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Topological Quantum Fluids*

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Abstract

We discuss quantum-mechanical many-body systems interacting with a topological field, known as topological quantum fields. Several topics on the theory of quantum fluids are examined. First we establish the existence of topological gauge fields in high-T_c superconductors and in Heisenberg quantum antiferromagnets. Then we consider typical topological quantum fluids, known as quantum Hall fluids, which are systems exhibiting the fractional quantum Hall effect (FQHE). A theoretical model of these fluids is described in detail. We also discuss the long distance physics of topological quantum fluids, their topological order parameter and possible experimental tests of the theory.

1. Introduction

In connection with theoretical studies of the high-T_c superconductors with the electron–electron antiferromagnetic interaction mechanism (Baskaran and Anderson 1988; Affleck and Marston 1988; Affleck *et al.* 1988; Dagotto *et al.* 1988; Wu *et al.* 1988; Marston and Affleck 1989; Hieu and Son 1989) and the fractional quantum Hall effect (FQHE) (Tsui *et al.* 1982; Laughlin 1983; Hadane 1983; Halperin 1983; Girvin 1984; MacDonald and Murray 1985; Prange and Girvin 1987; Fisher and Lee 1989; Jain 1989; Greiter and Wilczek 1990), a new type of quantum fluid has been proposed: the topological quantum fluid. A quantum fluid is a topological one if it is a quantum-mechanical many-body system interacting with a topological field (Witten 1988). A vector gauge field is topological if its Lagrangian contains a topological term which does not depend on the metric tensor $g^{\mu\nu}$ of the space–time and therefore is stable against the perturbations of $g^{\mu\nu}$. An example of the topological gauge field is the Abelian Cher–Simons (CS) gauge field a_μ in the $2+1$-dimensional (curved, in general) space–time with the Lagrangian

$$L_{\mathrm{CS}} = \alpha\varepsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho\,, \tag{1}$$

where α is some constant. For comparison we note that the Maxwell gauge field A_μ has the Lagrangian

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$$L_{\mathrm{M}} = \tfrac{1}{4} F_{\mu\nu} F_{\lambda\rho} g^{\mu\lambda} g^{\nu\rho}, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,, \tag{2}$$

which contains the metric tensor $g^{\mu\nu}$.

The order parameters of ordinary quantum fluids are related to the broken symmetries of the many-body system. A topological quantum fluid is characterised by an order parameter of the topological origin—the topological order parameter. It is connected with the degeneracy of the ground state of the system and depends on the topology of the space in which the many-body system exists. That is why topological quantum fluids are also called topologically ordered quantum fluids.

In this paper we discuss several topics on the theory of quantum fluids. First we establish the existence of topological gauge fields in high-T_{c} superconductors and in Heisenberg quantum antiferromagnets. Then we consider typical topological quantum fluids—the quantum Hall fluids. These are the systems exhibiting the FQHE. A theoretical model of these fluids is then described in detail. We also discuss briefly the long distance physics of topological quantum fluids, their topological order parameter and the possibilities for experimental tests of the theory.

2. Topological Gauge Fields in High-T_{c} Superconductors

Gauge fields in condensed matter have been proposed in various theoretical works on high-T_{c} superconductors with the Heisenberg quantum antiferromagnetic interaction of itinerant electrons (or holes) (Baskaran and Anderson 1988; Affleck and Marston 1988; Affleck *et al.* 1988; Dagotto *et al.* 1988; Wu *et al.* 1988; Marston and Affleck 1989; Hieu and Son 1989; Ioffe and Larkin 1989; Wen *et al.* 1990; Nagaska and Lee 1992; Zou *et al.* 1992; Ubbens and Lee 1992; Libby *et al.* 1992; Ubbens *et al.* 1993; Chen *et al.* 1993; Simon and Halperin 1993; Tikofsky *et al.* 1994).

