ is given by (10). The fluid pressure p_r is determined as

$$8\pi p_\mathrm{r} = \frac{Au(3-k)+B[u(3-k)\log(u+\sqrt{u^2-1})+(k-1)\sqrt{u^2-1}]}{R^2(u^2-1)(k-1)[Au+B\{u\log(u+\sqrt{u^2-1})-\sqrt{u^2-1}\}]}. \tag{17}$$

From (17) it is clear that p_r always remains finite. Clearly, we get

$$8\pi p_\perp = 8\pi p_\mathrm{r} + \frac{\beta r^2}{R^4(1+kr^2/R^2)^2}. \tag{18}$$

Any physically acceptable solution must satisfy the following boundary conditions:

(i) At the surface of the sphere ($r=a$), it shuold match with the Schwarzschild exterior solution given by the metric

$$\mathrm{d}s^2 = (1-2m/r)\mathrm{d}t^2 - (1-2m/r)^{-1}\,\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2), \tag{19}$$

where m is a constant representing the total mass of the sphere.

(ii) The radial pressure p_r must be finite at the centre $r = 0$ and it must vanish at the boundary $r = a$ of the sphere.

Applying these boundary conditions to our interior solution, we can determine the constants m, A and B. They are given by

$$\frac{2m}{a} = \frac{a^2/R^2}{u_a^2 - 1}, \tag{20}$$

$$B = \frac{k-1}{2\sqrt{k}} \frac{u_a}{u_a^2 - 1}, \tag{21}$$

$$A = \frac{1}{\sqrt{k}\sqrt{u_a^2 - 1}} - B\left(\log(u_a + \sqrt{u_a^2 - 1}) - \frac{\sqrt{u_a^2 - 1}}{u_a}\right), \tag{22}$$

where $u_a^2 = \{k/(k-1)\}(1 + a^2/R^2)$.

The explicit form of the line-element of our interior solution is

$$\begin{aligned} ds^2 &= [Au + B\{u \log(u + \sqrt{u^2 - 1}) - \sqrt{u^2 - 1}\}]^2 \, dt^2 \\ &\quad - \frac{1 + kr^2/R^2}{1 + r^2/R^2} \, dr^2 - r^2(d\theta^2 + \sin^2\theta \, d\phi^2), \end{aligned} \tag{23}$$

where u is defined by $u^2 = \{k/(k-1)\}(1 + r^2/R^2)$ and the constants A and B are given by (21) and (22). We have also verified that the solution given by the metric (23) is not conformally flat.

4. Discussion

In order to study several physical features of the above solution, we shall discuss numerical estimates of its various physical parameters. It can be easily checked that the central density ρ_0 and the central radial pressure p_{r0} are given by

$$8\pi\rho_0 = 3(k-1)/R^2, \tag{24}$$

$$8\pi p_{r0} = \frac{Au_0(3-k) + B[u_0(3-k)\log(u_0 + \sqrt{u_0^2 - 1}) + (k-1)\sqrt{u_0^2 - 1}]}{R^2(u_0^2 - 1)(k-1)[A_0 + B\{u_0 \log(u_0 + \sqrt{u_0^2 - 1}) - \sqrt{u_0^2 - 1}\}]}, \tag{25}$$

where u_0 is given by $u_0^2 = k/(k-1)$. The positivity of p_{r0} implies that the function

$$H(A, B) = \frac{Au_0(3-k) + B[u_0(3-k)\log(u_0 + \sqrt{u_0^2 - 1}) + (k-1)\sqrt{u_0^2 - 1}]}{Au_0 + B[u_0 \log(u_0 + \sqrt{u_0^2 - 1}) - \sqrt{u_0^2 - 1}]} \tag{26}$$

should be positive. Also $\rho_0 - p_{r0}$ should be positive. Clearly, we have

$$8\pi(\rho_0 - p_{r0}) = (1/R^2)f(A,\, B)\,, \tag{27}$$

where $f(A,\, B)$ is defined by

$$\begin{aligned} f(A,\, B)[Au_0 + B\{u_0 \log(u_0 + \sqrt{u_0^2 - 1}) - \sqrt{u_0^2 - 1}\}] \\ = 2(2k-3)Au_0 + 2B[(2k-3)u_0 \log(u_0 + \sqrt{u_0^2 - 1}) - 2(k-1)\sqrt{u_0^2 - 1}]\,. \end{aligned} \tag{28}$$

Therefore, for a physically viable model we must have

$$f(A,\, B) \geq 0\,. \tag{29}$$

At the boundary $r = a$ of the sphere the density ρ_a is given by

$$8\pi\rho_a = \frac{3(k-1)}{R^2}\left(1 + \frac{ka^2}{3R^2}\right)\left(1 + \frac{ka^2}{R^2}\right)^{-2}\,. \tag{30}$$

