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Planck's Radiation Law: A Many Body Theory Perspective

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Abstract

An exposition of Planck's Law of Radiation is presented within the context of modern quantum many body theory. In particular the generality of the Planck radiation law is demonstrated to be valid to all orders of perturbation theory for an interacting system, regardless of the underlying statistics for the particles comprising the matter field and the precise nature of the interaction mechanism. This exposition justifies the use of Planck's law with minor modifications via the refractive index, in the optical properties of semiconductors and insulators. We conclude by clarifying when Planck's law is likely to fail.

1. Introduction

The radiation spectrum of a large cavity was successfully explained by Planck in 1900. Its explanation required a fundamental shift in our viewpoint of nature, the consequences of which form the basis of modern quantum theory. In 1917, Einstein formulated a phenomenological argument which employs the statistical theorem of detailed balance to derive Planck's law.

Planck's law has application to a diverse range of systems, many of which have a non-negligible coupling between radiation and matter. For example, it has been used (with minor modifications) to describe the equilibrium radiation spectrum of semiconductors via a detailed balance argument for the creation and annihilation rates of electron–hole pairs (van Roosbroeck and Shockley 1954), and recently for 2-d electron systems in semiconductor heterojunctions (Hirakwa *et al.* 1993).

In this paper we re-analyse Planck's radiation law and to a lesser extent the detailed balance argument from the perspective of finite temperature many body theory. By applying these ideas to a dispersive medium we justify the form of Planck's law used in bulk semiconductors.

Although there exist elegant statistical, thermodynamic and semi-classical quantum mechanical arguments to derive Planck's law, many are macroscopic and unable to deal with specific microscopic mechanisms through which matter and radiation interact. The apparatus of many body theory provides a consistent method in which to do this.

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We believe this deeper exposition of the Planck radiation law is important for its teaching in the higher undergraduate and graduate curriculum and also in research areas involving the interaction of matter and radiation.

2. Planck's Radiation Law

To introduce a more general form of Planck's radiation law which remains intact when interactions are included, we first derive Planck's original law in a slightly different manner from the traditional method of integrating over the directions of the photon modes and converting the wave vector differential to a frequency differential (Eisberg and Resnick 1985). What is novel about our approach is that the manifold of frequencies that a particle can have is independent of reciprocal $\boldsymbol{k}$-space. In the non-interacting case there exists a correspondence between many points in $\boldsymbol{k}$-space and a single point in frequency-space, whereas in the interacting case there is no definite correspondence between a particle's wave vector and its frequency. From an experimental perspective, motivation is further gained by realising that a typical 'probe' will measure the frequency of single photons and not their quantum numbers.

(2a) The Non-interacting Case

Consider a homogenous system of non-interacting electrons and photons. In the second quantised formalism (Mahan 1990), the non-interacting part of the Hamiltonian will have the general form*

$$\hat{H}_0 = \sum_{\boldsymbol{k},\lambda} E_e(\boldsymbol{k},\lambda)\hat{c}^\dagger_{\boldsymbol{k},\lambda}\hat{c}_{\boldsymbol{k},\lambda} + \sum_{\boldsymbol{q},\sigma} E_p(\boldsymbol{q},\sigma)\hat{a}^\dagger_{\boldsymbol{q},\sigma}\hat{a}_{\boldsymbol{q},\sigma}\,, \tag{1}$$

where $\hat{c}_{\boldsymbol{k},\lambda}$ ($\hat{c}^\dagger_{\boldsymbol{k},\lambda}$) is the fermion destruction (creation) operator for an electron of momentum $\hbar\boldsymbol{k}$ and spin λ, and $\hat{a}_{\boldsymbol{q},\sigma}$ ($\hat{a}^\dagger_{\boldsymbol{q},\sigma}$) is the boson destruction (creation) operator for a photon of momentum $\hbar\boldsymbol{q}$ and spin σ. The dispersion relations for the electron and photon are $E_e(\boldsymbol{k},\lambda) = \hbar^2|\boldsymbol{k}|^2/2m$ and $E_p(\boldsymbol{q},\sigma) = c\hbar|\boldsymbol{q}|$. The number operator $\hat{a}^\dagger_{\boldsymbol{q},\sigma}\hat{a}_{\boldsymbol{q},\sigma}$ has eigenvalues which equal the number of photons in the mode $(\boldsymbol{q},\sigma)$; i.e. $\hat{a}^\dagger_{\boldsymbol{q},\sigma}\hat{a}_{\boldsymbol{q},\sigma}|n\rangle_{\boldsymbol{q},\sigma} = n_{\boldsymbol{q},\sigma}|n\rangle_{\boldsymbol{q},\sigma}$, where $|n\rangle_{\boldsymbol{q},\sigma}$ is the corresponding photon occupation number state.

To determine the equilibrium energy density of photons contained in a volume V at a temperature T, we take the thermal average of $\hat{H}_0/V$ with respect to the total electron–photon wave function $|\Psi\rangle_T$. For the non-interacting case $|\Psi\rangle_T = |\Psi_{\text{el.}}\rangle_T \otimes |\Psi_{\text{ph.}}\rangle_T$. In what follows we shall ignore the electronic part and consider only the energy density due to photons. Thus

$${}_T\langle\Psi_{\text{ph.}}|\hat{H}_0/V|\Psi_{\text{ph.}}\rangle_T = \frac{1}{V}\sum_{\boldsymbol{q},\sigma} E_p(\boldsymbol{q},\sigma)\,{}_T\langle\Psi_{\text{ph.}}|\hat{a}^\dagger_{\boldsymbol{q},\sigma}\hat{a}_{\boldsymbol{q},\sigma}|\Psi_{\text{ph.}}\rangle_T\,. \tag{2}$$

* All second quantised operators will be written with a hat.