The total Hamiltonian of the system has the form

$$H = H_0 + H_{\mathrm{int}} \,, \tag{3}$$

where H_0 is bilinear in the destruction and creation operators for the electron, $\hat{c}_{i\alpha}$ and $\hat{c}^+_{i\alpha}$, respectively, i and α denote the sites and the electron spin projections. Here H_{int} is the interaction Hamiltonian

$$\hat{H}_{\mathrm{int}} = \frac{J}{4} \sum_{\langle ij \rangle} \left(\hat{c}^+_i \,\boldsymbol{\sigma}\, \hat{c}_i \right) \left(\hat{c}^+_j \,\boldsymbol{\sigma}\, \hat{c}_j \right) , \tag{4}$$

where $\langle ij \rangle$ denote the pairs of nearest-neighbour sites i and j. The partition function of the system at the temperature T is

$$Z_c = T_r e^{-\beta H} = \int \left[Dc_{i\alpha} \right] \left[Dc^*_{i\alpha} \right] \, e^{-\int_0^\beta \left[\sum_i c^+_i \frac{d}{d\tau} c_i + H_0 + \frac{J}{4} \sum_{\langle ij \rangle} \left(c^+_i \,\sigma\, c_i \right) \left(c^+_j \,\sigma\, c_j \right) \right] d\tau} . \tag{5}$$

In the r.h.s. of equation (5) we have the functional integrals over the Grassmann variables $c_{i\alpha}$ and $c^*_{i\alpha}$ corresponding to the quantum operators $\hat{c}_{i\alpha}$ and $\hat{c}^+_{i\alpha}$. They are the functions of the imaginary time τ in the interval $0 < \tau < \beta$, $\beta = 1/T$.

Let us transform the expression on the r.h.s. of equation (5). The Fierz rearrangement gives

$$\left(c_i^+ \boldsymbol{\sigma}\, c_i\right)\left(c_j^+ \boldsymbol{\sigma}\, c_j\right) = -2\left(c_i^+ c_j\right)\left(c_j^+ c_i\right) - \left(c_i^+ c_i\right)\left(c_j^+ c_j\right). \tag{6}$$

Therefore the partition function becomes

$$Z_c = \int \left[Dc_{i\alpha}\right]\left[Dc^*_{i\alpha}\right] e^{-\int_0^\beta \left[\sum_i c_i^+ \frac{d}{d\tau} c_i + H_0\right] d\tau}$$
$$\times\, e^{\frac{J}{2}\int_0^\beta \sum_{\langle ij\rangle}\left(c_i^+ c_j\right)\left(c_j^+ c_i\right) d\tau}\, e^{\frac{J}{4}\int_0^\beta \sum_{\langle ij\rangle}\left(c_i^+ c_i\right)\left(c_j^+ c_j\right) d\tau}. \tag{7}$$

The last two factors on the r.h.s. of (7) can be transformed into new forms which contain only bilinear expressions of the electron quantum operators in the exponents. For this purpose we introduce functional integrals over the bilocal complex functions ξ_{ij} and the local real functions φ_i

$$Z_\xi = \int \left[D\xi_{ij}\right]\left[D\xi^*_{ij}\right] e^{-\int_0^\beta \sum_{\langle ij\rangle} \xi^*_{ij}\xi_{ij} d\tau}, \tag{8}$$

$$Z_\varphi = \int \left[D\varphi_i\right] e^{-\int_0^\beta \sum_{\langle ij\rangle} \varphi_i\varphi_j d\tau}. \tag{9}$$

By means of the Hubbard–Stratonovich transformation it can be shown that

$$e^{\frac{J}{2}\int_0^\beta \sum_{\langle ij\rangle}\left(c_i^+ c_j\right)\left(c_j^+ c_i\right) d\tau}$$
$$= \frac{1}{Z_\xi}\int \left[D\xi_{ij}\right]\left[D\xi^*_{ij}\right] e^{-\int_0^\beta \sum_{\langle ij\rangle}\left\{\xi^*_{ij}\xi_{ij} + \sqrt{\frac{J}{2}}\left[\xi^*_{ij}\left(c_i^+ c_j\right) + \xi_{ij}\left(c_j^+ c_i\right)\right]\right\} d\tau}, \tag{10}$$

$$e^{\frac{J}{4}\int_0^\beta \sum_{\langle ij\rangle}\left(c_i^+ c_i\right)\left(c_j^+ c_j\right) d\tau}$$
$$= \frac{1}{Z_\varphi}\int \left[D_\varphi\right] e^{-\int_0^\beta \sum_{\langle ij\rangle}\left\{\varphi_i\varphi_j + \sqrt{\frac{J}{4}}\left[\varphi_i\left(c_j^+ c_j\right) + \varphi_j\left(c_i^+ c_i\right)\right]\right\} d\tau}. \tag{11}$$

