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Collision Integrals for Tensorial Hermite Polynomials in Mixtures of Rarefied Gases and Plasmas

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Abstract

The collision integral of Maxwell's balance equation (equation of change) for tensorial Hermite polynomials is calculated with velocity distribution functions represented as orthogonal expansions of local Maxwellians with respect to these polynomials. Closed expressions are obtained for the tensorial coefficients in the expansion of the collision integral with respect to products of Hermitian moments, i.e. velocity averaged Hermite polynomials. The averages over the collisional kinetic energies are represented by transport collision frequencies with two superscripts, which form a null-sequence with increasing second superscript.

1. Introduction

To calculate the collision integral in Maxwell's (1867, equation 73) balance equation (equation of change) for a quantity of particles, assumptions must be made for the velocity distribution functions $f_j(\boldsymbol{c}_j)$ of all particle species j. In his first extensive paper on the dynamical theory of gases Maxwell (1867, equations 27, 48 and 56) used a shifted Maxwellian $f_j(\boldsymbol{c}_j) = f_j^M(m_j|\boldsymbol{c}_j - \overline{\boldsymbol{c}}_j|^2/2kT)$ with $\overline{\boldsymbol{c}}_j$ as the average velocity of particles j, but in his follow-up paper Maxwell (1879) introduced (without saying so) an expansion of a local Maxwellian in tensorial Hermite polynomials $He^{(n)}\{\boldsymbol{c}/(2kT/m)^{\frac{1}{2}}\}$, truncated after the third-order polynomial with $n = 3$. [Rearranging this truncated expansion with respect to tensorial powers $\boldsymbol{c}^0 = 1, \boldsymbol{c} = \boldsymbol{c}^1, \boldsymbol{cc} = \boldsymbol{c}^2, \boldsymbol{ccc} = \boldsymbol{c}^3$ yields immediately Maxwell's (1879) equations 11, 21 and 22.]

Seventy years later Grad (1949*b*, equations 4.8, 4.9 and 5.4) introduced this expansion for a simple gas with a local shifted Maxwellian as weight function. He not only neglected all polynomials with $n \geq 4$ but also the tracefree part of the third-order polynomial $He^{(3)}$, thus ending up with his 13 moment approximation, consisting of the number density, the average velocity, the temperature, the (tracefree symmetric) stress tensor, and the heat flux vector.

Instead of tensorial Hermite polynomials $He^{(n)}(\boldsymbol{x})$ Desloge (1964, equation 43), Weinert and Suchy (1977, equation A1), Lin *et al.* (1979, equation 9), Viehland and Lin (1979) and Viehland *et al.* (1981, equation 20) used products of three scalar Hermite polynomials $He_{n_1}(x_1)He_{n_2}(x_2)He_{n_3}(x_3)$. Their representation with Laguerre–Sonine polynomials and spherical harmonics (Weinert and Suchy 1977, equation A17 with A19 and A21) corresponds to our representation (75) with

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(76) below. Tensorial Hermite polynomials were used by Eu and Ohr (1994, equations 2.5, 3.2 and 3.4) to expand the logarithm of the velocity distribution function divided by a Maxwellian.

The most extensive treatment of the Hermite expansion was given in Balescu's (1988) book on transport processes in plasmas. Since plasmas are always mixtures, Balescu (1988, equations 4.3.11 and 4.3.16), like Maxwell (1867), introduced local shifted Maxwellians as weight functions for all species, but with species temperatures $T_j := (m_j/3k)\overline{|\boldsymbol{c}_j - \overline{\boldsymbol{c}}_j|^2}$ instead of Maxwell's mean temperature T. Furthermore Balescu (1988, equation 2.6.24) used the Landau collision integral, valid for long-range interactions only, instead of the Maxwell–Boltzmann integral (used by the other authors quoted) which is valid also for short-range interactions.

In contrast to Maxwell and Balescu we introduce tensorial Hermite polynomials in Section 2 for peculiar velocities $\boldsymbol{c}_j - \boldsymbol{v}$ (where $\boldsymbol{v}$ is the mean-mass velocity of the whole mixture) with the mean temperature T as additional parameter. This has been proved convenient for the calculation of the dynamical (differential) part of the balance equation for Hermite polynomials (Suchy 1995, equation 1.7) and is also useful for the decomposition of the peculiar velocities into relative velocities (between collision partners) and centre-of-mass velocities in Sections 3 and 4. This decomposition is necessary for the treatment of the deflection during collisions in Section 6. The necessary integrations over the centre-of-mass velocities are done in Section 5. In Section 7 the characteristics of the collisional (long- and short-range) interactions are condensed in transport collision frequencies, which are linear combinations of omega integrals used in the Chapman–Enskog approach (Chapman and Cowling 1970, equation 9.33,2).

2. Tensorial Hermite Polynomials and Hermitian Moments

For the treatment of transport processes Enskog's equation of change is very often used as the starting point. It is a balance equation for the velocity average

$$\overline{\phi_j}(\boldsymbol{r},t) := \frac{1}{n_j}\int d^3C_j f_j(\boldsymbol{r},t,\boldsymbol{C}_j)\phi_j(\boldsymbol{r},t,\boldsymbol{C}_j) \tag{1}$$

of a property $\phi_j(\boldsymbol{r},t,\boldsymbol{C}_j)$ of particles of species j with velocity distribution function $f_j(\boldsymbol{r},t,\boldsymbol{C}_j)$ and number density $n_j(\boldsymbol{r},t)$. The peculiar velocity $\boldsymbol{C}_j := \boldsymbol{c}_j - \boldsymbol{v}$ is defined as the particle velocity $\boldsymbol{c}_j$ relative to the mean-mass-velocity

$$\boldsymbol{v} := \frac{1}{\rho}\sum_j \rho_j\,\overline{\boldsymbol{c}}_j \tag{2}$$

with the mass densities

$$\rho_j := m_j n_j \quad \text{and} \quad \rho := \sum_j \rho_j\,. \tag{3}$$

With the convective (barycentric) derivative

$$\frac{D}{Dt} := \frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla \tag{4}$$

and the acceleration $\dot{\boldsymbol{c}}_j$ by external forces Enskog's equation of change reads (Chapman and Cowling 1970, equation 3.13,6)

$$\frac{\partial}{\partial t} n_j \overline{\phi_j} + \nabla \cdot (n_j (\boldsymbol{v} + \overline{\boldsymbol{C}_j) \phi_j}) - n_j \overline{\frac{D\phi_j}{Dt}} - n_j \, \overline{\boldsymbol{C}_j \cdot \nabla \phi_j}$$

$$+ n_j \, \overline{\boldsymbol{C}_j \cdot (\nabla \boldsymbol{v}) \cdot \frac{\partial}{\partial \boldsymbol{C}_j} \phi_j} + n_j \overline{\left(\frac{D\boldsymbol{v}}{Dt} - \dot{\boldsymbol{c}}_j \right) \cdot \frac{\partial}{\partial \boldsymbol{C}_j} \phi_j} = n_j \sum_k \overline{\left(\frac{\delta \phi_j}{\delta t} \right)_k^{\text{coll}}} \,. \tag{5}$$

In the rate of change by collisions with particles of species k (Chapman and Cowling 1970, equation 3.51,3), viz.

$$n_j \overline{\left(\frac{\delta \phi_j}{\delta t} \right)_k^{\text{coll}}} = \int d^3 C_j \int d^3 C_k f_j f_k c_{jk} \oint d^2 \hat{\boldsymbol{c}}'_{jk} \sigma_{jk} (\phi'_j - \phi_j), \tag{6}$$

the dashed quantities are taken with velocities after a collision, the undashed before a collision, and $\boldsymbol{c}_{jk} := \boldsymbol{c}_j - \boldsymbol{c}_k$ is the relative velocity (with conserved modulus $c'_{jk} = c_{jk}$ for elastic collisions). The surface element $d^2 \hat{\boldsymbol{c}}'_{jk}$ on the unit sphere about the particle k is the solid angle element $d\epsilon d(\cos\,\chi)$ with the azimuthal angle ϵ and the deflection angle χ. The differential cross section $\sigma_{jk}(\chi, c_{jk})$ depends on χ and c_{jk} (for elastic collisions between pointlike particles).

If the particle property $\phi_j(\boldsymbol{r}, t, \boldsymbol{C}_j)$ varies with $\boldsymbol{r}, t$ only via the mean-mass-velocity $\boldsymbol{v}$ (2) and a mean temperature, defined with Boltzmann's constant k as (Chapman and Cowling 1970, equation 2.5,10)

$$\tfrac{3}{2} kT := \frac{1}{n} \sum_j \frac{\rho_j}{2} \overline{C_j^2} \quad \text{with} \quad n := \sum_j n_j \,, \tag{7}$$

then the first two negative terms in Enskog's equation of change (5) can be specialised as

$$-n_j \overline{\frac{D\phi_j}{Dt}} - n_j \overline{\boldsymbol{C}_j \cdot \nabla \phi_j} = -n_j \frac{DT}{Dt} \overline{\frac{\partial \phi_j}{\partial T}} - n_j (\nabla T) \cdot \overline{\boldsymbol{C}_j \frac{\partial \phi_j}{\partial T}} \,. \tag{8}$$

Specialising the velocity dependence of ϕ_j to three-dimensional tensorial Hermite polynomials, Grad (1949*a*, 1949*b*) divided all velocities $\boldsymbol{C}_j$ by the square-root of kT/m_j to obtain a dimensionless vector $\boldsymbol{x}$ as the argument of the Hermite polynomials. The explicit appearance of derivatives with respect to $\boldsymbol{C}_j$ and T in the general balance equation (5) with (8) suggests the generalisation of Grad's (1949*a*, equation 13) nth order symmetric dimensionless Hermite polynomials

$$He^{(n)}(\boldsymbol{x}) := \exp(x^2/2) \left(\frac{-\partial}{\partial \boldsymbol{x}} \right)^n \exp(-x^2/2) \tag{9}$$

to (Suchy 1995, equation 1.7)

$$\begin{aligned} He^{(n)}(j) &:= He^{(n)}(\tau_j; \boldsymbol{C}_j) := \exp(C_j^2/2\tau_j) \left(\frac{-\tau_j \partial}{\partial \boldsymbol{C}_j}\right)^n \exp(-C_j^2/2\tau_j) \\ &= \tau_j^{(n/2)} He^{(n)} \left(\boldsymbol{C}_j/\tau_j^{\frac{1}{2}}\right) \qquad \text{with} \qquad \tau_j := kT/m_j\,. \end{aligned} \tag{10}$$