Here ${}_T\langle\Psi_{\text{ph.}}|\ldots|\Psi_{\text{ph.}}\rangle_T$ represents the ensemble average (Huang 1987),

$$\frac{\sum_n \langle n|\exp(\hat{H}_0/k_bT)\ldots|n\rangle}{\sum_n \langle n|\exp(\hat{H}_0/k_bT)|n\rangle},$$

where the summations are over all possible photons states and numbers. From equilibrium statistical mechanics the ensemble average of the photon number operator is given by the Bose–Einstein distribution (see page 187 of Huang),

$${}_T\langle\Psi_{\text{ph.}}|\hat{a}^\dagger_{\boldsymbol{q},\sigma}\hat{a}_{\boldsymbol{q},\sigma}|\Psi_{\text{ph.}}\rangle_T = n_b(E(\boldsymbol{q},\sigma)), \tag{3}$$

where

$$n_b(E(\boldsymbol{q},\sigma)) = \frac{1}{\exp(E(\boldsymbol{q},\sigma)/k_bT)-1} \tag{4}$$

and k_b is the Boltzmann constant. This gives the photon energy density as

$${}_T\langle\Psi_{\text{ph.}}|\hat{H}_0/V|\Psi_{\text{ph.}}\rangle_T = \frac{1}{V}\sum_{\boldsymbol{q},\sigma} E(\boldsymbol{q},\sigma)n_b(E(\boldsymbol{q},\sigma)), \tag{5}$$

which implies that the average energy density due to photons of a particular mode is

$$\frac{E(\boldsymbol{q},\sigma)n_b(E(\boldsymbol{q},\sigma))}{V}. \tag{6}$$

It follows then that the energy density due to photons of energy E can be given by

$$\sum_{\boldsymbol{q},\sigma} \frac{E(\boldsymbol{q},\sigma)n_b(E(\boldsymbol{q},\sigma))}{V}\ \delta_{E(\boldsymbol{q},\sigma),E}, \tag{7}$$

where $\delta_{E(\boldsymbol{q},\sigma),E}$ is the finite volume Kronecker delta function which is equal to unity if $E(\boldsymbol{q},\sigma) = E$ and zero otherwise. Planck's radiation law is the differential (with respect to energy) of this quantity.

To convert equation (7) to a differential, we suitably define the (finite time) energy differential ΔE. Observing a photon of a particular frequency requires an appropriate time interval t which has a corresponding minimum frequency $\Delta\nu = 1/t$, giving $t\Delta\nu = 1$ or (using $E = h\nu$) $t\Delta E/h = 1$. Accordingly we have the differential quantity

$$\sum_{\boldsymbol{q},\sigma} \frac{E(\boldsymbol{q},\sigma)n_b(E(\boldsymbol{q},\sigma))}{V}\ \delta_{E(\boldsymbol{q},\sigma),E}\ \frac{t}{h}\Delta E = En_b(E)\ \frac{1}{V}\sum_{\boldsymbol{q},\sigma}\frac{t}{h}\delta_{E(\boldsymbol{q},\sigma),E}\ \Delta E. \tag{8}$$

Using the (finite volume) momentum space normalisation condition, $V\Delta^3\boldsymbol{q}/(2\pi)^3 = 1$, we have

$$En_b(E) \ \frac{1}{(2\pi)^3}\sum_{\boldsymbol{q},\sigma}\frac{t}{h}\delta_{E(\boldsymbol{q},\sigma),E}\Delta^3\boldsymbol{q}\Delta E\,. \tag{9}$$

In the thermodynamic limit V becomes infinite giving $\Delta^3\boldsymbol{q} \to d^3\boldsymbol{q}$, the summation over $\boldsymbol{q}$ becoming an integral. In addition, the period of observation t becomes macroscopic; i.e. $t/h \gg \Delta E$, resulting in $(t/h)\delta_{E(\boldsymbol{q},\sigma),E}$ limiting to the continuous Dirac delta function $\delta(E(\boldsymbol{q},\sigma) - E)$ and $\Delta E \to dE$. Together, these limits entail

$$En_b(E) \ \sum_{\sigma}\frac{1}{(2\pi)^3}\int \delta(E(\boldsymbol{q},\sigma) - E)d^3\boldsymbol{q}\ dE\,. \tag{10}$$

Using $E(\boldsymbol{q},\sigma) = c\hbar|\boldsymbol{q}|$, we obtain Planck's law of radiation:

$$En_b(E)\ \frac{8\pi E^2}{(2\pi)^2(c\hbar)^3}\,dE\,, \tag{11}$$

which in terms of frequency is

$$h\nu\ n_b(h\nu)\frac{8\pi\nu^2}{c^3}\,d\nu\,. \tag{12}$$

By anticipating that the inclusion of interactions reduces the effectiveness of the non-interacting quantum numbers in describing the state of the system, we generalise equation (10) with minimal reference to the non-interacting quantum numbers of the free photon by re-writing Planck's law as

$$En_b(E)\mathcal{N}(E)\,dE\,, \tag{13}$$

where $(n = (\boldsymbol{q},\sigma))$

$$\mathcal{N}(E) = \frac{1}{V}\sum_{n}\delta(E - E(n)) \tag{14}$$

reflects the density of modes per unit volume with energy E and is commonly termed the photon density of states.

We now have a general and concise definition of Planck's law in equation (13) which has no explicit reference to the free particle quantum numbers and therefore retains its form when interactions are included.

(2b) The Inclusion of Interactions

Within the perturbative many body framework the interaction term contains the mechanism for allowing transitions between the non-interacting states of the

system. Consider the addition of an interaction term to our non-interacting Hamiltonian (equation 1):

$$\hat{H} = \hat{H}_0 + \sum_{mnl} V^{\text{em.}}_{mnl} \hat{c}^\dagger_n \hat{a}^\dagger_l \hat{c}_m + V^{\text{ab.}}_{mnl} \hat{c}^\dagger_n \hat{a}_l \hat{c}_m \,, \tag{15}$$

where the labels m and n represent the quantum numbers for the electrons, $(\boldsymbol{k}, \lambda)$, and l the quantum number of the photons, $(\boldsymbol{q}, \sigma)$. Inspection of the interaction term indicates that the electron can change its state via the emission or absorption of a photon.