Setting the expressions (10) and (11) into the r.h.s. of (7), permuting the order of the integrations over $c_{i\alpha}$, $c^*_{i\alpha}$ and of those over ξ_{ij}, ξ^*_{ij} and φ_i, after the integration over $c_{i\alpha}$ and $c^*_{i\alpha}$ we obtain an expression for the partition function of the system in the form of a functional integral over the scalar fields ξ_{ij}, ξ^*_{ij} and φ_i:

$$Z_c = \int \left[D\xi_{ij}\right]\left[D\xi^*_{ij}\right]\left[D\varphi_i\right] e^{-S\left[\xi_{ij},\xi^*_{ij},\varphi_i\right]}. \tag{12}$$

The scalar functions ξ_{ij}, ξ^*_{ij}, φ_i represent the effective fields of the bosonic collective excitations in the system.

Let us separate the magnitudes and phases of the bilocal functions

$$\xi_{ij} = \eta_{ij} e^{i\theta_{ij}}, \quad \xi^*_{ij} = \eta_{ij} e^{-i\theta_{ij}}, \tag{13}$$

η_{ij} and θ_{ij} being the real functions of τ. Under the gauge transformation

$$c_i \to c_i e^{i\alpha_i}, \quad c^+_j \to c^+_j e^{-i\alpha_j} \tag{14}$$

the phase θ_{ij} changes in the following manner:

$$\theta_{ij} \to \theta_{ij} + \alpha_j - \alpha_i \,. \tag{15}$$

It can be expressed in terms of a vector field $\boldsymbol{a}(\boldsymbol{r}, \tau)$

$$\theta_{ij} = \int_{\boldsymbol{r}_i}^{\boldsymbol{r}_j} \boldsymbol{a}(\boldsymbol{r}, \tau) d\boldsymbol{r} \,, \tag{16}$$

where $\boldsymbol{r}_i$ is the coordinate vector of the site i. Let $\alpha(\boldsymbol{r})$ be some function with the value α_i at the site i

$$\alpha(\boldsymbol{r}_i) = \alpha_i \,.$$

The corresponding gauge transformation of $\boldsymbol{a}$ is

$$\boldsymbol{a}(\boldsymbol{r}, \tau) \to \boldsymbol{a}(\boldsymbol{r}, \tau) + \nabla\alpha(\boldsymbol{r}) \,. \tag{17}$$

The effective action $S[\xi_{ij}, \xi^*_{ij}, \varphi_i]$ in the r.h.s. of (12), in general, contains the time derivatives

$$\frac{d}{d\tau}\xi_{ij}, \quad \frac{d}{d\tau}\xi^*_{ij} \,.$$

In order to have the invariance under the τ-dependent gauge transformations of the form (14) and (15),

$$\alpha_i = \alpha(\boldsymbol{r}_i, \tau) \,,$$

it is necessary to introduce also the time component $a_0(\boldsymbol{r}, \tau)$ of the gauge field together with the spatial ones $\boldsymbol{a}(\boldsymbol{r}, \tau)$, and to replace

$$\begin{aligned} \frac{d}{d\tau}\xi_{ij} &\longrightarrow \left\{\frac{d}{d\tau} - i\big[a_0(\boldsymbol{r}_j, \tau) - a_0(\boldsymbol{r}_i, \tau)\big]\right\}\xi_{ij} \,, \\ \frac{d}{d\tau}\xi^*_{ij} &\longrightarrow \left\{\frac{d}{d\tau} + i\big[a_0(\boldsymbol{r}_j, \tau) - a_0(\boldsymbol{r}_i, \tau)\big]\right\}\xi^*_{ij} \,. \end{aligned} \tag{18}$$

The effective action is invariant under the gauge transformations

$$\boldsymbol{a}(\boldsymbol{r}, \tau) \longrightarrow \boldsymbol{a}(\boldsymbol{r}, \tau) + \nabla\alpha(\boldsymbol{r}, \tau), \quad a_0(\boldsymbol{r}, \tau) \longrightarrow a_0(\boldsymbol{r}, \tau) + \frac{\partial\alpha(\boldsymbol{r}, \tau)}{\partial\tau}, \tag{19}$$

with any function $\alpha(\boldsymbol{r}, \tau)$. In the expression for the partition function we must include the functional integration over the component $a_0(\boldsymbol{r}, \tau)$ and add the gauge fixing term to the effective action.