The first polynomials are

$$\begin{gathered} He^{(0)}(j) = 1 \quad He^{(1)}(j) = \boldsymbol{C}_j \quad He^{(2)}(j) = \boldsymbol{C}_j^2 - \tau_j \underline{\underline{I}} \\ He^{(3)}(j) = \boldsymbol{C}_j^3 - 3\tau_j \widehat{\boldsymbol{C}_j \underline{\underline{I}}} \quad He^{(4)}(j) = \boldsymbol{C}_j^4 - 6\tau_j \widehat{\boldsymbol{C}_j^2 \underline{\underline{I}}} + 3\tau_j^2 \widehat{\underline{\underline{I}}\,\underline{\underline{I}}} \end{gathered} \tag{11}$$

with tensorial powers $\boldsymbol{C}_j^2 := \boldsymbol{C}_j \boldsymbol{C}_j$ etc., the second order unit tensor $\underline{\underline{I}} := \boldsymbol{g}_i \boldsymbol{g}^i$, and the symmetry symbol for a nth order tensor $T^{(n)}$ is

$$\widehat{T^{(n)}} := \frac{1}{\text{number of } \pi} \sum_{\pi} T^{(n)}\,, \tag{12}$$

where

$\pi :=$ significant permutations of either the base vectors $\boldsymbol{g}_i, \boldsymbol{g}^i$ or the indices.

For example we have

$$\widehat{\boldsymbol{abb}} = \tfrac{1}{3}(\boldsymbol{abb} + \boldsymbol{bab} + \boldsymbol{bba}) \tag{13}$$

Their velocity averages (1), called Hermitan moments (Balescu 1988, Section 4.3), are partly related to macroscopic quantities:

$$\begin{gathered} \overline{He^{(1)}(j)} = \overline{\boldsymbol{C}}_j\,, \\ \rho_j \overline{He^{(2)}(j)} = \underline{\underline{p}}_j - n_j kT \underline{\underline{I}}\,, \\ \tfrac{1}{2} \text{ trace } \rho_j \overline{He^{(3)}(j)} = \boldsymbol{q}_j - \tfrac{5}{2} n_j kT \overline{\boldsymbol{C}}_j\,, \end{gathered} \tag{14}$$

where

$$\underline{\underline{p}}_j := \rho_j \overline{\boldsymbol{C}_j \boldsymbol{C}_j} \quad \text{and} \quad \boldsymbol{q}_j := \frac{\rho_j}{2} \overline{C_j^2 \boldsymbol{C}_j} \tag{15}$$

are the partial pressure tensor and the partial energy flux vector, respectively.

The introduction of a mean temperature T in (7) does not require the equality of the temperatures

$$T_j := \frac{m_j}{3k} \overline{|\boldsymbol{c}_j - \overline{\boldsymbol{c}_j}|^2} \tag{16}$$

of the different species j. With the definition (7) we can write

$$T = \sum_j \frac{n_j}{n}\left(T_j + \frac{m_j}{3k}|\overline{C}_j|^2\right) \tag{17}$$

and obtain with $\overline{|C_j|^2} = \overline{|c_j - \overline{c}_j|^2} + |\overline{C}_j|^2$ the relation

$$\sum_j \mathrm{tr}\ \rho_j \overline{He^{(2)}(j)} = \sum_j \{\rho_j |\overline{C}_j|^2 + 3kn_j(T_j - T)\} = 0\,. \tag{18}$$

This is analogous to $\sum_j \rho_j \overline{He^{(1)}(j)} = \sum_j \rho_j(\overline{c}_j - \boldsymbol{v}) = 0$ which follows from the definition (2) of the mean-mass velocity $\boldsymbol{v}$ and therefore justifies the introduction of the mean temperature T in (7) for the width of the local Maxwellian n_j exp $(-m_j C_j^2/2kT)$ (10) in parallel to its centre at the mean-mass velocity $\boldsymbol{v}$.

The balance equations (equations of change) for the traces of $\rho_j \overline{He^{(2)}(j)}$ are evolution equations for the differences $T_j - T$, whereas their sum over all species j is the evolution equation for the mean temperature T.

3. Representation of the Collision Integral by Hermitian Moments

Since the left-hand (dynamical) side of the general balance equation (5) with (8) has been calculated for the generalised tensorial Hermite polynomials $He^{(n)}(j)$ (10) (Suchy 1995, equation 4.2) we will do the same for the collisional integral (6), viz.

$$n_j \overline{\left(\frac{\delta He^{(n)}(j)}{\delta t}\right)_k^{\mathrm{coll}}} = \int d^3C_j \int d^3C_k f_j f_k c_{jk} \oint d^2\hat{\mathbf{c}}'_{jk}\ \sigma_{jk}\left\{He^{(n)}(j') - He^{(n)}(j)\right\}\,. \tag{19}$$

The first step is the expansion of the velocity distribution functions f_j and f_k in an infinite series of orthogonal (generalised tensorial) Hermite polynomials (10) as (Grad 1949a, equation 4.8)

$$f_j = n_j w(j) \sum_{p=0}^{\infty} \frac{1}{p!} \frac{\overline{He^{(p)}(j)}}{\tau_j^{p/2}} \underset{p}{\cdot} \frac{He^{(p)}(j)}{\tau_j^{p/2}}\,, \tag{20}$$

$$f_k = n_k w(k) \sum_{q=0}^{\infty} \frac{1}{q!} \frac{\overline{He^{(q)}(k)}}{\tau_k^{q/2}} \underset{q}{\cdot} \frac{He^{(q)}(k)}{\tau_k^{q/2}}\,. \tag{21}$$

Here $\underset{p}{\cdot}$ means the p fold contraction (scalar product) with the innermost indices contracted first (McCourt *et al.* 1991, Section 20.4.1), and

$$w(j) := w(\tau_j; \boldsymbol{C}_j) := \frac{\exp(-C_j^2/2\tau_j)}{(2\pi\tau_j)^{\frac{3}{2}}}\,, \quad \text{analogous } w(k)\,, \tag{22}$$

is the local Maxwellian as weight function in the orthogonality relation (Erdéyi *et al.* 1953, equation 12.9.1)

$$\int d^3C_j w(j) He^{(p)}(j) He^{(q)}(j) \underset{q}{\cdot} T^{(q)} = \delta_{pq} \tau_j^p p! \widehat{T^{(q)}} \tag{23}$$

with $T^{(q)}$ as an arbitrary tensor of order q. This yields for the collision integral (19) the infinite series of Hermitian moments (Weinert and Suchy 1977, equation 2.10)

$$n_j \overline{\left(\frac{\delta He^{(n)}(j)}{\delta t}\right)_k^{\text{coll}}} = \sum_p \sum_q B^{(n+p+q)}(j,k) \underset{p+q}{\cdot} \frac{\overline{He^{(p)}(j)}}{p!\tau_j^{p/2}} \frac{\overline{He^{(q)}(k)}}{q!\tau_k^{q/2}} \tag{24}$$

with the Maxwell–Boltzmann tensor of order $n+p+q$

$$B^{(n+p+q)}(j,k) := n_j n_k \int d^3C_j w(j) \int d^3C_k w(k) c_{jk}$$

$$\times \oint d^2\hat{c}'_{jk} \sigma_{jk} \left\{He^{(n)}(j') - He^{(n)}(j)\right\} \frac{He^{(q)}(k)}{\tau_k^{q/2}} \frac{He^{(p)}(j)}{\tau_j^{p/2}} . \tag{25}$$

Since the Hermitian moments in (24) are already macroscopic quantities there remains the calculation of the Maxwell–Boltzmann tensor (25).

For the treatment of binary collisions as a two-body problem the introduction of the relative velocity $\boldsymbol{c}_{jk} := \boldsymbol{c}_j - \boldsymbol{c}_k$ and the centre-of-mass velocity relative to the mean-mass velocity $\boldsymbol{v}$ in (2)

$$\boldsymbol{C}_{jk} := \frac{m_j \boldsymbol{C}_j + m_k \boldsymbol{C}_k}{m_j + m_k} = \frac{m_j \boldsymbol{c}_j + m_k \boldsymbol{c}_k}{m_j + m_k} - \boldsymbol{v} \tag{26}$$

is advantageous. With the reduced mass $\mu_{jk} := m_j m_k/(m_j + m_k)$ there holds

$$\boldsymbol{C}_j = \boldsymbol{C}_{jk} + \frac{\mu_{jk}}{m_j}\boldsymbol{c}_{jk} , \qquad \boldsymbol{C}_k = \boldsymbol{C}_{jk} - \frac{\mu_{jk}}{m_k}\boldsymbol{c}_{jk} . \tag{27}$$

This leads to the Jacobian

$$\frac{\partial(\boldsymbol{C}_j, \boldsymbol{C}_k)}{\partial(\boldsymbol{C}_{jk}, \boldsymbol{c}_{jk})} = -1 \tag{28}$$

and to (Kumar 1966, equations 84 and 87)

$$\frac{C_j^2}{2\tau_j}+\frac{C_k^2}{2\tau_k}=\frac{C_{jk}^2}{2T_{jk}}+\frac{c_{jk}^2}{2\tau_{jk}} \quad \text{with} \quad T_{jk}:=\frac{kT}{m_j+m_k} \quad \text{and} \quad \tau_{jk}:=\frac{kT}{\mu_{jk}} \tag{29}$$

$$\tau_j\tau_k=T_{jk}\tau_{jk} \qquad w(j)w(k)=w(C_{jk})w(c_{jk}), \tag{30}$$

where

$$w(C_{jk}):=\frac{\exp(-C_{jk}^2/2T_{jk})}{(2\pi T_{jk})^{\frac{3}{2}}} \qquad w(c_{jk}):=\frac{\exp(-c_{jk}^2/2\tau_{jk})}{(2\pi\tau_{jk})^{\frac{3}{2}}}. \tag{31}$$