The structure of V_{mnl} is dependent on the form of the non-interacting single particle solutions. If we consider the interaction of free electrons and photons, V_{mnl} is proportional to the fine structure constant and contains a momentum conserving Kronecker delta. On the other hand, if we consider the interaction of electrons in a solid with free photons, V_{mnl} is proportional to the product of the fine structure constant and the $\boldsymbol{q}$th (the momentum imparted to the solid) Fourier coefficient of the crystal potential — by virtue of the crystal potential being static, the particle momentum is no longer conserved. In this case, $\hat{c}_{\boldsymbol{k},\lambda}$ ($\hat{c}^\dagger_{\boldsymbol{k},\lambda}$) destroys (creates) a Bloch electron whose dispersion relationship is determined by the crystal band structure.

With interactions there is now the possibility that a particle (described by a non-interacting state) will be scattered into another (non-interacting) state — the states defined by a single particle quantum number will now have a finite lifetime which manifests itself as an uncertainty in the particle energy. Hence there is no longer a definite relationship between a particle's quantum number and its energy; i.e. the dispersion spectrum is smeared and is no longer delta-function-like (as in equation 14).

In a general interacting many body system it is the spectral density function which provides the link between the particles non-interacting quantum number and its energy. It gives the probability of a particle in the non-interacting quantum state n having energy E, and is defined as (see page 143 of Mahan 1990)

$$A(n, E) \propto \text{Im}[G_{ret}(n, E)] \,, \tag{16}$$

where $G_{ret}(n, E)$ is the Fourier transform of the particle's finite temperature interacting retarded Green function.*

The proportionality constant is determined by the normalisation condition

$$\int A(n, E) \frac{dE}{2\pi} = 1 \,. \tag{17}$$

The density of states can be defined with respect to the spectral density function:

$$\mathcal{N}(E) = \sum_n A(E, n) \,. \tag{18}$$

* For a general introduction to Green's functions consult the review article given by (Stedman 1968) and for a modern treatment of finite temperature Green's functions, see Chapter 3 of Mahan.

It is now clear that within many body theory, the generalised radiation energy density given by equation (13) requires no modification when interactions are included and any explicit deviations will arise purely from changes induced by the new definition of the density of states in equation (18).

3. The Principle of Detailed Balance

In the context of statistical mechanics, the principle of detailed balance is a condition for thermodynamic equilibrium (see page 121 of Van Kampen 1981) and is contained within the principle of microscopic reversibility which can be derived from non-equilibrium thermodynamics via the master equation (see page 189 of Huang 1987).

In its most transparent form, the principle of detailed balance entails a microscopic definition of thermal equilibrium between radiation and matter (Van Kampen 1981). Underlying this is that the net transition rate between any (atomic) states that the constituents of matter can enter is equal to that of the inverse transition. Using these ideas and a minimum of assumptions, Einstein was then able to derive Planck's law (Einstein 1917; Lewis 1973; Friedberg 1994).

In addition, the point has been made that Einstein's result could be obtained if one replace the 'classical' atomic states whose distribution follows that of Boltzmann statistics by that of particles which could be either fermions or bosons (Zitter and Hilborn 1987).

It is useful to view these derivations and their consequences from, firstly, Einstein's viewpoint and, secondly, the perspective of finite temperature many body theory. We perform the latter by explicitly applying the principle of detailed balance to our interacting Hamiltonian for the direct process of electron–hole creation and the corresponding inverse process of electron–hole annihilation.

(3a) Einstein's Derivation

Einstein gave the net transition rate from E_i to E_j as

$$d\Gamma_{ij}/dt = N(E_i)B_{ij}U(\nu,T)\,, \tag{19}$$

and for the inverse process, he wrote

$$d\Gamma_{ji}/dt = N(E_j)(B_{ji}U(\nu,T) + A_{ji})\,. \tag{20}$$

Here $U(\nu,T)$ is the energy density of a radiation field of frequency ν and temperature T. The proportionality coefficients A_{ji}, B_{ij} and B_{ji} are assumed to be dependent on the microscopic characteristics of the atom and the factor $N(E)$ represents the probability that the atom is in the state E.

The principle of detailed balance then entails equating the net direct and inverse transition rates, giving the radiation energy density as

$$U(\nu,T) = \frac{A_{ji}/B_{ji}}{\exp((E_j - E_i)/k_bT) - 1}\,, \tag{21}$$

where $B_{ij} = B_{ji}$ and $N(E) = \exp(-E/k_bT)$. The final assumption of Einstein was that the radiation energy density obeys Wien's Displacement Law; $U(\nu,T) \propto \nu^3 f(\nu/T)$. This requirement demands the proportionality relations,

$$\frac{A_{ji}}{B_{ji}} \propto \nu^3 , \tag{22}$$

$$E_j - E_i \propto \nu . \tag{23}$$

To obtain the proportionality constants, a theory of matter and radiation is needed, or alternatively as Einstein did, comparison with Planck's radiation law gives

$$\frac{A_{ji}}{B_{ji}} = \frac{8\pi\hbar\nu^3}{c^3} , \tag{24}$$

$$E_j - E_i = h\nu , \tag{25}$$

where h is Planck's constant and c the velocity of light in vacua.

This is indeed a remarkable result, for without the explicit use of quantum mechanics, Einstein was able to find the ratio between the spontaneous and stimulated transition rates as well as the form of the relationship between the frequency and energy of a photon.

This whole argument can be looked upon in two ways. Firstly, use of the above hypothesis can be seen as a derivation of Planck's law. Secondly, if one accepts Planck's radiation law, then Einstein's argument can be seen as a derivation of the spontaneous emission coefficient A_{ij} in terms of the induced (stimulated) coefficients B_{ij}.