Thus the existence of the gauge field $a_\mu(\boldsymbol{r},\tau)$ in the system with the interaction Hamiltonian (4) has been established. This gauge field has been investigated extensively (see the work cited at the beginning of this section; see also Narikio *et al.* 1990 and Fukuyama and Kuboki 1990). Wen *et al.* (1990) have studied the chiral spin states and shown that in this case the effective Lagrangian of the gauge field is the CS term of the form (1). This result was confirmed by the recent works of Libby *et al.* (1992) and Tikofsky *et al.* (1994) . The corresponding systems are called the chiral spin fluids. Chen *et al.* (1993) have considered the system with the interaction Hamiltonian (4) near the Mott transition and also derived the same CS effective Lagrangian.

3. Quantum Hall Fluids

Consider now another type of topological quantum fluid: the two-dimensional electron system exhibiting the FQHE. They are called Hall fluids. A simple field theory description of these systems has been proposed by Zhang *et al.* (1989). The model consists of a quantum electron field $\hat{\psi}$ interacting with a CS quantum gauge field $\hat{a}_\mu$ in the presence of the classical external electromagnetic field A_μ. The total Lagrangian of the system equals

$$\hat{L}_{\text{tot}} = \hat{L}_e(\hat{\psi},\hat{\psi}^+, g\hat{a}_\mu - eA_\mu) + \hat{L}_{\text{CS}}(\hat{a}_\mu)\,, \tag{20}$$

where $\hat{L}_e$ is the Lagrangian of the electron field $\hat{\psi}$ interacting with the vector fields $\hat{a}_\mu$ and A_μ:

$$\begin{aligned}\hat{L}_e(\psi,\hat{\psi}^+, g\hat{a}_\mu - eA_\mu) &= i\hat{\psi}^+\left(\frac{\partial}{\partial t} + ig\hat{a}_0 - ieA_0\right)\hat{\psi} \\ &\quad + \frac{1}{2M}\hat{\psi}^+\left(\nabla - ig\hat{\boldsymbol{a}} + ie\boldsymbol{A}\right)^2\hat{\psi}\,,\end{aligned} \tag{21}$$

$\hat{L}_{\text{CS}}$ is the CS Lagrangian for the quantum field $\hat{a}_\mu$:

$$\hat{L}_{\text{CS}}(\hat{a}_\mu) = \frac{g^2}{4\pi p}\varepsilon^{\mu\nu\rho}\hat{a}_\mu\partial_\nu\hat{a}_\rho\,. \tag{22}$$

Here g is the coupling constant of the gauge interaction of the field $\hat{a}_\mu$ with the electron, and p is some integer. We choose p to be positive. The equation of motion with respect to the time component $\hat{a}_0$ reads

$$\varepsilon^{ij}\partial_i\hat{a}_j = p\frac{2\pi}{g}\hat{\rho}\,, \tag{23}$$

where $\hat{\rho} = \hat{\psi}^+\hat{\psi}$ is the electron number density. Due to the interaction of the two fields $\hat{\psi}$ and $\hat{a}_\mu$ each electron always carries a flux of the magnetic-like field with the vector potential equal to the expectation value of $\hat{a}_j$. From equation (23) it follows that each electron is attached by p unit quantum fluxes. This is the physical picture of the model of composite fermions in quantum Hall fluids (Jain 1989; Greiter and Wilczek 1990).

From the Lagrangian determined by equations (20)–(22) it is straightforward to derive the expression for the Hall conductivity. We choose to work in the gauge $\hat{a}_0 = A_0 = 0$. The functional integral of the system is

$$Z = \int [D\psi]\,[D\psi^*]\,[D\mathbf{a}]\,e^{i\int L_{\text{tot}}dt}\,. \tag{24}$$

It determines completely the physical consequences of the theory. By a shift of the functional integration variable

$$\boldsymbol{a} \longrightarrow \boldsymbol{a} + \frac{e}{g}\boldsymbol{A}$$

we can rewrite Z in a new form

$$Z = \int [D\psi]\,[D\psi^*]\,[D\mathbf{a}]\,e^{i\int L_{\text{eff}}dt}\,, \tag{25}$$

with the effective Lagrangian

$$L_{\text{eff}} = L_e(\psi, \psi^*, g\boldsymbol{a}) + L_{CS}\Big(\boldsymbol{a} + \frac{e}{g}\boldsymbol{A}\Big)\,. \tag{26}$$