The first integrations for the Maxwell–Boltzmann tensor (25) are now transformed as

$$\int d^3C_j\int d^3C_k w(j)w(k)=\int d^3C_{jk}\int d^3c_{jk}w(C_{jk})w(c_{jk}). \tag{32}$$

4. Decomposition of the Maxwell–Boltzmann Tensor

To replace the (generalised) Hermite polynomials $He^{(p)}(j):=He^{(p)}(\tau_j;\boldsymbol{C}_j)$ and $He^{(q)}(k):=He^{(q)}(\tau_k;\boldsymbol{C}_k)$(10) by Hermite polynomials in the new variables $T_{jk},\boldsymbol{C}_{jk}$ and $\tau_{jk},\boldsymbol{c}_{jk}$ (26) and (29) we write (Kumar 1966, equation 83)

$$\frac{\boldsymbol{C}_j}{\tau_j^{\frac{1}{2}}}=r_j\frac{\boldsymbol{C}_{jk}}{T_{jk}^{\frac{1}{2}}}+r_k\frac{\boldsymbol{c}_{jk}}{\tau_{jk}^{\frac{1}{2}}}, \qquad \frac{\boldsymbol{C}_k}{\tau_k^{\frac{1}{2}}}=r_k\frac{\boldsymbol{C}_{jk}}{T_{jk}^{\frac{1}{2}}}-r_j\frac{\boldsymbol{c}_{jk}}{\tau_{jk}^{\frac{1}{2}}} \tag{33}$$

with the mass ratios

$$r_j^2:=\frac{m_j}{m_j+m_k}, \quad r_k^2:=\frac{m_k}{m_j+m_k}, \quad r_j^2+r_k^2=1. \tag{34}$$

With (10) this yields

$$\frac{He^{(p)}(j)}{\tau_j^{p/2}}=He^{(p)}\left(\boldsymbol{C}_j/\tau_j^{\frac{1}{2}}\right)=He^{(p)}\left(r_j\frac{\boldsymbol{C}_{jk}}{T_{jk}^{\frac{1}{2}}}+r_k\frac{\boldsymbol{c}_{jk}}{\tau_{jk}^{\frac{1}{2}}}\right),$$

$$\frac{He^{(q)}(k)}{\tau_k^{q/2}}=He^{(q)}\left(\boldsymbol{C}_k/\tau_k^{\frac{1}{2}}\right)=He^{(q)}\left(r_k\frac{\boldsymbol{C}_{jk}}{T_{jk}^{\frac{1}{2}}}-r_j\frac{\boldsymbol{c}_{jk}}{\tau_{jk}^{\frac{1}{2}}}\right). \tag{35}$$

The addition theorem (Erdélyi *et al.* 1953, equation 10.13.40)

$$He^{(n)}(\alpha\boldsymbol{x}+\beta\boldsymbol{y})=\sum_{m=0}^{n}\binom{n}{m}\alpha^m\beta^{n-m}\,\overbrace{He^{(m)}(\boldsymbol{x})He^{(n-m)}(\boldsymbol{y})} \tag{36}$$

$$\text{for } \alpha^2+\beta^2=1$$

together with (10) leads to the desired result

$$\frac{He^{(p)}(j)}{\tau_j^{p/2}}=\sum_{t=0}^{p}\binom{p}{t}\left(\frac{r_j}{T_{jk}^{\frac{1}{2}}}\right)^t\left(\frac{r_k}{\tau_{jk}^{\frac{1}{2}}}\right)^{p-t}\overbrace{He^{(t)}(\boldsymbol{C}_{jk})He^{(p-t)}(\boldsymbol{c}_{jk})},$$

$$\frac{He^{(q)}(k)}{\tau_k^{q/2}}=\sum_{u=0}^{q}\binom{q}{u}\left(\frac{r_k}{T_{jk}^{\frac{1}{2}}}\right)^u\left(\frac{-r_j}{\tau_{jk}^{\frac{1}{2}}}\right)^{q-u}\overbrace{He^{(u)}(\boldsymbol{C}_{jk})He^{(q-u)}(\boldsymbol{c}_{jk})}, \quad (37)$$

where we have written for short

$$He^{(n)}(\boldsymbol{C}_{jk}):=He^{(n)}(T_{jk};\boldsymbol{C}_{jk}) \qquad He^{(n)}(\boldsymbol{c}_{jk}):=He^{(n)}(\tau_{jk};\boldsymbol{c}_{jk}). \quad (38)$$

The insertion of the expressions (37) and an anologous one for $He^{(n)}(j')-He^{(n)}(j)$ into the Maxwell–Boltzmann tensor (25) would lead to products of three Hermite polynomials of the same variable. To avoid this the following ansatz is made (Kumar 1966, equation 88; Weinert and Suchy 1977, equation 2.14)

$$\frac{He^{(q)}(k)}{\tau_k^{q/2}}\ \frac{He^{(p)}(j)}{\tau_j^{p/2}}=\sum_M\sum_\mu\frac{He^{(M)}(\boldsymbol{C}_{jk})}{T_{jk}^{M/2}}\ \frac{He^{(\mu)}(\boldsymbol{c}_{jk})}{\tau_{jk}^{\mu/2}}\ \underset{\mu+M}{\cdot}\ T^{(\mu+M+q+p)}, \quad (39)$$

$$\frac{He^{(n)}(j)}{\tau_j^{n/2}}=\sum_N\sum_\nu T^{(n+0+N+\nu)}\ \underset{\nu+N}{\cdot}\ \frac{He^{(\nu)}(\boldsymbol{c}_{jk})}{\tau_{jk}^{\nu/2}}\ \frac{He^{(N)}(\boldsymbol{C}_{jk})}{T_{jk}^{N/2}}, \quad (40)$$

with the dimensionless transformation tensors (Kumar 1966, equation 90; Weinert und Suchy 1977, equations 2.18 and 2.19)

$$T^{(\mu+M+q+p)}:=\int d^3c_{jk}w(c_{jk})\int d^3C_{jk}w(C_{jk}) \times\frac{He^{(\mu)}(\boldsymbol{c}_{jk})}{\mu!\tau_{jk}^{\mu/2}}\ \frac{He^{(M)}(\boldsymbol{C}_{jk})}{M!T_{jk}^{M/2}}\ \frac{He^{(q)}(k)}{\tau_k^{q/2}}\ \frac{He^{(p)}(j)}{\tau_j^{p/2}}, \quad (41)$$

$$T^{(n+0+N+\nu)}:=\int d^3c_{jk}w(c_{jk})\int d^3C_{jk}w(C_{jk})\frac{He^{(n)}(j)}{\tau_j^{n/2}}\frac{He^{(N)}(\boldsymbol{C}_{jk})}{N!T_{jk}^{N/2}}\ \frac{He^{(\nu)}(\boldsymbol{c}_{jk})}{\nu!\tau_{jk}^{\nu/2}}, \quad (42)$$

where the orthogonality relation (23) has been used. The range of the summations for non-negative integer M,μ,N,ν will be restricted later (61) to $M+\mu=p+q$ and $N+\nu=n$.

The addition theorem (37) will be used later in (46).

Insertion of the ansatz (39) and (40) into the Maxwell–Boltzmann tensor (25) leads to (Weinert and Suchy 1977, equation 2.20)

$$B^{(n+p+q)}(j,k) = \tau_j^{n/2} \sum_N \sum_\nu T^{(n+0+N+\nu)} \underset{\nu+N}{\cdot} \sum_M \sum_\mu V^{(\nu+N+M+\mu)} \underset{M+\mu}{\cdot} T^{(\mu+M+q+p)} \tag{43}$$

with the collision tensor

$$V^{(\nu+N+M+\mu)} := n_j n_k \int d^3 c_{jk} w(c_{jk}) c_{jk} \int d^3 C_{jk} w(C_{jk}) \tag{44}$$
$$\times \oint d^2 \hat{c}'_{jk}\, \sigma_{jk} \frac{He^{(\nu)}(\mathbf{c}'_{jk}) He^{(N)}(\mathbf{C}'_{jk}) - He^{(\nu)}(\mathbf{c}_{jk}) He^{(N)}(\mathbf{C}_{jk})}{\nu! \tau_{jk}^{\nu/2} N! T_{jk}^{N/2}}$$
$$\times \frac{He^{(M)}(\mathbf{C}_{jk})}{M! T_{jk}^{M/2}} \frac{He^{(\mu)}(\mathbf{c}_{jk})}{\mu! \tau_{jk}^{\mu/2}} .$$

The transformation tensors (41) and (42) merely depend on the properties of the Hermite polynomials, whereas the collision tensor (44) depends in addition on the properties of the particle interactions.