(3b) The Many Body Argument

To apply Einstein's principle of detailed balance argument to our interacting Hamiltonian (equation 15) we would calculate the lowest order direct and inverse transition rate between two energy levels E_n and E_m. These processes are represented by the finite temperature Feynman diagrams shown in Fig. 1*a* and are calculated by Fermi's Golden Rule (Stedman 1971). In the case of the interaction between free electrons and free photons such a first order process cannot occur since both momentum and energy conservation cannot be satisfied — a free electron cannot emit a free photon, the allowed interactions (e.g. Compton scattering) beginning at second order. Therefore we implicitly assume that electrons are in a solid for the remainder of this section. Equating the direct and inverse transition rates via detailed balance results in a Bose–Einstein distribution for

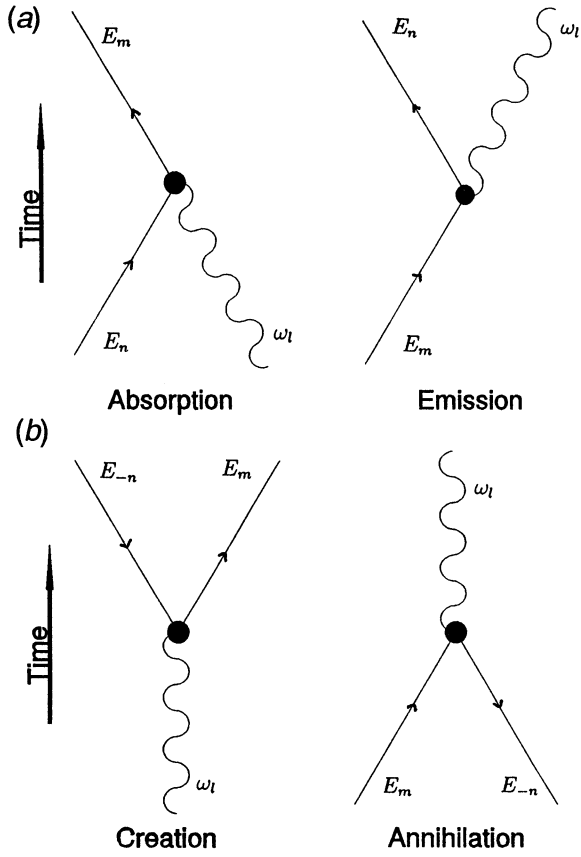

Fig. 1. First order finite temperature Feynman diagrams for (a) an electron absorbing or emitting a photon and (b) the electron–hole creation/annihilation process. As the electron line represents a Bloch electron, momentum is not conserved and these diagrams constitute allowed transitions.

the radiation field which is independent of the statistics of the matter field. This result can easily be shown to all orders in the interaction.

We wish however to apply detailed balance to the process of electron–hole creation and annihilation. This is another means by which radiation can be absorbed and emitted by matter. To do this, we simplify our Hamiltonian (equation 15) by assuming* $V^{\rm ab}_{mnl} = V^{\rm em}_{mnl} = \mu$ and introduce the hole construct by allowing the summations over the matter particle quantum numbers to include the negative domain and interpreting $\hat{c}^{\dagger}_{-m}$ ($\hat{c}_{-m}$) as the creation (destruction) of a hole. Our Hamiltonian (equation 15) now has the facility to create/annihilate an electron–hole pair. The direct (creation) and inverse (annihilation) processes are shown in Fig. 1b and the corresponding transition rates using Fermi's Golden

* This is a significant simplification which nevertheless gives the correct answer. The same result can be achieved for a realistic interaction vertex, if all the appropriate Feynman diagrams are included.

Rule are

$$\Gamma^{\text{dir.}}_{mnl} = \mu^2(1 - n_f(E_m))(1 - n_f(E_{-n}))N(\omega_l)\delta(E_m + E_{-n} - \omega_l),$$
$$\Gamma^{\text{inv.}}_{mnl} = \mu^2 n_f(E_m)n_f(E_{-n})(N(\omega_l) + 1)\delta(E_m + E_{-n} - \omega_l)\,, \tag{26}$$

where we have assumed that the matter particles are fermions. Equating these via detailed balance and solving for $N(\omega_l)$ results in

$$N(\omega_l) = \frac{1}{\exp(\omega_l/k_bT) - 1}\,. \tag{27}$$

On the other hand, if the statistics of the matter field are Bose–Einstein, the direct and inverse transition rates now correspond to

$$\Gamma^{\text{dir.}}_{mnl} = \mu^2(n_b(E_m) + 1)(n_b(E_{-n}) + 1)N(\omega_l)\delta(E_m + E_{-n} - \omega_l),$$
$$\Gamma^{\text{inv.}}_{mnl} = \mu^2 n_b(E_m)n_b(E_{-n})(N(\omega_l) + 1)\delta(E_m + E_{-n} - \omega_l)\,, \tag{28}$$

which via detailed balance again gives rise to equation (27). Again this result is correct to all orders in the pair creation/annihilation process.

Thus within the apparatus of many body theory, the application of the principle of detailed balance demands that the statistics of the radiation field be that of Bose–Einstein and this is independent of the type of interaction and the statistics of the 'matter' field. That this result is correct to all orders in the interaction is simply a consequence of the finite temperature many body formalism — an underlying assumption of perturbative many body theory is that the non-interacting particle statistics (with respect to energy) survive interactions.

From the many body viewpoint, the detailed balance argument does not give the radiation energy density. To obtain it, the density of states $E\mathcal{N}(E)$ needs to be calculated. Similarly in Einstein's original application of detailed balance, a precise knowledge of B_{mn}/A_{mn} could not be obtained unless Wien's displacement law or Planck's radiation law was assumed.

