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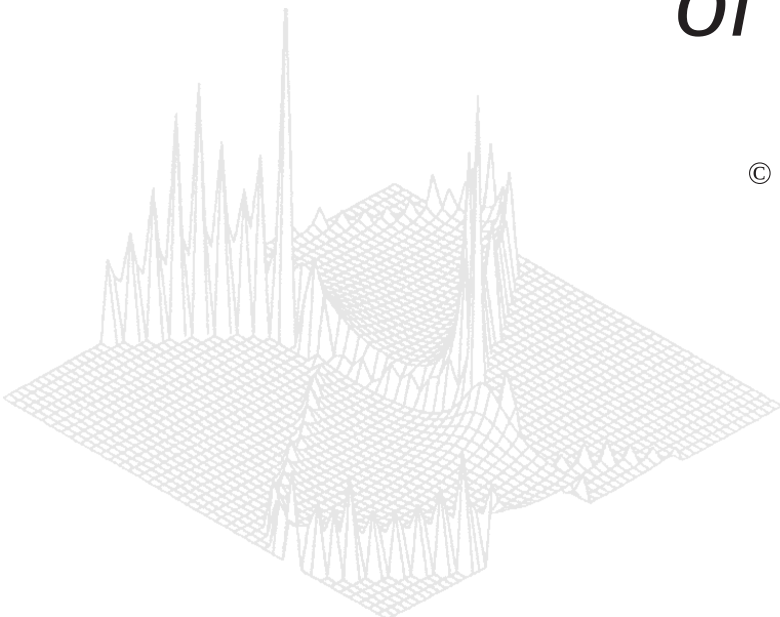

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Stability of a Modulated Ion Acoustic Wave in a Magnetised Plasma

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Abstract

A nonlinear Schrödinger equation which governs the nonlinear interaction of the ion-acoustic wave with the quasistatic plasma slow response in a magnetised plasma is deduced. The magnetic field is assumed to be constant. It is observed that the coefficient of the nonlinear term in the derived nonlinear Schrödinger equation turns out to be complex, in contrast to the usual situation. The condition for modulational stability is derived and it is found to be somewhat different. In the final section such a condition is discussed graphically. Our NLS equation goes back to that of the unmagnetised case if ω_c is put to zero.

1. Introduction

Modulational stability of the ion-acoustic wave in a nonlinear and dispersive plasma has attracted the attention of plasma physicists for a long time (Schimizu and Ichikawa 1972; Kako and Hasegawa 1976; Chabra and Sharma 1986). Various types of physical conditions involving harmonic generated nonlinearity and parallel and oblique modulation have been discussed in the literature (Yashvir *et al.* 1985; Kakutani and Sugimoto 1974). On the other hand, the topic of a slow quasi-static plasma response to ion-acoustic waves leading to the modulation of the latter was considered by Shukla (1986) and Bharathrum and Shukla (1987). An important point which should be mentioned is that in almost all the above-quoted references the plasma was considered to be unmagnetised. In the present paper we consider the plasma to be in a constant magnetic field and derive the nonlinear Schrödinger equation (NSE) which governs the oblique modulation of the ion-acoustic wave due to its interaction with plasma slow response in the presence of a constant ambient magnetic field. The derived NSE is different from the usual case because the coefficient of the nonlinear term turns out to be complex. We have rederived the condition of modulational stability of such an equation ab initio. Lastly, such a condition is analysed graphically. It may also be noted that in the absence of the magnetic field we get the result of Mishra *et al.* (1990). Furthermore, it may be mentioned that a similar approach has been adopted by Zakharov and Kuznetsov (1998) and by Vladimirov and Krivitsky (1983) to analyse, respectively, the existence of optical solitons and the modulational instability in general. In the former it has been shown that both solitons and quasi-solitons can exist and how the idea of stability relates to them. A general discussion about the stability of solitary-type excitations in

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both plasma and hydrodynamics can be found in the review by Kuznetsov *et al.* (1986).