5. Calculation of the Transformation Tensor

To evaluate the transformation tensor $T^{(\mu+M+q+p)}$ in (41) in the decomposition (43) of the Maxwell–Boltzmann tensor $B^{(n+p+q)}$ we observe that the latter is always $p+q$ fold contracted with the symmetric Hermitian moments $\overline{He^{(p)}(j)}\ \overline{He^{(q)}(k)}$ in (24). Therefore we calculate the contracted transformation tensor

$$T^{(\mu+M+q+p)} \underset{p+q}{\cdot} S^{(p)} S^{(q)}$$
$$= \int d^3 c_{jk} w(c_{jk}) \int d^3 C_{jk} w(C_{jk}) \frac{He^{(\mu)}(\mathbf{c}_{jk})}{\mu! \tau_{jk}^{\mu/2}} \frac{He^{(M)}(\mathbf{C}_{jk})}{M! T_{jk}^{M/2}}$$
$$\times \frac{He^{(q)}(k) \underset{q}{\cdot} S^{(q)}}{\tau_k^{q/2}} \frac{He^{(p)}(j) \underset{p}{\cdot} S^{(p)}}{\tau_j^{p/2}} , \tag{45}$$

where $S^{(p)}$ and $S^{(q)}$ are arbitrary symmetric tensors of order p and q, respectively. From the addition theorem (37) we obtain

$$\frac{He^{(p)}(j)}{\tau_j^{p/2}} \underset{p}{\cdot} S^{(p)} = \sum_{t=0}^{p} \binom{p}{t} \left(\frac{r_j}{T_{jk}^{\frac{1}{2}}}\right)^t \left(\frac{r_k}{\tau_{jk}^{\frac{1}{2}}}\right)^{p-t} He^{(t)}(\mathbf{C}_{jk}) \underset{t}{\cdot} S^{(p)} \underset{p-t}{\cdot} He^{(p-t)}(\mathbf{c}_{jk}) ,$$

$$\frac{He^{(q)}(k)}{\tau_k^{q/2}} \underset{q}{\cdot} S^{(q)} = \sum_{u=0}^{q} \binom{q}{u} \left(\frac{r_k}{T_{jk}^{\frac{1}{2}}}\right)^u \left(\frac{-r_j}{\tau_{jk}^{\frac{1}{2}}}\right)^{q-u} He^{(u)}(\mathbf{C}_{jk}) \underset{u}{\cdot} S^{(q)} \underset{q-u}{\cdot} He^{(q-u)}(\mathbf{c}_{jk}) , \tag{46}$$

where the contracted products of Hermite polynomials are scalar quantities and therefore commute with tensors. This property will be used in the next step, when (46) is inserted into (45):

$$
\begin{aligned}
T^{(\mu+M+q+p)} \underset{p+q}{\cdot} S^{(p)} S^{(q)} &= \sum_{t=0}^{p} \binom{p}{t} r_j^t\, r_k^{p-t} \sum_{u=0}^{q} \binom{q}{u} r_k^u (-r_j)^{q-u} \left(\frac{1}{\tau_{jk}^{\frac{1}{2}}}\right)^{p-t} \left(\frac{1}{\tau_{jk}^{\frac{1}{2}}}\right)^{q-u} \\
&\times \int d^3 c_{jk} w(c_{jk}) U^{(M)} \frac{He^{(\mu)}(\boldsymbol{c}_{jk})}{\mu!\tau_{jk}^{\mu/2}} \qquad (47)
\end{aligned}
$$

with the tensor

$$
\begin{aligned}
U^{(M)} =: & \int d^3 C_{jk} w(C_{jk}) \\
& \times He^{(p-t)}(\boldsymbol{c}_{jk}) \underset{p-t}{\cdot} S^{(p)} \underset{t}{\cdot} \frac{He^{(t)}(\boldsymbol{C}_{jk})}{T_{jk}^{t/2}} \frac{He^{(M)}(\boldsymbol{C}_{jk})}{M!T_{jk}^{M/2}} \\
& \times \frac{He^{(u)}(\boldsymbol{C}_{jk})}{T_{jk}^{u/2}} \underset{u}{\cdot} S^{(q)} \underset{q-u}{\cdot} He^{(q-u)}(\boldsymbol{c}_{jk}) \,. \qquad (48)
\end{aligned}
$$

For the integration over $\boldsymbol{C}_{jk}$ we use the generalised orthonormality relation (Erdélyi *et al.* 1954, equation 16.5.15)

$$
T^{(l)} \underset{l}{\cdot} \int d^3 y \frac{\exp(-y^2/2\alpha)}{(2\pi\alpha)^{\frac{3}{2}}} \frac{He^{(l)}(\alpha;\boldsymbol{y})}{\alpha^{l/2}} \frac{He^{(m)}(\alpha;\boldsymbol{y})}{\alpha^{m/2}} \frac{He^{(n)}(\alpha;\boldsymbol{y})}{\alpha^{n/2}} \underset{n}{\cdot} T^{(n)}
$$

$$
= \begin{cases} \widehat{T^{(l)}} \underset{\frac{l-m+n}{2}}{\cdot} \widehat{T^{(n)}} \dfrac{l!m!n!}{[l,m,n]} & \text{for} \quad l-m+n \quad \text{even} \\ 0^{(m)} \quad \text{for} \quad l-m+n \quad \text{odd} \end{cases} \qquad (49)
$$

with

$$
\frac{1}{[l,m,n]} := \frac{1}{\frac{l-m+n}{2}!\frac{m-n+l}{2}!\frac{n-l+m}{2}!}\,, \qquad (50)
$$

where $T^{(l)}$ and $T^{(n)}$ are arbitrary tensors of order l and n, respectively, and $[l,m,n]$ is invariant with respect to any interchange of two or three of the numbers l,m,n. The quoted formula gives only the factor $l!m!n!/[l,m,n]$ in (49). The other factor with the $(l-m+n)/2$ fold contraction of the two symmetric tensors $\widehat{T^{(l)}}$ and $\widehat{T^{(n)}}$ can be explained as follows: An l-fold contraction of an arbitrary tensor $T^{(l)}$ with a symmetric tensor $He^{(l)}$ cancels all nonsymmetric parts of $T^{(l)}$, leaving only contributions of the symmetric part $\widehat{T^{(l)}}$ (Suchy 1964, equation

A2.12). The only possibility to construct an mth-order tensor with $\widehat{T^{(l)}}$ and $\widehat{T^{(n)}}$ is their $(l-m+n)/2$ fold contraction. For $l-m+n$ odd, also $l+m+n$ is odd. Since the Hermite polynomials $He^{(n)}(\boldsymbol{x})$ in (11) contain even/odd tensorial powers of $\boldsymbol{x}$ for n even/odd, the integral in (49) vanishes for $l+m+n$ odd and therefore for $l-m+n$ odd.

Calculating the factorials in the definition (50) of $[l,m,n]$ for $l+m+n$ even, one obtains

$$\begin{aligned}\frac{1}{[l,m,n]} &= \frac{\delta_{l+m,n}}{l!m!} \quad \text{for} \quad l \le m \le n \\ &= \frac{\delta_{m+n,l}}{m!n!} \quad \text{for} \quad m \le n \le l \\ &= \frac{\delta_{n+l,m}}{n!l!} \quad \text{for} \quad n \le l \le m\,.\end{aligned} \tag{51}$$

With $\boldsymbol{y} = \boldsymbol{C}_{jk}$ and $\alpha = T_{jk}$ in the expression (49) we obtain for (48)

$$U^{(M)} = \frac{t!u!}{[t,M,u]} He^{(p-t)}(\boldsymbol{c}_{jk}) \underset{p-t}{\cdot} S^{(p)} \underset{\frac{t-M+u}{2}}{\cdot} S^{(q)} \underset{q-u}{\cdot} He^{(q-u)}(\boldsymbol{c}_{jk}) \tag{52}$$

with the

$$\text{first selection rule:} \quad t-M+u \quad \text{even.} \tag{53}$$

For the next integration in (47), viz.

$$\begin{aligned}&\frac{1}{\tau_{jk}^{(p-t)/2}} \frac{1}{\tau_{jk}^{(q-u)/2}} \int d^3c_{jk} w(c_{jk}) U^{(M)} \frac{He^{(\mu)}(\boldsymbol{c}_{jk})}{\mu! \tau_{jk}^{\mu/2}} \\ &= \frac{t!u!}{[t,M,u]} \int d^3c_{jk} w(c_{jk}) \frac{He^{(p-t)}(\boldsymbol{c}_{jk})}{\tau_{jk}^{(p-t)/2}} \underset{p-t}{\cdot} S^{(p)} \underset{\frac{t-M+\mu}{2}}{\cdot} \\ &\quad \times S^{(q)} \underset{q-u}{\cdot} \frac{He^{(q-u)}(\boldsymbol{c}_{jk})}{\tau_{jk}^{(q-u)/2}} \frac{He^{(\mu)}(\boldsymbol{c}_{jk})}{\mu! \tau_{jk}^{\mu/2}} \\ &= \frac{t!u!}{[t,M,u]} \frac{(p-t)!(q-u)!}{[p-t,q-u,\mu]} S^{(p)} \underset{\frac{p+q-M-\mu}{2}}{\cdot} S^{(q)}\end{aligned} \tag{54}$$

with the

$$\text{second selection rule:} \quad p-t-\mu+q-u \text{ even,} \tag{55}$$

we have used again the generalised orthonormality relation (49) with (50) with $l = p - t, n = q - u, m = \mu$, which resulted in the contraction number

$$\frac{t - M + u}{2} + \frac{p - t - \mu + q - u}{2} = \frac{p + q - M - \mu}{2} \tag{56}$$

of the two symmetric tensors $S^{(p)}$ and $S^{(q)}$.