Thus the ratio $\mu = \rho_a/\rho_0$ is given by

$$\mu = \frac{3\; + ka^2/R^2}{3(1\; + ka^2/R^2)^2}\,. \tag{31}$$

As ρ is a decreasing function of r, μ is always less than 1. Equation (31) determines a^2/R^2 in terms of k and μ as

$$\frac{a^2}{R^2} = \frac{1 - 6\mu + \sqrt{1 + 24\mu}}{6\mu k}\,. \tag{32}$$

Using the scheme outlined above, we take the matter density on the boundary of the star as $\rho_a = 2{\times}10^{14}$ g cm^{-3}. Again we choose different values for the ratio μ and for each chosen value of μ and the assumed value of ρ_a we compute ρ_0. We assign a particular value to the constant k. We take $\beta = 0{\cdot}21$ so that $k = 2{\cdot}1$. Equation (22) then gives R. Equation (32) then gives us an estimate of the radius a of the star. Equation (20) can then be used to find the mass m in km. The mass M of the star in grams can be obtained by $M = mc^2/G$. Equations (21) and (22) give us values of the constants A and B. Consequently $H(A,\, B)$ and $f(A,\, B)$ can be calculated from equations (26) and (28) respectively. It is easier to express the mass of the star as a multiple of one solar mass $M_\odot = 1{\cdot}475$ km. The results of the calculations for various values of μ are given in Table 1.

From Table 1 it is clear that R and A are increasing functions of μ. Also, $f(A,\, B)$ and $H(A,\, B)$ are always positive. Therefore, the requirements $p_0 \geq 0$ and $\rho_0 \geq p_0$ are satisfied. From Table 1 it is also clear that m is a decreasing function of μ. The maximum possible mass for the configuration is $1{\cdot}74889M$ which is reached when the radius is $2{\cdot}48307$ km. Thus all values of μ give a series of physically viable anisotropic star models.

Table 1. Results of the calculation for various values of μ

μ	R	a/R	a	m/a	m	$m/M_\odot$	B	A	ρ_0	$f(A,B)$	$H(A,B)$
0·90	28·06243	0·1766696	4·957779	0·0161107	7·987329E−02	5·415138E−02	−0·4497912	0·3856234	3·333334E−04	4·180191E−03	1·028487E−05
0·85	27·27178	0·2216173	6·043898	2·448722E−02	0·1479982	0·1003378	−0·4382175	0·3789956	3·529412E−04	4·42028E−03	1·668824E−05
0·80	26·45751	0·2624748	6·944428	3·310209E−02	0·2298751	0·1558475	−0·4262788	0·3720558	3·750001E−04	4·690112E−03	2·417485E−05
0·75	25·61738	0·3014777	7·723068	4·197686E−02	0·3241901	0·2197899	−0·4139387	0·3647689	0·0004	4·995578E−03	3·299361E−05
0·70	24·74874	0·3399021	8·412148	5·113661E−02	0·4301687	0·2916398	−0·4011546	0·357092	4·285715E−04	5·344288E−03	4·346834E−05
0·65	23·84848	0·37866	9·030464	6·061067E−02	0·5473425	0·3710797	−0·3878756	0·3489741	4·615385E−04	5·746165E−03	5·603268E−05
0·60	22·91288	0·418533	9·589793	7·043385E−02	0·6754461	0·4579295	−0·3740405	0·3403526	5·000001E−04	6·214442E−03	7·127338E−05
0·55	21·93741	0·4602982	10·09775	0·0806478	0·8143612	0·5521093	−0·3595746	0·3311502	5·454547E−04	6·767137E−03	9·000769E−05
0·50	20·9165	0·5048263	10·5592	9·130322E−02	0·9640888	0·6536195	−0·3443851	0·3212692	6·000002E−04	7·429459E−03	1·133998E−04
0·45	19·84313	0·5531876	10·97698	0·1024629	1·124732	0·7625304	−0·3283539	0·3105839	6·666669E−04	8·237789E−03	1·431666E−04
0·40	18·70828	0·606801	11·3522	0·114206	1·296489	0·8789758	−0·3113282	0·2989277	7·500004E−04	9·246644E−03	1·819314E−04
0·35	17·5	0·6676767	11·68434	0·1266348	1·479644	1·003148	−0·2931027	0·286074	8·571432E−04	1·054161E−02	2·339019E−04
0·30	16·20185	0·7388619	11·97093	0·1398857	1·674562	1·135296	−0·2733922	0·2717012	1·000001E−03	0·0122652	3·062339E−04
0·25	14·79019	0·8253433	12·20699	0·1541473	1·881674	1·275711	−0·2517801	0·2553297	1·200001E−03	1·467363E−02	4·120767E−04
0·20	13·22875	0·9361185	12·38368	0·1696935	2·101429	1·424698	−0·2276161	0·2361945	1·500001E−03	1·827888E−02	5·78282E−04
0·15	11·45643	1·089819	12·48544	0·1869502	2·334156	1·582487	−0·1997827	0·2129495	2·000000E−03	2·427404E−02	8·688419E−04
0·10	9·354136	1·334497	12·48307	0·2066489	2·579612	1·74889	−0·1660486	0·1828382	3·000005E−03	3·623453E−02	1·479822E−03