4. Planck's Radiation Law to Higher Orders

To obtain the photon density of states we require the spectral density function which in turn requires the finite temperature photon retarded Green function. The latter is found by an analytical continuation of the Fourier transform of the photon Matsubara Green function.*

We consider the photon Green function for the interaction term $e^2\boldsymbol{A}\cdot\boldsymbol{A}/2m$, which arises from the expansion of the minimal coupled Hamiltonian for the

* That is, the retarded frequency Green function is obtained from the photon Matsubara (frequency) Green function $D_{\mu\nu}(n; i\omega_n)$ where $D_{\mu\nu}(n;\tau) = (1/\beta)\sum_{i\omega_n} D_{\mu\nu}(n; i\omega_n)\exp(-i\omega_n\tau)$, via the substitution $i\omega_n \to \omega + i\delta$, see page 138 of Mahan (1990).

interaction between matter and radiation, $(\boldsymbol{P}-e\boldsymbol{A})^2/2m$. In the second quantised form this is

$$\hat{H}^{\text{self}}(\tau) = \frac{e^2}{2m} \sum_{\boldsymbol{k}',\boldsymbol{k},\boldsymbol{q}'\neq\boldsymbol{q}} \sum_{\lambda,\sigma,\sigma'} \frac{\varepsilon^{\mu}(\boldsymbol{q}',\sigma')\varepsilon_{\mu}(\boldsymbol{q},\sigma)}{\sqrt{\omega_{\boldsymbol{q}'}\omega_{\boldsymbol{q}}}} \hat{c}^{\dagger}_{\boldsymbol{k},\lambda}(\tau)\hat{c}_{\boldsymbol{k},\lambda}(\tau)$$

$$\times \left(\hat{a}^{\dagger}_{\boldsymbol{q}',\sigma}(\tau)\hat{a}_{\boldsymbol{q},\sigma}(\tau)\delta_{\boldsymbol{q}'-\boldsymbol{q}+\boldsymbol{k}'-\boldsymbol{k},0} + \hat{a}_{\boldsymbol{q}',\sigma'}(\tau)\hat{a}^{\dagger}_{\boldsymbol{q},\sigma}(\tau)\delta_{-\boldsymbol{q}'+\boldsymbol{q}+\boldsymbol{k}'-\boldsymbol{k},0}\right), \qquad (29)$$

which has a different structure to equation (15); the *free* electron now undergoes a transition via the scattering of a photon.

Formally (see pages 94 and 125 of Mahan) the interacting polarised photon Matsubara Green function is given by

$$D_{\mu\nu}(\boldsymbol{k};\tau-\tau') = \sum_{\sigma''} \frac{-i_T\langle T_\tau \hat{A}_{\mu}(\boldsymbol{k},\sigma'',\tau)\hat{A}_{\nu}(-\boldsymbol{k},\sigma'',\tau')\hat{S}(0,\beta)\rangle_T}{{}_T\langle \hat{S}(0,\beta)\rangle_T}, \qquad (30)$$

where

$$\hat{A}_{\mu}(\boldsymbol{k},\sigma'',\tau) = \varepsilon_{\mu}(\boldsymbol{k},\sigma'')\left(\frac{1}{V\omega_{\boldsymbol{k}}}\right)^{\frac{1}{2}}\left[\hat{a}^{\dagger}_{\boldsymbol{k},\sigma''}(\tau) + \hat{a}_{-\boldsymbol{k},\sigma''}(\tau)\right], \qquad (31)$$

is the vector potential field operator for a photon normalised to a volume V with momentum $\boldsymbol{q}$ and polarisation ε_{μ}, and

$$\hat{S}(\tau,\tau') = T_\tau \exp\left(-i\int_{\tau'}^{\tau} \hat{H}_I(\tau_1)\,d\tau_1\right) \qquad (32)$$

is the system's time evolution operator. In both equations (30) and (32), T_τ is the complex time ordering operator. In equation (30), $H_I(\tau_1)$ is the interaction part of the Hamiltonian and all creation and destruction operators are in the interaction picture (Stedman 1971; Mandl and Shaw 1988).

Expanding equation (30) to first order via the time ordered exponential (equation 30), the photon Matsubara Green function becomes

$$D^{(1)}_{\mu\nu}(\boldsymbol{k};\tau-\tau') = D^{(0)}_{\mu\nu}(\boldsymbol{k};\tau-\tau') - \sum_{\sigma''}\int_0^{\beta} d\tau_1 \langle T_\tau \hat{A}_{\mu}(\boldsymbol{k},\sigma'',\tau)\hat{A}_{\nu}(\boldsymbol{k},\sigma'',\tau')\hat{H}_i(\tau_1)\rangle. \qquad (33)$$

Upon evaluation of the above time ordered product via the use of Wick's theorem (see page 95 of Mahan) we obtain, using our interaction Hamiltonian term (equation 29),

$$D^{(1)}_{\mu\nu}(\boldsymbol{k};\tau-\tau') = D^{(0)}_{\mu\nu}(\boldsymbol{k};\tau-\tau') + \frac{e^2 n_0}{m}\int_0^{\beta} d\tau_1 \times \sum_{\eta} D^{(0)}_{\mu\eta}(\boldsymbol{k};\tau-\tau_1)D^{(0)}_{\eta\nu}(\boldsymbol{k};\tau_1-\tau'), \qquad (34)$$

where n_0 represents the total electron density, obtained from

$$(1/V)\sum_{\boldsymbol{q},\lambda} {}_T\langle\Psi_{\text{el.}}|c^{\dagger}_{\boldsymbol{q},\lambda}c_{\boldsymbol{q},\lambda}|[\Psi_{\text{el.}}\rangle_T].$$

We require the Fourier transform of equation (34);

$$D^{(1)}_{\mu\nu}(\boldsymbol{k};i\omega_n) = D^{(0)}_{\mu\nu}(\boldsymbol{k};i\omega_n) + \sum_{\eta,\lambda} D^{(0)}_{\mu\eta}(\boldsymbol{k};i\omega_n)\Pi_{\eta\lambda}(\boldsymbol{k};i\omega_n)D^{(0)}_{\eta\nu}(\boldsymbol{k};i\omega_n)\,, \tag{35}$$

where

$$\Pi_{\eta\lambda}(\boldsymbol{k};i\omega_n) = \delta_{\eta\lambda}\Pi^{(1)} = \delta_{\eta\lambda}e^2n_0/m \tag{36}$$

is termed the photon self energy. This photon Green function is graphically represented in Fig. 2.