The two selection rules (53) and (55) together give the

$$\text{combined selection rule:} \quad p + q - M - \mu \quad \text{even} \tag{57}$$

with the consequence

$$p + q \geq M + \mu\,. \tag{58}$$

Until now only a post-contracted transformation tensor $T^{(\mu+M+q+p)}$ in (45) has been considered. But in the decomposition (43) of the Maxwell-Boltzmanan tensor $B^{(n+p+q)}$ the transformation tensor $T^{(\mu+M+q+p)}$ appears $M + \mu$ fold pre-contracted. Modifying the previous results for this case with arbitrary symmetric tensors $S^{(\mu)}$ and $S^{(M)}$ we are led to

$$S^{(M)} S^{(\mu)} \underset{M+\mu}{\cdot} T^{(\mu+M+q+p)} \begin{cases} \sim S^{(M)} \underset{\frac{M+\mu-p-q}{2}}{\cdot} S^{(\mu)} & \text{for} \quad M + \mu - p - q \quad \text{even} \\ = 0^{(q+p)} \quad \text{for} \quad M + \mu - p - q \quad \text{odd} \end{cases} \tag{59}$$

with the consequence

$$M + \mu \geq p + q\,. \tag{60}$$

The combination of (58) and (60) gives the

$$\text{selection rule:} \quad p + q = M + \mu\,. \tag{61}$$

Application of the selection rule (61) to (47) with (54) and (59) results in

$$\begin{aligned} T^{(\mu+M+q+p)} \underset{p+q}{\cdot} S^{(p)} S^{(q)} &\sim S^{(p)} \underset{0}{\cdot} S^{(q)} = S^{(p)} S^{(q)}\,, \\ S^{(M)} S^{(\mu)} \underset{M+\mu}{\cdot} T^{(\mu+M+q+p)} &\sim S^{(M)} \underset{0}{\cdot} S^{(\mu)} = S^{(M)} S^{(\mu)}\,. \end{aligned} \tag{62}$$

Therefore the transformation tensor $T^{(\mu+M+q+p)}$ can be taken as proportional to the symmetrising tensor $I^{(\mu+M|q+p)}$ which selects and reproduces products of symmetric tensors $S^{(M)} S^{(\mu)}$ and $S^{(p)} S^{(q)}$ if $M + \mu$ times precontracted or $p + q = M + \mu$ times postcontracted, respectively:

$$T^{(\mu+M+q+p)} \sim \delta_{\mu+M,q+p}\, I^{(\mu+M|q+p)}\,. \tag{63}$$

Examples are, cf. (12) and (13),

$$I^{(1|1)} = \boldsymbol{g}_i \boldsymbol{g}^i , \quad I^{(2|2)} = \boldsymbol{g}_i \boldsymbol{g}_j \overparen{\boldsymbol{g}^j \boldsymbol{g}^i} = \overparen{\boldsymbol{g}_i \boldsymbol{g}_j}\, \boldsymbol{g}^j \boldsymbol{g}^i = \overparen{\boldsymbol{g}_i \boldsymbol{g}_j}\, \overparen{\boldsymbol{g}^j \boldsymbol{g}^i} ,$$

$$I^{(3|3)} = \boldsymbol{g}_i \boldsymbol{g}_j \boldsymbol{g}_k \overparen{\boldsymbol{g}^k \boldsymbol{g}^j \boldsymbol{g}^i} = \overparen{\boldsymbol{g}_i \boldsymbol{g}_j \boldsymbol{g}_k}\, \boldsymbol{g}^k \boldsymbol{g}^j \boldsymbol{g}^i = \overparen{\boldsymbol{g}_i \boldsymbol{g}_j \boldsymbol{g}_k}\, \overparen{\boldsymbol{g}^k \boldsymbol{g}^j \boldsymbol{g}^i} ,$$

where Einstein's summation convention is used.

Combining the results (47), (52), (54), (61) and (63) we can write the transformation tensor as

$$T^{(\mu+M+q+p)} = \delta_{\mu+M,q+p} T_{jk}(M,\mu;p,q) I^{(\mu+M|q+p)} \tag{64}$$

with the scalar transformation factor

$$\begin{aligned} T_{jk}(M,\mu;p,q) &:= p!q! \sum_{t=0}^{p} \sum_{u=0}^{q} \frac{(-1)^{q-u}\, r_j^{q+t-u}\, r_k^{p-t+u}}{[t,u,M][p-t,q-u,\mu]} \\ &= (-1)^q T_{kj}(\mu,M;p,q) . \end{aligned} \tag{65}$$

For Burnett functions [i.e. products of scalar spherical harmonics and Laguerre–Sonine polynomials, cf. (75) and (76)] instead of tensorial Hermite polynomials, expressions containing Talmi coefficients (Kumar 1966, Sections V and VI) are the equivalent to the transformation tensors (64) with (65). The symmetry relations of the Talmi coefficients have their counterparts presumably in the properties of the symmetrising tensor $I^{(\mu+M|q+p)}$.

For the calculation of the double sum in the expression (65) of the transformation factor $T_{jk}(M,\mu;p,q)$ the result (51) for the factorial products $[l,m,n]$ (50) is used together with the selection rule (61). The result is

$$\begin{aligned} T_{jk}(M,\mu;p,q) &= p!q! \sum_{t=0}^{p} \sum_{u=0}^{q} (-1)^{q-u}\, r_j^{q-u+t}\, r_k^{p-t+u} \frac{\delta_{M,t+u}}{t!u!} \frac{\delta_{p+q,\mu+M}}{(p-t)!(q-u)!} \\ &= \delta_{p+q,\mu+M} (-r_j)^q r_k^p \sum_{t=0}^{p} \sum_{u=0}^{q} \delta_{M,t+u} (-1)^u \binom{p}{t} \binom{q}{u} \left(\frac{r_j}{r_k}\right)^{t-u} \\ &= \delta_{M+\mu,p+q} (-r_j)^q r_k^p \left(-\frac{r_k}{r_j}\right)^M \sum_{t=0}^{M} \binom{p}{t} \binom{q}{M-t} \left(-\frac{r_j^2}{r_k^2}\right)^t \\ &= (-1)^q T_{kj}(M,\mu;q,p) \\ &= \delta_{M+\mu,p+q} \,(-r_j)^q\, r_k^p \left(\frac{r_k}{r_j}\right)^M P_M^{(p-M,q-M)}\left(r_j^2 - r_k^2\right) \end{aligned} \tag{66}$$

with the Jacobi polynomial $P_M^{(p-M,q-M)}(r_j^2 - r_k^2)$ (Erdelyi *et al.* 1953, equation 10.8.12), recalling $r_j^2 + r_k^2 = 1$ in (34).

Finally we have to insert the expression (64) for the two transformation tensors $T^{(n+0+N+\nu)}$ and $T^{(\mu+M+q+p)}$ in the decomposition (43) of the Maxwell–

Boltzmann tensor $B^{(n+p+q)}(j,k)$ (25). The symmetrising tensors $I^{(n+0|N+\nu)}$ and $I^{(\mu+M|q+p)}$ reproduce products of symmetric tensors $S^{(\nu)}S^{(N)}$ and $S^{(M)}S^{(\mu)}$ if $\nu+N$ times postcontracted and $M+\mu$ times precontracted, respectively. Since the collision tensor $V^{(\nu+N+M+\mu)}$ in (44) contains symmetric Hermite tensors $He^{(\nu)}, He^{(N)}, He^{(M)}$ and $He^{(\mu)}$ from (10) we can therefore write

$$I^{(n+0|N+\nu)} \underset{\nu+N}{\cdot} V^{(\nu+N+M+\mu)} \underset{M+\mu}{\cdot} I^{(\mu+M|q+p)} = V^{(\nu+N|m+\mu)}\,, \tag{67}$$

where $V^{(\nu+N|M+\mu)}$ equals $V^{(\nu+N+M+\mu)}$ but with the additional property of symmetrising when $\nu+N$ fold precontracted or $M+\mu$ fold postcontracted. But during the calculations of the following Section 6 this property becomes overgrown with superior symmetries. Therefore we continue to write $V^{(\nu+N+M+\mu)}$.

Now the insertion of (64) into the decomposition (43) of the Maxwell–Boltzmann tensor yields

$$B^{(n+p+q)}(j,k) \tag{68}$$

$$= \tau_j^{n/2} \sum_N \sum_\nu \delta_{N+\nu,n} T_{jk}(N,\nu;n,0) \sum_M \sum_\mu \delta_{\mu+M,p+q} T_{jk}(M,\mu;p,q) V^{(\nu+N+M+\mu)}\,.$$

All that is left of the two transformation tensors in (43) are the two scalar transformation factors $T_{jk}(N,\nu;n,0)$ and $T_{jk}(M,\mu;p,q)$ (66) and the Kronecker symbols $\delta_{N+\nu,n}\ \delta_{M+\mu\ ,p+q}$ representing the selection rule (61).

6. Collision Tensor and Collision Rates

We now turn to the evaluation of the collision tensor $V^{(\nu+N+M+\mu)}$ in (44). The conservation of the total momentum during a collision leads with (26) to

$$\boldsymbol{C}'_{jk} = \boldsymbol{C}_{jk}\,. \tag{69}$$

For the integration over $\boldsymbol{C}_{jk}$ we make use of the orthogonality relation (23) of the Hermite polynomials and obtain

$$V^{(\nu+N+M+\mu)} \underset{M+\mu}{\cdot} S^{(p)}S^{(q)} = n_j n_k \int d^3 c_{jk} w(c_{jk}) c_{jk}$$

$$\times \oint d^2\hat{\boldsymbol{c}}'_{jk} \sigma_{jk} \frac{\delta_{MN}}{M!} \frac{He^{(\nu)}(\boldsymbol{c}'_{jk}) - He^{(\nu)}(\boldsymbol{c}_{jk})}{\nu!\tau_{jk}^{\nu/2}} \overbrace{\frac{He^{(\mu)}(\boldsymbol{c}_{jk})}{\mu!\tau_{jk}^{\mu/2}}} \underset{\mu}{\cdot} S^{(p)}S^{(q)}\,. \tag{70}$$

For the integration over the solid angle $d^2\hat{\boldsymbol{c}}'_{jk} = d\epsilon d(\cos\chi)$ we expand the differential cross section $\sigma_{jk}(c_{jk},\chi)$ for point-like particles as (Kumar 1966, equation 96)

$$\sigma_{jk}(c_{jk},\chi) = \sum_{l=0}^{\infty} \sigma_l(c_{jk}) P_l(\cos\chi) \quad \text{with} \quad \cos\chi = \hat{\boldsymbol{c}}'_{jk} \cdot \hat{\boldsymbol{c}}_{jk}\,. \tag{71}$$

The orthogonality of the Legendre polynomials $P_l(\cos\chi)$ yields for the coefficient (Weinert and Suchy 1977, equation 2.23)

$$\sigma_l(c_{jk}) = \frac{2l+1}{2}\int_{-1}^{+1} d(\cos\chi)\sigma_{jk}\, P_l(\cos\chi)\,. \tag{72}$$

With the connection (Ikenberry 1961, equation 5)

$$P_l(\hat{\boldsymbol{c}}'_{jk}\cdot\hat{\boldsymbol{c}}_{jk}) = \frac{(2l-1)!!}{l!}Y^{(l)}(\hat{c}'_{jk})\underset{l}{\cdot} Y^{(l)}(\boldsymbol{c}_{jk}) \tag{73}$$

between the Legendre polynomials P_l and Ikenberry's (1961, equation 1) symmetric and tracefree tensorial spherical harmonics $Y^{(l)}$ we can write the expansion (71) as

$$\sigma_{jk}(c_{jk},\hat{\boldsymbol{c}}'_{jk}\cdot\hat{\boldsymbol{c}}_{jk}) = \sum_{l=0}^{\infty}\frac{(2l-1)!!}{l!}\sigma_l(c_{jk})Y^{(l)}(\hat{\boldsymbol{c}}'_{jk})\underset{l}{\cdot} Y^{(l)}(\hat{\boldsymbol{c}}_{jk}) \tag{74}$$

with $(2l-1)!! := (2l-1)(2l-3)...1$ as the double factorial of an odd number.