This effective Lagrangian describes the system as well as the original one L_{tot} given by equations (20)–(22). Note that in this new form of Z the external electromagnetic vector potential A_μ enters only into the CS term

$$\begin{aligned} L_{\text{CS}}\Big(\boldsymbol{a} + \frac{e}{g}\boldsymbol{A}\Big) = \frac{1}{4\pi p}\varepsilon^{ij}\big[&g^2 a_i\partial_t a_j + ega_i\partial_t A_j \\ &+ egA_i\partial_t a_j + e^2 A_i\partial_t A_j\big]\,. \end{aligned} \tag{27}$$

By definition the electromagnetic current operator equals

$$\hat{J}_i^e = \frac{\delta}{\delta A_i}\int L_{\text{eff}}dt = \frac{1}{2\pi p}\varepsilon^{ij}\big[e^2\partial_t A_j + eg\partial_t\hat{a}_j\big]\,. \tag{28}$$

We write $\sigma^{(1)}_{\text{Hall}}$ for the contribution of the first term in the r.h.s. of (28) to the Hall conductivity. We have immediately

$$\sigma^{(1)}_{\text{Hall}} = \frac{1}{p}\frac{e^2}{2\pi}\,. \tag{29}$$

However, $\hat{a}_i$ is a quantum field interacting with the external one A_i through the interaction Lagrangian

$$L_{\text{int}}(\hat{\boldsymbol{a}}, \boldsymbol{A}) = \frac{eg}{4\pi p}\varepsilon^{jk}\big(\hat{a}_j\partial_t A_k + A_j\partial_t\hat{a}_k\big)\,, \tag{30}$$

and the expectation value of the last term in the r.h.s. of (28) does not vanish.

Let us calculate the expectation value of the current operator (28) in the presence of the external time dependent perturbation of the form (30). We have

$$\overline{J_i^e(\boldsymbol{r},t)} = \langle \hat{J}_i^e(\boldsymbol{r},t)\rangle = \frac{1}{2\pi p}\varepsilon^{ij}\left[e^2\partial_t A_j(\boldsymbol{r},t) + eg\partial_t\langle\hat{a}_j(\boldsymbol{r},t)\rangle\right]. \tag{31}$$

To first order with respect to this perturbation we obtain

$$\langle\hat{a}_j(\boldsymbol{r},t)\rangle = i\frac{eg}{4\pi p}\varepsilon^{\ell k}\int dt'\int d\boldsymbol{r}'\Big\{\Big\langle T\Big[\hat{a}_j(\boldsymbol{r},t)\hat{a}_\ell(\boldsymbol{r}',t')\Big]\Big\rangle_0 \frac{\partial}{\partial t'}A_k(\boldsymbol{r}',t') + A_\ell(\boldsymbol{r}',t').$$

$$\frac{\partial}{\partial t'}\Big\langle T\Big[\hat{a}_j(\boldsymbol{r},t)\hat{a}_k(\boldsymbol{r}',t')\Big]\Big\rangle_0\Big\} = \frac{eg}{2\pi p}\varepsilon^{\ell k}\frac{\partial}{\partial t}\int dt'\int d\boldsymbol{r}' D_{j\ell}(\boldsymbol{r}-\boldsymbol{r}',t-t')A_k(\boldsymbol{r}',t'), \tag{32}$$

where

$$D_{j\ell}(\boldsymbol{r}-\boldsymbol{r}',t-t') = i\big\langle T\big[\hat{a}_j(\boldsymbol{r},t)\hat{a}_\ell(\boldsymbol{r}',t')\big]\big\rangle_0 \tag{33}$$

is the two-point Green function of the CS gauge field interacting with the electrons through the Lagrangian (21) in the absence of the electromagnetic field. We have studied this elsewhere (Hieu and Son 1991; Hieu 1992, 1993). From equations (31) and (32) it follows that the contribution of the second term in the r.h.s. of (31) is of the same order with respect to the electromagnetic coupling constants e as that of the first term and therefore cannot be neglected. Calculations give

$$\sigma_{\mathrm{Hall}} = \frac{1}{p}\frac{e^2}{2\pi}\frac{\Omega}{\Omega-\omega_c}, \tag{34}$$

$$\Omega = \frac{2\pi\rho}{M}p, \tag{35}$$

where ρ is the electron number density

$$\rho = \langle\hat{\rho}\rangle_0,$$

and ω_c is the cyclotron frequency of the electron in the effective magnetic-like field with strength