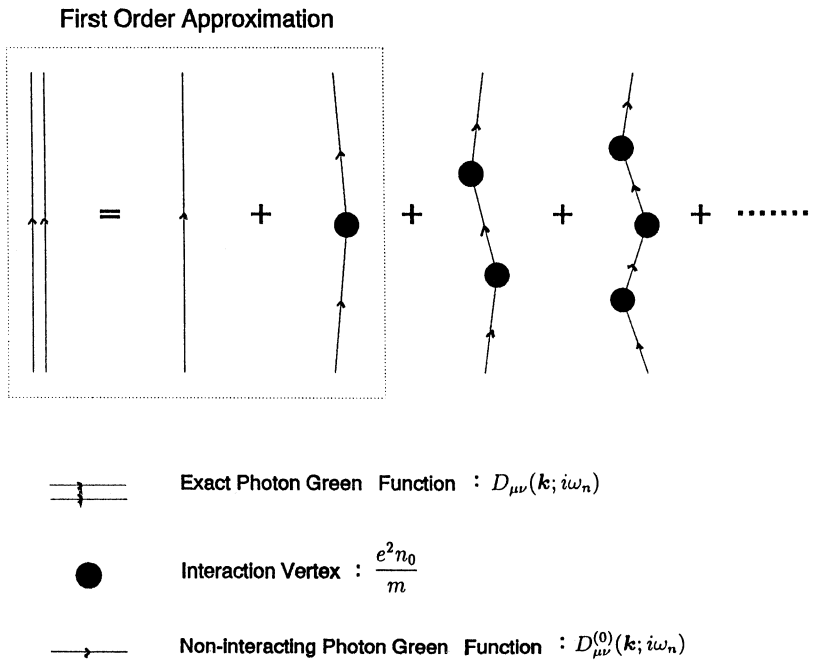

Fig. 2. Feynman diagrams for the interacting photon Green function for the non-local plasmon–photon interaction.

For the interaction which we consider, higher order corrections are constructed from products of equation (36), which via the Dyson equation,* can be summed analytically using the Fourier transform of the non-interacting photon Matsubara Green function

$$D_{\mu\nu}(\boldsymbol{k}; i\omega_n) = -\frac{4\pi(\delta_{\mu\nu} - k_\mu k_\nu/|\boldsymbol{k}|^2)}{\omega_n^2 + \omega_{\boldsymbol{k}}^2}. \tag{37}$$

Doing this we obtain the exact photon Green function

$$D_{\mu\nu}(\boldsymbol{k}; i\omega_n) = -\frac{4\pi(\delta_{\mu\nu} - k_\mu k_\nu/|\boldsymbol{k}|^2)}{\omega_n^2 + \omega_{\boldsymbol{k}}^2 + 4\pi\Pi^{(1)}}. \tag{38}$$

Equation (38) can be rewritten in terms of the dielectric function $\epsilon[i\omega_n]$,

$$\begin{aligned} D_{\mu\nu}(\boldsymbol{k}; i\omega_n) &= -\frac{4\pi(\delta_{\mu\nu} - k_\mu k_\nu/|\boldsymbol{k}|^2)}{\epsilon[i\omega_n]\omega_n^2 + \omega_{\boldsymbol{k}}^2} \\ &= -\frac{4\pi(\delta_{\mu\nu} - k_\mu k_\nu/|\boldsymbol{k}|^2)}{-\epsilon[i\omega_n](i\omega_n)^2 + \omega_{\boldsymbol{k}}^2}, \end{aligned} \tag{39}$$

where $\epsilon[i\omega_n] = 1 - -4\pi\Pi^{(1)}/(i\omega_n)^2$. Assuming that $\epsilon[i\omega_n] > 0$ (indicating dispersion), the effect of the interaction is to renormalise the frequency of the photon by the factor $\sqrt{\epsilon[i\omega_n]}$. The (renormalised) retarded photon Green function is found using the substitution $i\omega_n \to \omega + i\delta$ in equation (39),

$$\begin{aligned} D^{\mathrm{ret}}_{\mu\nu}(\boldsymbol{k}; \omega) = &\frac{4\pi(\delta_{\mu\nu} - k_\mu k_\nu/|\boldsymbol{k}|^2)}{-2\omega_{\boldsymbol{k}}} \\ &\times \left(\frac{1}{\omega_{\boldsymbol{k}} - n[\omega]\omega - i\delta} + \frac{1}{\omega_{\boldsymbol{k}} + n[\omega]\omega + i\delta}\right). \end{aligned} \tag{40}$$

Here $n[\omega]$ is the refractive index of the medium and is defined as

$$n[\omega] = \sqrt{\epsilon[\omega]} = \sqrt{1 - \left(\frac{\omega_p}{\omega}\right)^2}, \tag{41}$$

where $\omega_p = 4\pi e^2 n_0/m$ is the plasmon angular frequency.

* The Dyson method entails an iterative equation which for the exact photon Green function has the general form $D_{\mu\nu} = D^0_{\mu\nu} + \sum_{\lambda\delta} D^0_{\mu\lambda}\Pi_{\lambda\delta}D_{\lambda\nu}$. This equation is exact and contains the summation of all possible Feynman diagrams constructed from the non-interacting Green function to infinite order. An analytic solution is however only possible when certain approximations are made. This involves summing only a certain class of diagrams to infinity. The most common approximation is termed the random phase approximation (RPA) and involves summing only those Feynman diagrams which contribute to the linearised form of the interactions. In our case $\Pi_{\lambda\delta} = \Pi\delta_{\lambda,\delta}$ and the RPA solution is trivial.