Thus, we have obtained an exact relativistic model for an anisotropic superdense star which permits a density of the order of $2\times10^{14}\,\mathrm{g\,cm^{-3}}$, radii of the order of a few kilometres and masses up to $1\cdot75$ times the solar mass. Though the numerical calculations have been carried out for the exact solution corresponding $k = 2\cdot1$, the method is quite general and can be used for any $k > 1$. At the centre $(r = 0)$, the radial and tangential pressures are equal. Hence $\lim(p_\mathrm{r} - p_\perp)/r = 0$ holds (i.e. the gradient $\mathrm{d}p_\mathrm{r}/\mathrm{d}r$ is finite at $r = 0$).

The conservation law $T^i_{k;i} = 0$ for the anisotropic fluid distribution gives

$$(\rho + p_\mathrm{r})\nu'/2 = -p_\mathrm{r}{}' + (2/r)(p_\perp - p_\mathrm{r})\,.$$

These laws for a isotropic fluid distribution give

$$(\rho + p_\mathrm{r})\nu'/2 = -p_\mathrm{r}{}'\,.$$

On comparing these equations we see that there is an additional term $(2/r)(p_\perp - p_\mathrm{r})$ in the equation for the anisotropic case, representing the 'force' which is due to the anisotropic nature of the fluid. This force is directed outwards when $p_\perp > p_\mathrm{r}$ and inward when $p_\perp < p_\mathrm{r}$. The existence of a repulsive force (in the case $p_\perp > p_\mathrm{r}$) allows the construction of a more compact distribution, when using an anisotropic fluid rather than an isotropic fluid.

Finally, we discuss the physical requirement $(\mathrm{d}p_\mathrm{r}/\mathrm{d}\rho) < 1$ at the centre and the boundary. Now $(\mathrm{d}p_\mathrm{r}/\mathrm{d}\rho)_{r=a}$ can be simplified to the form

$$\left(\frac{\mathrm{d}p_\mathrm{r}}{\mathrm{d}\rho}\right)_a = \frac{(1 + ka^2/R^2)(3-k)[(k+1) + (3-k)ka^2/R^2]}{4k(k-1)(1 + a^2/R^2)(5 + ka^2/R^2)}\,.$$

For the assumed value $k = 2\cdot1$, it can be verified that $(\mathrm{d}p_\mathrm{r}/\mathrm{d}\rho)_\mathrm{a} < 1$. One can easily see that

$$\left(\frac{\mathrm{d}p_\mathrm{r}}{\mathrm{d}\rho}\right)_0 = \frac{2\pi R^2(\rho_0 + p_{\mathrm{r}0})[(k-1) + 8\pi p_{\mathrm{r}0}R^2]}{5k(k-1)}\,.$$

Values of $(\mathrm{d}p_\mathrm{r}/\mathrm{d}r)_0$ for various values of μ are given in Table 2.

Table 2. Values of $(\mathrm{d}p_\mathrm{r}/\mathrm{d}\rho)_0$, z_a and z_i

μ	$(\mathrm{d}p_\mathrm{r}/\mathrm{d}\rho)_0$	z_a	z_i
0·90	0·01196	0·01651	0·03231
0·80	0·01154	0·03484	0·06997
0·70	0·01110	0·05542	0·11470
0·60	0·01058	0·07887	0·16972
0·50	0·00999	0·10607	0·23915
0·40	0·00928	0·13843	0·33173
0·30	0·00841	0·17832	0·46541
0·20	0·00732	0·23034	0·68577
0·10	0·00574	0·26379	0·86950
		0·30554	1·77223

Now we discuss the effect of anisotropy on the surface redshift. The redshift is given by

$$z = \left(1 - \frac{2m}{a}\right)^{-1/2} - 1\,,$$

where a is the boundary radius. For the isotropic case z is denoted by z_{i} and for the anisotropic case by z_{a}. Table 2 gives values of z_{a} and z_{i} for various values of μ. It is clear that z_{i} is always greater than z_{a}. Thus the introduction of anisotropy in the pressure gives rise to a decrease in the surface redshift.

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