Next we need the representation of the Hermite polynomials in terms of spherical harmonics $Y^{(m)}$ and Laguerre–Sonine polynomials $L^{m+1/2}_{(n-m)/2}$ (Ikenberry 1962, equations 2, 5, 11 and 31)

$$He^{(n)}(\alpha;\boldsymbol{x}) = n!\alpha^{n/2}\sum_{m=0,1}^{n}{}_2\, L(n,m;x^2/2\alpha)\,\overbrace{\underline{\underline{I}}^{(n-m)/2}Y^{(m)}(\hat{\boldsymbol{x}})} \tag{75}$$

with the polynomials

$$L(n,m;x^2/2\alpha) := (-1)^{(n-m)/2}\frac{(2m+1)!!2^{m/2}}{m!(n+m+1)!!}\left(\frac{x^2}{2\alpha}\right)^{(m/2)} L^{m+\frac{1}{2}}_{\frac{n-m}{2}}\left(\frac{x^2}{2\alpha}\right) \tag{76}$$

and $\underline{\underline{I}}^{(n-m)/2}$ as the $(n-m)/2$ fold tensorial power of $= I^{(2)}$, the unit tensor of second order. The sum over m means $m = 0, 2, ...n$ for n even and $m = 1, 3...n$ for n odd. Hence $n - m$ is always even.

With complex basis-vectors composed of cartesian ones Ikenberry's tensors $Y^{(l)}(\hat{\boldsymbol{x}})$ can be represented with conventional spherical harmonics $Y_{lm}(\hat{\boldsymbol{x}})$ (McCourt *et al.* 1991, equation 20.6–9a). The products of these with $x^l \exp(-x^2)L^{l+\frac{1}{2}}_{\frac{n-l}{2}}(x^2)$ are often called Burnett functions and used as basis functions for the calculation of collision integrals (Weinert 1978, equations 3.3 and 3.4).

With the expansion (74) for the differential cross section σ_{jk} and the use of (75) with (76) for $He^{(\nu)}$ and $He^{(\mu)}$ in (70) we write the collision tensor (44) in the form

$$V^{(\nu+N+M+\mu)} = n_j n_k\frac{\delta_{MN}}{M!}\sum_{l=0}^{\infty}\frac{(2l-1)!!}{l!}\int_0^{\infty} dc_{jk}c^3_{jk}w\sigma_l$$

$$\times\sum_{u=0,1}^{\nu}{}_2\, L\left(\nu,u;\frac{c^2_{jk}}{2\tau_{jk}}\right)\sum_{t=0,1}^{\mu}{}_2\, L\left(\mu,t;\frac{c^2_{jk}}{2\tau_{jk}}\right)W^{(\nu+2M+\mu)}(l,u,t) \tag{77}$$

with $\nu - u$ even and $\mu - t$ even and with the angular part

$$W^{(\nu+2M+\mu)}(l,u,t) \underset{M+\mu}{\cdot} S^{(p)}S^{(q)} \tag{78}$$

$$:= \oint d^2\hat{\mathbf{c}}_{jk} \oint d^2\hat{\mathbf{c}}'_{jk} Y^{(l)}(\hat{\mathbf{c}}'_{jk}) \underset{l}{\cdot} Y^{(l)}(\hat{\mathbf{c}}_{jk}) \widehat{\underline{\underline{I}}^{(\nu-u)/2} Y^{(u)}(\hat{\mathbf{c}}'_{jk})}\, \widehat{\widehat{\underline{\underline{I}}^{(\mu-t)/2} Y^{(t)}(\hat{\mathbf{c}}_{jk})} \underset{\mu}{\cdot} S^{(p)}S^{(q)}}$$

$$- \oint d^2\hat{\mathbf{c}}_{jk} \oint d^2\hat{\mathbf{c}}'_{jk} Y^{(l)}(\hat{\mathbf{c}}'_{jk}) \underset{l}{\cdot} Y^{(l)}(\hat{\mathbf{c}}_{jk}) \widehat{\underline{\underline{I}}^{(\nu-u)/2} Y^{(u)}(\hat{\mathbf{c}}_{jk})}\, \widehat{\widehat{\underline{\underline{I}}^{(\mu-t)/2} Y^{(t)}(\hat{\mathbf{c}}_{jk})} \underset{\mu}{\cdot} S^{(p)}S^{(q)}} \,.$$

For the integrations over the solid angles $d^2\hat{\mathbf{c}}_{jk}$ and $d^2\hat{\mathbf{c}}'_{jk}$ we use the orthonormality of Ikenberry's spherical harmonics (McCourt *et al.* 1991, equation 20.6–9c)

$$\oint d^2\hat{\boldsymbol{x}} Y^{(l)}(\hat{\boldsymbol{x}}) Y^{(m)}(\hat{\boldsymbol{x}}) = \delta_{lm} \frac{l!4\pi}{(2l+1)!!} D^{(2l)} \,. \tag{79}$$

The 'detracer' (Pirani 1965, equation 2.44; Thorne 1980, equation 2.2)

$$D^{(2l)} = \sum_{m=0}^{[l/2]} (-1)^m \frac{\binom{l}{m}\binom{l}{2m}}{\binom{2l}{2m}} D_m^{(2l)} = \sum_{m=0}^{[l/2]} (-1)^m \frac{l!(2l-2m-1)!!}{(l-2m)!(2m)!!(2l-1)!!} D_m^{(2l)} \tag{80}$$

with

$$\begin{aligned} D_m^{(2l)} \underset{l}{\cdot} S^{(l)} &:= \widehat{\underline{\underline{I}}^m \mathrm{tr}^m S^{(l)}} := \widehat{\underline{\underline{I}}^{2m} \underset{2m}{\cdot} S^{(l)}} = I^{(l|l)} \underset{l}{\cdot} \left(\underline{\underline{I}}^{2m} \underset{2m}{\cdot} S^{(l)} \right) \\ &= \left(I^{(l|l)} \underset{2m}{\cdot} \underline{\underline{I}}^{2m} \right) \underset{l}{\cdot} S^{(l)} \end{aligned} \tag{81}$$

sorts out the tracefree part (deviator) of the completely symmetric tensor $S^{(l)}$. The double factorial of an even number is $(2m)!! = 2^m m!$. The m fold trace of $S^{(l)}$ is defined as the $2m$ fold convolution of $S^{(l)}$ with the mth tensorial power $\underline{\underline{I}}^m$ of the unit tensor $\underline{\underline{I}}$. The symmetrising symbol $\widehat{\ \ }$ was replaced by the contraction with the symmetriser $I^{(l|l)}$ (63). Then the distributive law

$$T^{(l)} \underset{l}{\cdot} \left(U^{(2n)} \underset{n}{\cdot} V^{(l)} \right) = \left(T^{(l)} \underset{n}{\cdot} U^{(2n)} \right) \underset{l}{\cdot} V^{(l)} \quad \text{with} \quad n \leq l \tag{82}$$

for multiple contractions was employed. Examples are

$$
\begin{aligned}
&D^{(0)} = 1, \quad D^{(2)} = I^{(1|1)} = \underline{\underline{I}}, \quad D^{(4)} = I^{(2|2)} - \tfrac{1}{3}\underline{\underline{I}}\,\underline{\underline{I}}, \\
&D^{(6)} = I^{(3|3)} - \tfrac{3}{5} I^{(3|3)} \underset{2}{\cdot} \underline{\underline{I}}^2, \\
&D^{(8)} = I^{(4|4)} - \tfrac{6}{7} I^{(4|4)} \underset{2}{\cdot} \underline{\underline{I}}^2 + \tfrac{3}{35} I^{(4|4)} \underset{4}{\cdot} \underline{\underline{I}}^4, \\
&D^{(10)} = I^{(5|5)} - \tfrac{10}{9} I^{(5|5)} \underset{2}{\cdot} \underline{\underline{I}}^2 + \tfrac{5}{21} I^{(5|5)} \underset{4}{\cdot} \underline{\underline{I}}^4 .
\end{aligned}
\tag{83}
$$

The integration over $d^2\hat{\mathbf{c}}'_{jk}$ in (78) yields with (79), recalling that Ikenberry's spherical harmonics $Y^{(l)}$ are already tracefree,

$$
\begin{aligned}
&W^{(\nu+2M+\mu)}(l,u,t) \underset{M+\mu}{\cdot} S^{(p)}S^{(q)} \\
&= \oint d^2\hat{\mathbf{c}}_{jk}\delta_{lu}\frac{l!4\pi}{(2l+1)!!}\overbrace{\overbrace{\underline{\underline{I}}^{(\nu-u)/2}Y^{(l)}(\hat{\mathbf{c}}_{jk})}\,\overbrace{\underline{\underline{I}}^{(\mu-t)/2}Y^{(t)}(\hat{\mathbf{c}}_{jk})}} \underset{\mu}{\cdot} S^{(p)}S^{(q)} \\
&\quad - \oint d^2\hat{\mathbf{c}}_{jk}\delta_{l0}\frac{l!4\pi}{(2l+1)!!}Y^{(l)}\,\overbrace{\overbrace{\underline{\underline{I}}^{(\nu-u)/2}Y^{(u)}(\hat{\mathbf{c}}_{jk})}\;\overbrace{\underline{\underline{I}}^{(\mu-t)/2}Y^{(t)}(\hat{\mathbf{c}}_{jk})}} \underset{\mu}{\cdot} S^{(p)}S^{(q)}
\end{aligned}
\tag{84}
$$

with $0!Y^{(0)}/1!! = 1$.