2. Formulation

We assume that the plasma consists of ions and electrons, and a hydrodynamic description is possible, where the electrons actually form the background. The equations describing the plasma can be written as

$$\frac{\partial n_i}{\partial t} + \vec{\nabla}.(n_i \vec{v}_i) = 0\,, \tag{1}$$

$$\frac{\partial \vec{v}_i}{\partial t} + (\vec{v}_i.\vec{\nabla})\vec{v}_i = -\vec{\nabla}\phi + \omega_c(\vec{v}_i \times \vec{z})\,, \tag{2}$$

$$\vec{\nabla}\phi = \frac{1}{n_e}\vec{\nabla} n_e\,, \tag{3}$$

$$\nabla^2\phi = n_e - n_i\,, \tag{4}$$

where the plasma is assumed to be immersed in a constant magnetic field, B_0 is in the z-direction and $\omega_c = (eB_0/m_i) \ll \omega$. In the above equations $\vec{v}_i$ is the ion velocity, n_i is the ion density, n_e is the electron density, and ϕ represents the electrostatic potential. The densities are normalised with respect to the unperturbed plasma density n_0, the velocity with respect to the ion-acoustic velocity $C_s = (T_e/m_i)^{\frac{1}{2}}$, the electrostatic potential with respect to the electron thermal potential, and time with respect to the inverse of the ion plasma frequency ω_{pi}^{-1}.

Since we are interested in investigating the slow response of the quasi-static plasma to the ion-acoustic wave, we write the field quantities as

$$n_j = 1 + n_j^h + n_j^l, \qquad \vec{v}_j = \vec{v}_j^h + \vec{v}_j^l, \qquad \phi = \phi^h + \phi^l\,, \tag{5}$$

where for each variable n_j, v_j and ϕ, the superscripts h and l represent respectively the high and low frequency parts of each and $j = e, I$.

Substituting in the basic equations we get from (1)–(4)

$$n_e^h = (1 + n_e^l)\phi^h, \qquad n_e^l = \phi^l \tag{6}$$

and the same set of equations also yields

$$n_i^h = n_e^h - \nabla^2\phi^h\,. \tag{7}$$

The quasi-neutrality and quasi-static nature of the plasma is expressed through $n_i^l = n_e^l$ and $v_i^l \approx v_e^l \approx 0$.

We now assume that the obliquely modulated ion-acoustic wave is propagating in the xy plane in the magnetised plasma. In the absence of any nonlinearity we can write the dispersion relation as

$$\omega^2 = \omega_c^2 + \frac{k^2}{1+k^2}\,, \tag{8}$$

where $k^2 = k_x^2 + k_y^2$, and where k_x, k_y are respectively the x and y components of the wave vector. The modulation group velocities are

$$v_{gx} = \frac{\omega^3}{k^3}\left(1 - \frac{\omega_c^2}{\omega^2}\right)^2 \cos\theta, \qquad v_{gy} = \frac{\omega^3}{k^3}\left(1 - \frac{\omega_c^2}{\omega^2}\right)^2 \sin\theta\,, \tag{9}$$

where θ is the angle between the wave vector of the acoustic wave and the x axis.

Now from equations (1)–(6) we obtain

$$\begin{aligned} R\phi^h &+ \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) n_e^l \phi^h \\ &+ \omega_c \frac{\partial}{\partial x}((1+n_e^l)v_{ix}^h) - \frac{\partial}{\partial y}((1+n_e^l)v_{iy}^h) = 0\,, \end{aligned} \tag{10}$$

where

$$R = \frac{\partial^2}{\partial t^2}\left(1 - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\,.$$

On the other hand we have

$$v_{iy}^h = \frac{\omega k_y - i k_x \omega_c}{\omega^2 - \omega_c^2}\phi^h, \qquad v_{ix}^h = \frac{\omega k_x + i k_y \omega_c}{\omega^2 - \omega_c^2}\phi^h\,. \tag{11}$$

Now considering the slow component of equation (2) we obtain

$$(\vec{v}_i^h . \vec{\nabla})\vec{v}_i^h = -\vec{\nabla}\phi^l\,. \tag{12}$$

If we further assume that the modulational amplitude and v_{iy}^h, v_{ix}^h vary slowly with respect to y, we get at once

$$n_e^l \approx -\tfrac{1}{2}\langle v_{ix}^h \rangle^2\,, \tag{13}$$

where the angle brackets denote the average of the high frequency parts. So, finally we get from equation (10)

$$\left[\frac{\partial^2}{\partial t^2}\left(1 - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right]\phi^h + N_1 + N_2 = 0\,, \tag{14}$$

where

$$N_1 = -\beta\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)|\phi_h^2|\phi^h\,,$$

$$N_2 = +\gamma\omega_c \frac{\partial}{\partial x}((1-\beta|\phi^h|^2)\phi^h) - \delta\omega_c \frac{\partial}{\partial y}((1-\beta|\phi^h|^2)\phi^h)\,,$$

$$\beta = \frac{(1+k^2)^2}{2k^2}[(\omega^2+\omega_c^2)\mathrm{cos}^2\theta - \omega_c^2 + i\omega\omega_c \mathrm{sin}2\theta]\,,$$

$$\gamma = \frac{\omega k_y - ik_x\omega_c}{\omega^2-\omega_c^2}, \qquad \delta = \frac{\omega k_x + ik_y\omega_c}{\omega^2-\omega_c^2}\,. \tag{15}$$