$$B_{\mathrm{eff}} = B - \frac{eH}{g}, \tag{36}$$

$$\omega_c = \frac{gB_{\mathrm{eff}}}{M}. \tag{37}$$

Here H is the magnetic field created by the external electrical current and B is the magnetic-like field created by the electron in the system according to equation (23)

$$B = \frac{2\pi\rho}{g}p. \tag{38}$$

The same result was also derived directly from the original Lagrangian (20). We introduce the integral filling factor of the electrons in the effective magnetic-like field,

$$n = \frac{2\pi\rho}{gB_{\text{eff}}} . \tag{39}$$

We have then

$$\omega_c = \frac{2\pi\rho}{M}\frac{1}{n} \tag{40}$$

and therefore

$$\frac{\Omega}{\Omega - \omega_c} = \frac{p}{p - \dfrac{1}{n}} . \tag{41}$$

The fractional filling factor of the electrons in the magnetic field H is

$$\nu = \frac{2\pi\rho}{eH} . \tag{42}$$

From equations (36), (38), (39) and (42) we obtain

$$\frac{1}{n} = p - \frac{1}{\nu} . \tag{43}$$

Thus the fractional filling factor ν is expressed in terms of the integers p and n in the following manner:

$$\nu = \frac{1}{p - \dfrac{1}{n}} . \tag{44}$$

Setting expression (41) into the r.h.s. of (34) and using the relation (44), we finally obtain the expression for the Hall conductivity

$$\sigma_{\text{Hall}} = \frac{1}{p - \dfrac{1}{n}}\,\frac{e^2}{2\pi} = \nu\frac{e^2}{2\pi} . \tag{45}$$

This is consistent with the definition of the filling factor ν. The model of Zhang *et al.* (1989) is adequate for a theoretical description of the FQHE. It has been generalised by Wen, Zee and others (Wen 1989; Blok and Wen 1990; Wen and Niu 1990; Wen and Zee 1989, 1991) in order to describe the hierarchical states with filling factors of the form

$$\nu = \cfrac{1}{p_1 - \cfrac{1}{p_2 - \cfrac{1}{\ddots - \cfrac{1}{p_\ell}}}} . \tag{46}$$

4. Long Distance Physics of Topological Quantum Fluids

In the preceeding section we have presented a model of the fractional quantum Hall system as an integral quantum Hall system of composite fermions—electrons interacting with the CS gauge field. The integral filling factor n of the last system equals the number of filled lowest Landau levels. Generalising this model of Zhang *et al.* (1989), Wen, Zee and others (Wen 1989; Blok and Wen 1990; Wen and Niu 1990; Wen and Zee 1989, 1991) have proposed describing the electrons in each separate filled Landau level by a fermion field $\hat{\psi}_I$, $I = 1, 2, \ldots, n$, then replacing each fermion field by a boson one $\hat{\Phi}_I$ interacting with a corresponding CS gauge field $\hat{a}^I_\mu$, which converts the boson into the fermion degree of freedom (Wilczek and Zee 1983). Including the possible self-interaction potentials of the boson fields also, these authors obtained the following effective Lagrangian:

$$L = \sum_{I=1}^{n} \Big\{ i\hat{\Phi}_I^+ \Big[\frac{\partial}{\partial t} + i(g\hat{a}_0 + \hat{a}_0^I - eA_0)\Big]\hat{\Phi}_I + \frac{1}{2M}\hat{\Phi}_I^+ \left[\nabla - i(g\hat{\mathbf{a}} + \hat{\mathbf{a}}^I - e\mathbf{A})\right]^2 \hat{\Phi}_I$$

$$+ V(\hat{\Phi}_I) + \frac{1}{4\pi}\varepsilon^{\mu\nu\rho}\hat{a}^I_\mu \partial_\nu \hat{a}^I_\rho \Big\} + \frac{g^2}{4\pi p}\varepsilon^{\mu\nu\rho}\hat{a}_\mu \partial_\nu \hat{a}_\rho \,. \tag{47}$$

In the broken symmetric phase each complex scalar field has a gap-less Goldstone–Nambu excitation η^I and a gap-full vortex excitation. Since the dimension of the space–time is $2+1$, the scalar massless fields η^I are expressed in terms of the massless gauge fields ξ^I_μ according to the relation

$$\partial^\mu \eta^I = \text{const} \times \varepsilon^{\mu\nu\rho}(\partial_\nu \xi^I_\rho - -\partial_\rho \xi^I_\nu) \,. \tag{48}$$