For the application of the orthogonality relation (79) on the integration over $d^2\hat{\mathbf{c}}_{jk}$ we observe that the μ-fold post-contraction of the μth order tensor $\overbrace{\underline{\underline{I}}^{(\mu-t)/2}Y^{(t)}}$ has 'scalarised' that tensor. Now we can combine it with the first symmetrised tensor product, as required in the orthogonality relation (79). We obtain, recalling the selection rule $p+q = M+\mu$ in (61),

$$
\begin{aligned}
&W^{(\nu+2M+\mu)}(l,u,t) \underset{M+\mu}{\cdot} S^{(p)}S^{(q)} \\
&\qquad = \delta_{ut}\frac{(4\pi)^2 t!}{(2t+1)!!}\left(\delta_{lt}\frac{l!}{(2l+1)!!} - \delta_{l0}\right)\overbrace{\overbrace{\underline{\underline{I}}^{(\nu+\mu)/2-t}D^{(2t)}}} \underset{\mu}{\cdot} S^{(p)}S^{(q)}
\end{aligned}
\tag{85}
$$

with $\nu+\mu$ even.

With this result for the angular part $W^{(\nu+2M+\mu)}$ in the expression (77) for the collision tensor the infinite sum over the l-dependent factors reduces to the single term $\sigma_t/(2t+1) - \sigma_0$. In the resulting expression we change t to l, define the 'transfer collision frequencies' (Suchy and Rawer 1971, equation 2.2), cf. (72),

$$
\nu_{jk}^{(l)}(c_{jk}) := n_k c_{jk} 4\pi\left(\sigma_0 - \frac{\sigma_l}{2l+1}\right) = n_k c_{jk}\oint d^2\hat{\mathbf{c}}'_{jk}\sigma_{jk}\left\{1 - P_l(\cos\chi)\right\}
\tag{86}
$$

and introduce the collision rates

$$K_{jk}(\nu,\mu,l) := \frac{l!4\pi}{(2l+1)!!}\int_0^\infty dc_{jk}c_{jk}^2 wL\left(\nu,l;\frac{c_{jk}^2}{2\tau_{jk}}\right)L\left(\mu,l;\frac{c_{jk}^2}{2\tau_{jk}}\right)\nu_{jk}^{(l)}(c_{jk})$$
$$= K_{jk}(\mu,\nu,l)\,. \tag{87}$$

Then we can write the collision tensor (77) as

$$V^{(\nu+N+M+\mu)}\underset{M+\mu}{\cdot}S^{(p)}S^{(q)} = -n_j\frac{\delta_{MN}}{M!}\sum_{l=1,2}^{\mu}{}^{2}\,K_{jk}(\nu,\mu,l)\,\overbrace{\overbrace{\underline{\underline{I}}^{(\nu+\mu)/2-l}D^{(2l)}}\underset{\mu}{\cdot}S^{(p)}S^{(q)}} \tag{88}$$

with $\mu - l$ even and $\nu + \mu$ even.

The last task is the evaluation of the collision rates (87).

7. Calculation of the Collision Rates

The collision rates (87) contain products of two polynomials $L(\nu,l;\epsilon_{jk})L(\mu,l;\epsilon_{jk})$ (76), where we have introduced with (29) the normalised kinetic collision energy

$$\epsilon_{jk} := \frac{c_{jk}^2}{2\tau_{jk}} = \frac{\mu_{jk}c_{jk}^2}{2kT}\,. \tag{89}$$

The most important factor in the calculation of $L(\nu,l;\epsilon_{jk})L(\mu,l;\epsilon_{jk})$ is the product of two Laguerre polynomials

$$L_m^\alpha(\epsilon_{jk})L_n^\alpha(\epsilon_{jk}) = \sum_{s=0}^{m+n} a(m,n;\alpha,s)L_s^\alpha(\epsilon_{jk}) \tag{90}$$

with the coefficients (Weinert 1975, equations A3.3, A3.15 and A3.17)

$$a(m,n;\alpha,s)$$
$$:= (-1)^{m+n+s}\frac{\binom{n+\alpha}{n}}{\binom{s+\alpha}{s}}\sum_{k=|m-n|}^{m+n}{}^{2}\binom{m+\alpha}{\frac{m-n+k}{2}}\binom{n}{\frac{m+n-k}{2}}\binom{m+n-k}{s-k}$$
$$= a(n,m;\alpha,s) \quad \text{for non-negative } m \text{ and } n\,. \tag{91}$$

The summation index k runs through all even/odd numbers between $|m-n|$ and $m+n$ if $m+n$ is even/odd. Examples are

$$a(0,n;\alpha,s) = \delta_{ns}\,, \qquad a(1,n;\alpha,s) = \frac{\delta_{n-1,s}}{n+\alpha-1} + \frac{(n+1)(\alpha+1)}{n+\alpha+1}\delta_{n+1,s}\,. \tag{92}$$

Now we can write the product of the two polynomials $L(\nu,l;\epsilon_{jk})L(\mu,l;\epsilon_{jk})$ in (76) in the integrand of the collision rate (87) as

$$L(\nu,l;\epsilon_{jk})L(\mu,l;\epsilon_{jk}) = (-1)^{(\nu+\mu/2)-l}\left(\frac{(2l+1)!!}{l!}\right)^2 \frac{(2\epsilon_{jk})^l}{(\nu+l+1)!!(\mu+l+1)!!}$$
$$\times \sum_{s=0}^{(\mu+\nu/2)-l} a\left(\frac{\nu-l}{2},\frac{\mu-l}{2};l+\tfrac{1}{2},s\right) L_s^{l+\frac{1}{2}}(\epsilon_{jk}) \qquad (93)$$

with $\mu-l,\nu-l$ even and non-negative.

With the definitions (89) for ϵ_{jk} and (31) for $w(c_{jk})$ we transform the integration in (87) as

$$\int_0^\infty dc_{jk}c_{jk}^2 w(c_{jk}) = \frac{2^{\frac{1}{2}}}{(2\pi)^{\frac{3}{2}}}\int_0^\infty d\epsilon_{jk}\epsilon_{jk}^{\frac{1}{2}}\exp(-\epsilon_{jk}), \qquad (94)$$

introduce 'transport collision frequencies' (Suchy and Rawer 1971, equations 3.5 and 3.6)

$$\nu_{jk}^{(ls)}(\tau_{jk}) := \frac{(-1)^s s!}{(l+s+\frac{1}{2})!}\int_0^\infty d\epsilon_{jk}\epsilon_{jk}^{l+\frac{1}{2}}\exp(-\epsilon_{jk})L_s^{l+\frac{1}{2}}(\epsilon_{jk})\nu_{jk}^{(l)}\left\{(2\tau_{jk}\epsilon_{jk})^{\frac{1}{2}}\right\} \qquad (95)$$

as coefficients in the orthogonal expansion of the transfer collision frequencies (86)

$$\nu_{jk}^{(l)}\left\{(2\tau_{jk}\epsilon_{jk})^{\frac{1}{2}}\right\} = \sum_{s=0}^{\infty}(-1)^s\nu_{jk}^{(ls)}(\tau_{jk})L_s^{l+\frac{1}{2}}(\epsilon_{jk}) \qquad (96)$$

and obtain for the collision rate (87) with (93), (94) and (96) the expression

$$K_{jk}(\nu,\mu,l) = \sum_{s=0}^{(\nu+\mu/2)-l} \alpha(\nu,\mu,l,s)\nu_{jk}^{(ls)}(\tau_{jk}). \qquad (97)$$

The coefficients are calculated as

$$\alpha(\nu,\mu,l,s)$$
$$:= \frac{(2l+1)!!}{(\nu+l+1)!!(\mu-l)!!l!}\sum_{k=|\frac{\nu-\mu}{2}|}^{(\nu+\mu/2)-l}{}^{2}\binom{\frac{\nu+l+1}{2}}{\frac{\nu-\mu}{4}+\frac{k}{2}}\binom{\frac{\mu-l}{2}}{\frac{\nu+\mu}{4}-\frac{l+k}{2}}\binom{\frac{\nu+\mu}{2}-l-k}{s-k}$$
$$= \alpha(\mu,\nu,l,s) \qquad (98)$$

with $\mu-l,\nu-l$ even and non-negative. The third binomial coefficient requires a lower bound $|\nu-\mu|/2$ for s. Together with the upper bound $\{(\mu+\nu)/2\}-l$ in the expression (97) we obtain the restriction

$$\left|\frac{\nu-\mu}{2}\right| \le s \le \frac{\nu+\mu}{2}-l \qquad (99)$$

for the summation over s in (97).