The spectral density function for the photon is

$$B_{\mu\nu}(\boldsymbol{k},\omega) = C\mathrm{Im}[D^{\mathrm{ret}}_{\mu\nu}(\boldsymbol{k};\omega)]\,, \tag{42}$$

where the proportionality factor C can be determined from the normalisation condition (equation (17)); i.e.

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$$C^{-1} = \int \mathrm{Im}[D^{\mathrm{ret}}_{\mu\nu}(\boldsymbol{k};\omega)]\frac{d\omega}{2\pi}\,. \tag{43}$$

Insertion of equation (40) into the above results in

$$C^{-1} = \frac{-1}{\omega_{\boldsymbol{k}}}\int \mathrm{Im}\left[\frac{1}{\omega_{\boldsymbol{k}} - n[\omega]\omega - i\delta} + \frac{1}{\omega_{\boldsymbol{k}} + n[\omega]\omega + i\delta}\right]d\omega\,, \tag{44}$$

where we have not included the term $\delta_{\mu\nu} - k_\mu k_\nu/|\boldsymbol{k}|^2$ because each nonzero component contributes a factor of unity to C.

The imaginary part is obtained using the well known delta function identity:

$$\frac{1}{x + i\delta} = \mathrm{P}\left(\frac{1}{x}\right) + i\pi\delta(x)\,, \tag{45}$$

giving

$$\begin{aligned} C^{-1} &= \frac{-\pi}{\omega_{\boldsymbol{k}}}\int\Big(\delta(\omega_{\boldsymbol{k}} - n[\omega]\omega) + \delta(\omega_{\boldsymbol{k}} + n[\omega]\omega)\Big)d\omega \\ &= -2\pi\int\delta(\omega^2_{\boldsymbol{k}} - (n[\omega]\omega)^2)d\omega \\ &= \frac{-(2\pi)^2}{2\omega_R n[\omega_R]\Big(n[\omega_R] + n'[\omega_R]\omega_R\Big)}\,. \end{aligned} \tag{46}$$

In the last line we have used the delta function transform identity* and $\omega_R = \sqrt{w^2_{\boldsymbol{k}} + \omega^2_p}$. Thus the spectral density function becomes

$$\begin{aligned} B_{\mu\nu}(\boldsymbol{k},\omega) = \left(\delta_{\mu\nu} - \frac{k_\mu(k_\nu)}{|\boldsymbol{k}|^2}\right)&\frac{\omega R n[\omega R](n[\omega R] + n'[\omega R]\omega R)}{\omega\boldsymbol{k}} \\ &\times(\delta(\omega\boldsymbol{k} = n[\omega]\omega) + \delta(\omega_{\mathrm{k}} + n[\omega]\omega))\,, \end{aligned}$$

and the density of states†

* To transform the argument in a delta function to a different variable we use the identity: $\delta(f(x)) = \sum_n \delta(x - x_n)/|f'(x_n)|$, where x_n are the simple roots of $f(x)$.

† In the case of the photon, the degrees of freedom associated with the spin indices have been replaced by the polarisation vector which corresponds to the summation over spin being replaced by a trace over the polarisation vector.

$$\mathcal{N}(\omega) = \frac{1}{V}\sum_{\boldsymbol{k}} \mathrm{Tr}[B_{\mu\nu}(\boldsymbol{k};\omega)]$$
$$= \frac{4\pi}{(2\pi)^3}\int d\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}}^2 \frac{\omega_R n[\omega_R](n[\omega_R]+n'[\omega_R]\omega_R)}{\omega_{\boldsymbol{k}}}$$
$$\times\,(\delta(\omega_{\boldsymbol{k}} - n[\omega]\omega) + \delta(\omega_{\boldsymbol{k}} + n[\omega]\omega))\,, \tag{47}$$

where we have integrated with respect to $d\Omega_{\boldsymbol{k}}$ (obtaining a factor of 4π), $\mathrm{Tr}[\delta_{\mu\nu} - k_\mu k_\nu/|\boldsymbol{k}|^2] = 2$ and used $d|\boldsymbol{k}||\boldsymbol{k}|^2 = d\omega_{\boldsymbol{k}}\omega_{\boldsymbol{k}}^2$. Performing the remaining angular frequency integral we obtain

$$\mathcal{N}(\omega) = \frac{8\pi\omega^2 n^2[\omega](n[\omega]+n'[\omega]\omega)}{c^3(2\pi)^2}\,, \tag{48}$$

where $\omega_R = \omega$. In terms of the frequency $\nu = \omega/2\pi$, this becomes

$$\mathcal{N}(\nu) = \frac{8\pi\nu^2 n^2[\nu](n[\nu]+n'[\nu]\nu/(2\pi))}{c^3}\,. \tag{49}$$

Taking into account the interaction of the photon with the electron, via equation (29), the radiation energy density becomes

$$U(\nu)\,d\nu = n_b(h\nu)h\nu\frac{8\pi\nu^2 n^2[\nu](n[\nu]+n'[\nu]\nu/(2\pi))}{c^3}\,d\nu\,. \tag{50}$$

The physical content of such a correction to the photon self energy and thus the refractive index of the medium (equation 41) is the interaction of the photon with the quantised collective excitation modes of the electron gas. Such non-local excitations are referred to as plasmons and correspond to an electron charge density wave where the plasmon frequency ν_p is the resonant frequency of the excitation.

For semiconductors and insulators, ν_p is in the infrared region ($< 1\,\mathrm{eV}$) and for optical radiation ($\sim 2{\cdot}0\,\mathrm{eV}$) the dielectric function (equation 41) is positive and equation (14) is applicable, inducing only a shift in the photon frequency; i.e. unattenuated propagation. For metals ν_p is typically in the ultraviolet region due to higher conduction electron densities and the dielectric function becomes negative for optical radiation. This corresponds to the absorption of optical radiation and effects the density of states by smearing out the non-interacting delta function structure—taking the imaginary part of equation (39) when the dielectric function is negative, results in a Lorentzian centred at $\sqrt{-\varepsilon}\omega_{\boldsymbol{k}}$ instead of a delta function.