Transforming the Lagrangian into the dual representation (Wen and Zee 1989; Fisher and Lee 1989), we obtain

$$\hat{L} = \sum_{I=1}^{n} \Big[\frac{2}{4\pi}\varepsilon^{\mu\nu\rho}(g\hat{a}_\mu + \hat{a}^I_\mu - eA_\mu)\partial_\nu \hat{\xi}^I_\rho + \frac{1}{4\pi}\varepsilon^{\mu\nu\rho}\hat{a}^I_\mu \partial_\nu \hat{a}^I_\rho + \hat{\xi}^I_\mu \hat{j}^\mu_{I\text{vortex}}\Big]$$

$$+ \frac{g^2}{4\pi p}\varepsilon^{\mu\nu\rho}\hat{a}_\mu \partial_\nu \hat{a}_\rho + \ldots \,. \tag{49}$$

Here $\hat{j}^\mu_{I\text{vortex}}$ are the currents of the vortices. In the r.h.s. of (49) we do not specify the terms which give no contribution in the long-distance limit. Writing

$$\varepsilon^{\mu\nu\rho}\int (a^I_\mu \partial_\nu a^I_\rho + 2a^I_\mu \partial_\nu \xi^I_\rho)\,drdt = \varepsilon^{\mu\nu\rho}\int \Big[(a^I_\mu + \xi^I_\mu)\partial_\nu(a^I_\rho + \xi^I_\rho) - \xi^I_\mu \partial_\nu \xi^I_\rho\Big]drdt$$

and integrating over the fields a^I_μ in the functional integral, we reduce the Lagrangian to the form

$$\hat{L} = \sum_{I=1}^{n} \Big[-\frac{1}{4\pi}\varepsilon^{\mu\nu\rho}\hat{\xi}^I_\mu \partial_\nu \hat{\xi}^I_\rho + \frac{2}{4\pi}\varepsilon^{\mu\nu\rho}(g\hat{a}_\mu - eA_\mu)\partial_\nu \hat{\xi}^I_\rho + \hat{\xi}^I_\mu \hat{j}^\mu_{I\text{vortex}}\Big]$$

$$+ \frac{g^2}{4\pi p}\varepsilon^{\mu\nu\rho}\hat{a}_\mu \partial_\nu \hat{a}_\rho \,. \tag{50}$$

Writing again

$$\varepsilon^{\mu\nu\rho}\int\Big(\frac{g^2}{p}a_\mu\partial_\nu a_\rho+2ga_\mu\partial_\nu\sum_I\xi^I_\rho\Big)drdt$$

$$=\varepsilon^{\mu\nu\rho}\int\Big[\frac{1}{p}\Big(ga_\mu+p\sum_I\xi^I_\mu\Big)\partial_\nu\Big(ga_\rho+p\sum_I\xi^I_\mu\Big)-p\sum_{I,J}\xi^I_\mu\partial_\nu\xi^J_\rho\Big]drdt\,,$$

and then integrating over the field a_μ, finally we obtain the effective Lagrangian in the new form

$$\begin{aligned}\hat{L}=&-\frac{1}{4\pi}\varepsilon^{\mu\nu\rho}\sum_I\hat{\xi}^I_\mu\partial_\nu\hat{\xi}^I_\rho-\frac{p}{4\pi}\varepsilon^{\mu\nu\rho}\sum_{I,J}\hat{\xi}^I_\mu\partial_\nu\hat{\xi}^J_\rho-\frac{2}{4\pi}e\varepsilon^{\mu\nu\rho}A_\mu\partial_\nu\sum_I\hat{\xi}^I_\rho\\&+\sum_I\hat{\xi}^I_\mu\hat{j}^\mu_{I\mathrm{vortex}}+\cdots\qquad(51)\\=&\frac{1}{4\pi}\varepsilon^{\mu\nu\rho}\sum_{I,J}\xi^I_\mu K_{IJ}\partial_\nu\xi^J_\rho-\frac{2}{4\pi}e\varepsilon^{\mu\nu\rho}A_\mu\partial_\nu\sum_I\hat{\xi}^I_\rho+\sum_I\hat{\xi}^I_\mu\hat{j}^\mu_{I\mathrm{vortex}}+\cdots\,,\end{aligned}$$

where K_{IJ} are the elements of the $n\times n$ matrix

$$K=-(I+pC)\,,$$

I being the unit matrix and C the matrix in which every element equals 1. By means of the time-reversal transformation we can change the sign of K. By combining different quantum Hall fluids we can construct a new one with some symmetric matrix K suitable for the description of the hierarchical state with the filling factor (46). If different components of the quantum Hall fluid have different charges t_I, then instead of the second term on the r.h.s. of (51) we should write

$$\frac{2}{4\pi}\varepsilon^{\mu\nu\rho}A_\mu\partial_\nu\sum_I t_I\hat{\xi}^I_\mu\,.$$