Since the transfer collision frequencies $\nu_{jk}^{(0)}$ in (86) vanish and therefore also the transport collision frequencies $\nu_{jk}^{(0s)}$ in (95) we need only the coefficients $\alpha(\nu,\mu,l,s)$ in (98) for $l \geq 1$. Examples are

$$\alpha(0,\mu,l,s)=0,\quad \alpha(1,\mu,l,s)=\frac{\delta_{l1}\delta_{2s,\mu-1}}{(\mu-1)!!},\quad \alpha(2,\mu,l,s)=\frac{\delta_{l2}\delta_{2s,\mu-2}}{2\cdot(\mu-2)!!}, \tag{100}$$

$$\alpha(3,\mu,l,s)=\frac{\delta_{l1}}{2}\left\{\frac{1}{5\cdot(\mu-3)!!}\binom{2}{s-\frac{\mu-3}{2}}+\frac{\delta_{2s,\mu+1}}{(\mu-1)!!}\right\}+\frac{\delta_{l3}}{3!}\frac{\delta_{2s,\mu-3}}{(\mu-3)!!}, \tag{101}$$

$$\alpha(4,\mu,l,s)=\frac{\delta_{l2}}{2\cdot 2!}\left\{\frac{1}{7\cdot(\mu-4)!!}\binom{2}{s-\frac{\mu-4}{2}}+\frac{\delta_{2s,\mu}}{(\mu-2)!!}\right\}+\frac{\delta_{l4}\delta_{2s,\mu-4}}{4!(\mu-4)!!}, \tag{102}$$

$$\begin{aligned}\alpha(5,\mu,l,s)&\\ =\frac{\delta_{l1}}{2\cdot 2!}&\left\{\frac{1}{2\cdot5\cdot7\cdot(\mu-5)!!}\binom{4}{s-\frac{\mu-5}{2}}+\frac{1}{5\cdot(\mu-3)!!}\binom{2}{s-\frac{\mu-1}{2}}+\frac{\delta_{2s,\mu+3}}{2\cdot(\mu-1)!!}\right\}\\ &+\frac{\delta_{l3}}{2\cdot 3!}\left\{\frac{1}{3\cdot3\cdot(\mu-5)!!}\binom{2}{s-\frac{\mu-5}{2}}+\frac{\delta_{2s,\mu-1}}{(\mu-3)!!}\right\}+\frac{\delta_{l5}\delta_{2s,\mu-5}}{5!(\mu-5)!!}.\end{aligned} \tag{103}$$

Because the transport collision frequencies $\nu_{jk}^{(ls)}$ in (95) are coefficients of an orthogonal expansion (96) they form a null sequence with increasing s. Its convergence is fastest for Maxwell interaction with transfer collision frequencies $\nu_{jk}^{(l)}$ in (86) independent of the kinetic collision energy. Because of the orthogonality property of the Laguerre–Sonine polynomials $L_s^{l+\frac{1}{2}}(\epsilon_{jk})$ the definition (95) of the transport collision frequencies yields

$$\nu_{jk}^{(ls)}=\delta_{s0}\nu_{jk}^{(l0)}\qquad\text{for Maxwell interaction}. \tag{104}$$

For rigid spheres one obtains (Suchy 1984, equations B.7a and B.7c)

$$\frac{\nu_{jk}^{(ls)}}{\nu_{jk}^{(l0)}}=\frac{(l+\frac{1}{2})!}{(l+\frac{1}{2}+s)!}\,\frac{\frac{1}{2}!}{(\frac{1}{2}-s)!}=\frac{\Gamma\left(l+\frac{3}{2}\right)}{\Gamma\left(l+\frac{3}{2}+s\right)}\,\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-s\right)}. \tag{105}$$

With increasing s the factor $(l+\frac{1}{2}+s)!$ in the denominator increases very fast, whereas $(\frac{1}{2}-s)!$ increases also and in addition changes sign. Hence for short-range interactions a fast convergence is secured with increasing s. For the long-range (screened) Coulomb interaction the expression for $\nu_{jk}^{(ls)}$ is much more involved (Suchy 1996, equation 7.10) and the convergence is much slower than for short-range interactions (Suchy 1984, figure 23).

The transport collision frequencies $\nu^{(ls)}$ in (95) are linear combinations of omega integrals $\Omega^{(l)}(r)$, introduced by Chapman and Cowling (1970, equation 9.33,2), as was shown by Weinert *et al.* (1978, equations A.16 and A.19). But the $\Omega^{(l)}(r)$ in general do not decrease with increasing r, and therefore the $\nu^{(ls)}$ are more suitable for estimates in connection with the truncation of the series expansion (24) for the Hermite polynomials.

Since the collision tensor $V^{(\nu+N+M+\mu)}$ in (88) is proportional to the Kronecker delta δ_{MN} the Maxwell–Boltzmann tensor $B^{(n+p+q)}(j,k)$ in (68) can now be written as

$$B^{(n+p+q)}(j,k) = \tau_j^{n/2} \sum_N \sum_\nu \delta_{N+\nu,n}\, T_{jk}(N,\nu;n,0) \times \sum_M \delta_{MN} \sum_\mu \delta_{\mu+M,p+q} T_{jk}(M,\mu;p,q) V^{(\nu+N+M+\mu)} . \tag{106}$$

The three Kronecker deltas allow the reduction of the fourfold sum to a single sum over N and the replacements

$$\nu = n - N \quad \text{and} \quad \mu = p + q - N . \tag{107}$$

The requirement $\nu + \mu$ even in (88) leads then to the requirement $n + p + q$ even in the double sum (24) for the collision integral of the Hermite polynomial $He^{(n)}(j)$.

Because of the vanishing of the coefficients $\alpha(0,\mu,l,s)$ and $\alpha(\nu,0,l,s)$, compare (100) with (98), the collision rates $K_{jk}(0,\mu,l,s)$ and $K_{jk}(\nu,0,l,s)$ in (97) also vanish and (88) leads to

$$V^{(0+N+M+\mu)} = 0 = V^{(\nu+N+M+0)} . \tag{108}$$

Then (107) yields the smaller of the numbers $n-1$ and $p+q-1$ as the upper limit in the sum over N :

$$0 \le N \le \min(n-1, p+q-1) . \tag{109}$$

The particular Maxwell–Boltzmann tensor $B^{(0+p+q)}(j,k)$ in (106) requires $N + \nu = 0$ and therefore contains only the vanishing collision tensors $V^{(0+\mu)}$ in (108). Therefore

$$B^{(0+p+q)}(j,k) = 0 . \tag{110}$$

Since $B^{(0+p+q)}(j,k)$ represents the collision integral (24) for the Hermite tensor $He^{(0)}(j) = 1$ of zeroth order, its vanishing expresses the conservation of particles of species j during (non-reactive) collisions, as it should be.

8. Concluding Remarks

According to (24) the Maxwell–Boltzmann tensors $B^{(n+p+q)}(j,k)$ in (106) are the expansion coefficients of the collision integral (19) for the Hermite polynomial

$He^{(n)}(j)$ in (10) with respect to the Hermitian moments $\overline{He^{(p)}(j)}\ \overline{He^{(q)}(k)}$ of the colliding particle species j and k. The first factor $\tau_j^{n/2}$ in (106) is a power of the ratio $\tau_j := kT/m_j$ in (10). The transformation factors $T_{jk}(N,\nu;n,0)T_{jk}(M,\mu;p,q)$ in (66) are (dimensionless) functions of the mass ratios $r_j^2 := m_j/(m_j+m_k)$ and $r_k^2 := m_k/(m_j+m_k)$ in (34). The collision tensor $V^{(\nu+N+M+\mu)}$ in (88) with (97) contains linear combinations of transport collision frequencies $\nu_{jk}^{(ls)}(\tau_{jk})$ in (95), depending on the ratio $\tau_{jk} := kT/\mu_{jk}$ in (29).

The appearence of Talmi coefficients was avoided due to the decomposition (37) of the Hermite polynomials for peculiar velocities into Hermite polynomials for centre-of-mass and relative velocities, in comparison to the corresponding decomposition of spherical harmonics (Kumar 1966, equation 88).

The use of the mean-mass-velocity $\boldsymbol{v}$ in (2) as the centre of the local Maxwellians (22) and the mean temperature T in (7) for its width was important for the achievement of a closed expression (106) with (66) and (88) for the Maxwell–Boltzmann tensor $B^{(n+p+q)}(j,k)$ in (25), as it was for a relatively short expression for the dynamical (left-hand) side of the balance equation for $\overline{He^{(n)}(j)}$ (Suchy 1995, equation 4.2). In contrast to the procedure with the velocity average $\overline{\boldsymbol{c}_j}$ as the centre of the local Maxwellian and the species temperature T_j in (16) for its width (Balescu 1988, equation 4.2.6), the (infinite) set of coupled balance equations (24) contains automatically a balance equation for the diffusion velocity $\overline{\boldsymbol{C}_j} = \overline{He^{(1)}(j)}$ in (14), whereas it has to be introduced additionally in the other procedure (Balescu 1988, equations 4.3.16 and 4.4.5).

In (14) and (18) it was illustrated that the traces of the Hermitian moments $\overline{He^{(2)}(j)}$ and $\overline{He^{(3)}(j)}$ play an important role to represent physical quantities. To extract them from the infinite hierarchy of coupled balance equations for the Hermitian moments $\overline{He^{(n)}(j)}$ (Suchy 1995, equation 4.2) one has to take (multiple) traces of these equations. To avoid this one could introduce these traces already in the expansion (20) of the velocity distribution function. This would lead to a double sum with a doubly infinite number of summands (Balescu 1988, equation 4.3.11). For the first members of this double sum the balance equations were calculated stepwise by Balescu (1988, Sections 3.4, 4.5 and 4.6).

Balescu (1988, Section 5.4) has shown that for long-range interactions Grad's (1949*b*, equation 5.4) 13 moment approximation is too crude for the calculation of the electrical conductivity of a plasma. The additional taking into account of the tracefree part of the single trace of $\overline{He^{(4)}}$ and the double trace of $\overline{He^{(5)}}$ in a 21 moment approximation (Balescu 1988, equation 4.3.21) improves the results considerably, whereas the further taking into account of the tracefree part of the double trace of $\overline{He^{(6)}}$ and the triple trace of $\overline{He^{(7)}}$ in a 29 moment approximation (Balescu 1988, equation 4.3.22) makes only marginal improvements. Therefore the 21 moment approximation seems to be a sufficiently accurate approximation for highly ionised plasmas. The neglect of the tracefree parts of $\overline{He^{(3)}}, \overline{He^{(4)}}$ and $\overline{He^{(5)}}$ can be understood with an investigation of the dynamical (left-hand) parts of the balance equations for $\overline{He^{(n)}}$ with $n \geq 3$ (Suchy 1995, Section 5).

A comparison of the rapidity of the convergence of Balescu's procedure and that presented here has not yet been made, because it would require considerable numerical computations. Since the difference between the two procedures rests mainly in the centring and widening of the local Maxwellians, it is difficult to assume that one procedure converges faster than the other. But it is very likely that for predominant short-range interactions the convergence is faster than for predominant long-range (Coulomb) interactions, cf. (104) and (105) and the following remarks, and Grad's 13 moment approximation is sufficient.

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