Equation (50) is a general result (independent of the precise form of $n[\nu]$), applying to any system where interactions manifest themselves solely via a dispersive effect. The dielectric function for a more realistic system will be more complex than equation (41), containing photon self energy terms arising from virtual electron–hole pairs (and their mutual Coulombic attraction) which surround the photon.

Our result has particular relevance for semiconductors where van Roosbroeck and Shockley (1954) assumed that at thermal equilibrium the rate of optical generation of electron–hole pairs is equal to their rate of radiative recombination (i.e. no net absorption only a net dispersion) and via a detailed balance argument obtained

$$U(\nu)\,d\nu = n_b(h\nu)h\nu\frac{8\pi\nu^2 n^3}{c^3}\,d\nu\,, \tag{51}$$

where the refractive index is a constant obeying the Moss formula (Pankove 1971) — a simple empirical rule between the refractive index (and thus the dielectric function) and the energy gap in a semiconductor: $n = (77/E_g(eV))^{\frac{1}{4}}$. The results of our many body calculation do in fact parallel the early work of determining the refractive index of semiconductors (Moss 1961).

The intuitive reasoning of van Roosbroeck and Shockley is vindicated by our formal derivation of equation (50) — which reduces to equation (51) for a constant refractive index. Furthermore their detailed balance argument is precise from the many body perspective because the type and order of interaction between matter and radiation in thermal equilibrium does not affect the end result, that the statistics of the radiation remain bosonic (Section 3*b*) and any modification of Planck's radiation law occurs through the photon density of states.

5. When Planck's Law is Likely to Fail

Returning to our specific calculation of the plasmon interaction and ignoring for the moment the exact result, we could propose that if the interaction strength $e^2 n_0/m$ is much less than unity, it would suffice to only include the lowest order correction to the photon Green function. Naively the perturbation series rapidly converges and the final radiation law is not that dissimilar to Planck's original law. However, returning to the exact result, we find that at a certain frequency ν_p the density of states is identically zero (equation 49) — because the refractive index is (zero). Thus the perturbation technique fails* even though the strength of the interaction is far less than unity. For more complex interactions, no exact solution exists as a guide and a perturbation calculation (with a weak interaction) may result in an incorrect answer when compared with experiment for a particular range of particle energies.

Physically this breakdown corresponds to a strong coupling between the photon and the plasmon (which often only manifests itself in the exact solution), so much so that a new elementary excitation is formed; the elementary excitations of the photon and plasmon no longer are separately definable. The photon concept breaks down and therefore a perturbation with respect to the (free) photon must similarly breakdown. Under such circumstances Planck's radiation law fails to describe the energy density of the excitation field.

* There is in fact a logical inconsistency here, for we have used the perturbation series (via Dyson's method) to obtain the dielectric function and yet at the same time we propose that the perturbation series fails. Ultimately this calculation is heuristic, for to calculate the series correctly we must perturb from the new non-interacting elementary excitations (which will involve a new type of interaction) formed by the strong coupling between the photon and plasmon. Only then does the perturbation series become precise and logically consistent. Another example is in superconductivity, where the pairing mechanism must be introduced and cannot be derived via a perturbation from the traditional fermion ground state.

Generally the failure of Planck's law corresponds to the photon strongly coupling with other elementary excitations of the system. Consider for example, the quantised lattice vibrations of a crystal where the elementary excitations are phonons. In some cases the coupling between the photon and phonon is so strong that a new elementary excitation must be defined: the phonon polariton. Again the perturbation series with respect to the non-interacting photon breaks down and Planck's radiation law fails, even though the lowest order interaction is weak. Similarly the photon can strongly couple to an exciton (a weakly interacting electron–hole pair in a semiconductor), forming an exciton polariton.

For radiation fields with frequencies above the Compton frequency ($> 0{\cdot}51\,\mathrm{MeV}$), quantum electrodynamics enters a new non-perturbative phase where the electron, positron and photon no longer constitute the non-interacting elementary excitations of the system (Caldi 1992). In this regime Planck's law fails, whilst below the Compton frequency it requires little modification (Barton 1990).

Superconductivity provides another example where the photon loses its traditional meaning as an elementary excitation and acquires an effective mass. The breakdown of Planck's law is clearly demonstrated via the Meissner effect where the electromagnetic field is completely expelled from a superconductor at and below a critical temperature (Feynman 1972).

6. Conclusion

In this paper we have viewed Planck's radiation law from the perspective of finite temperature many body theory. Einstein's detailed balance argument has also been investigated from this viewpoint. We have expressed Planck's law in a form which is independent of the specific detail of a system and have calculated the explicit form for a dispersive medium, justifying the van Roosboeck–Shockley modification to Planck's radiation law for semiconductors.

The generality of the law has been justified theoretically to all orders in the interaction between matter and radiation, failing only when the perturbation technique breaks down. Such a failure was demonstrated for the case of the photon–plasmon interaction and coincided with a breakdown of the photon concept — the effective coupling being so great that the photon no longer constituted a definable elementary excitation.

We emphasise that our results were derived entirely within the context of conventional perturbative many body theory. Indeed we further expect the generality of Planck's law to fail beyond this level as witnessed by the existence of non-bosonic excitations of the matter field, for example holons and renormalised fermionic matter fields such as spinons (Laughlin 1992). Their properties in the modern theory of interaction with exotic matter (e.g. the quantum Hall effect) is still an active area of research.

In conclusion, we have emphasised that the deep physical intuition that led to the original papers by Planck and Einstein carries through to the modern ideas of interacting systems as incorporated into finite temperature many body theory.

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