Thus, in the long distance limit, the physics of quantum Hall fluids is determined by the Lagrangian

$$\hat{L}=\frac{1}{4\pi}\varepsilon^{\mu\nu\rho}\sum_{I,J}\hat{\xi}^I_\mu K_{IJ}\partial_\nu\hat{\xi}^J_\rho+\frac{2}{4\pi}\varepsilon^{\mu\nu\rho}A_\mu\partial_\nu\sum_I t_I\hat{\xi}^I_\rho+\sum_I\hat{\xi}^I_\mu\hat{j}^\mu_{I\mathrm{vortex}}+\cdots\,. \qquad (52)$$

The matrix K has odd diagonal elements, and all other elements are integral numbers. It was shown (Wen 1989; Blok and Wen 1990; Wen and Niu 1990; Wen and Zee 1989, 1991; Frohlich and Zee 1991) that the filling factor of the system is

$$\nu=\sum_{I,J}\left(K^{-1}\right)_{IJ}\,, \qquad (53)$$

and the charges of the vortices are

$$q_I = \sum_J t_J (K^{-1})_{JI} \,. \tag{54}$$

The matrix K determines also the statistics of the vortices. The ground states of the system are described by the nontrivial configurations of the gauge fields ξ^I_μ. Their degeneracy equals

$$D = (\det K)^g \,,$$

where g is the genus of the two-dimensional closed surface in which the system exists.

Thus the matrix K contains all information on the physics of the topological quantum fluid. It is an order parameter of the topological character—a topological order parameter.

5. Discussion

Although the physics of topological quantum fluids with different structural origins can be described by a universal effective Lagrangian of the form (52) at long distances, these fluids have different elementary excitations and different physical properties. In conclusion we mention some topics of the theory of topological quantum fluids which are related with their microscopic structure. These problems are being widely studied in theoretical and experimental work:

(1) Physical properties of the flux-charge composite fermions in quantum Hall fluids, in particular their behaviour in external electromagnetic fields (Jain *et al.* 1993; Kang *et al.* 1993; Goldman *et al.* 1994).

(2) Quantum mechanics and field theory models of quantum Hall fluids, in particular their elementary excitations and energy spectra (Narikio *et al.* 1990; Fukuyama and Kuboki 1990; Zhang *et al.* 1989; Hieu 1992, 1993; Hieu and Son 1991, 1994; Wen 1989; Blok and Wen 1990; Wen and Niu 1990; Wen and Zee 1989, 1991; Wilczek and Zee 1983; Fisher and Lee 1989; Frohlich and Zee 1991; Hanna *et al.* 1989; Fetter and Hanna 1992; Hosotani and Chakravarty 1990; Hosotani 1993; Frohlich and Kerler 1991; Kol and Read 1993; Zang *et al.* 1994; Schemeltzer and Birman 1993; Chen *et al.* 1989; Iengo and Lechner 1991; Cristfano *et al.* 1990; Pinczuk *et al.* 1993; Kukushkin *et al.* 1994).

(3) Physical properties and physical parameters of topological quantum fluids, in particular their linear and nonlinear responses in an electromagnetic field; the Meissner effect and superconductivity (Fetter and Hanna 1992; Hosotani and Chakravarty 1990; Engel *et al.* 1993; Randjbar-Daemi *et al.* 1990; Banks and Lykken 1990; Panigrahi *et al.* 1990; Lykken *et al.* 1990; Hetrick *et al.* 1991; Ezawa and Iwazaki 1991; Balachandran *et al.* 1990; Fradkin 1989).

(4) Edge excitations of quantum Hall fluids (Halperin 1982; Wen 1990; Lee and Wen 1991; Stone 1990).

(5) Symmetries and algebraic aspects in the theory of the FQHE (Cappeli *et al.* 1993*a*, 1993*b*; Karabali 1994; Wen and Wu 1994).

All these problems require further investigation.

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