We now assume that the nonlinear interaction of the ion-acoustic wave with the slow response plasma gives rise to an envelope of waves whose amplitude varies on the time and space scales much more slowly than those of ion acoustic oscillations. So we set

$$\phi^h = \epsilon^{\frac{1}{2}}\phi^h(\xi,\tau)\mathrm{exp}(-i\omega t + ik_x x + ik_y y) + \mathrm{c.c.}\,,$$

$$\xi = \epsilon^{\frac{1}{2}}(x - v_g t), \qquad \tau = \epsilon t\,. \tag{16}$$

Substituting (16) in equation (14) we get

$$i\frac{\partial\phi^h}{\partial\tau} + P\frac{\partial^2\phi^h}{\partial\xi^2} + Q|\phi^h|^2\phi^h = 0\,, \tag{17}$$

where

$$Q = 3\omega(1+k^2)(\cos\theta + i\omega_c\sin\theta)^2/4 = Q_1 + iQ_2\,,$$

$$P = (\omega^3/2k^4)[1-(1+3\omega^2)\cos^2\theta]\,.$$

Equation (17) is the required nonlinear Schrödinger equation. Note that the coefficient of the nonlinear term is complex in contrast to the case of a non-magnetised plasma. Furthermore, if we set $\omega_c = 0$, we get back the result of Mishra *et al.* (1990). Also the coefficient of the dispersive term is the same even in the presence of a magnetic field. The complex part of Q also vanishes when $\theta = 0$, because the particles are moving parallel to B_0 and their gyromotion is to be neglected.

Modulational Stability Condition

Since the NSE has a complex coefficient, we deduce the stability condition ab initio. We set

$$\phi^h = \rho^{\frac{1}{2}}\mathrm{exp}\left(i\int^{\xi}\frac{\sigma}{2p}\partial\xi\right), \tag{18}$$

where ρ is the amplitude and the argument in the exponential is the phase. Hence we get

$$\frac{\partial\rho}{\partial\tau} + \frac{\partial}{\partial\xi}(\rho\sigma) = 0\,, \tag{19}$$

$$\frac{\partial\sigma}{\partial\tau}+\sigma\frac{\partial\sigma}{\partial\xi}=2PQ\frac{\partial\rho}{\partial\xi}+P^2\frac{\partial}{\partial\xi}\left(\rho^{-1/2}\frac{\partial}{\partial\xi}\left(\rho^{-1/2}\frac{\partial\rho}{\partial\xi}\right)\right). \tag{20}$$

We linearise by perturbation

$$\rho=\rho_0+\delta\rho\exp(i(k\xi-\omega t))=\rho_0+\delta\bar{\rho}\,,$$
$$\sigma=\sigma_0+\delta\sigma\exp(i(k\xi-\omega t))=\sigma_0+\delta\bar{\sigma}\,, \tag{21}$$

and hence we get

$$\frac{\partial}{\partial\tau}(\delta\bar{\rho})+\rho_0\frac{\partial}{\partial\xi}(\delta\bar{\sigma})+\sigma_0\frac{\partial}{\partial\xi}(\delta\bar{\rho})=0\,, \tag{22}$$

$$\frac{\partial}{\partial\tau}(\delta\bar{\sigma})+\sigma_0\frac{\partial}{\partial\xi}(\delta\bar{\sigma})=2PQ\frac{\partial}{\partial\xi}(\delta\bar{\rho})+P^2\rho_0^{-1}\frac{\partial^3}{\partial\xi^3}(\delta\bar{\rho})\,. \tag{23}$$

This immediately leads to

$$(\sigma_0k-\omega)=ki(2\rho_0)^{\frac{1}{2}}(P(Q_1+iQ_2))^{\frac{1}{2}}\,, \tag{24}$$

and hence we get $\omega=\sigma_0k+i\Gamma$ with

$$\Gamma=\pm k(\rho_0)^{\frac{1}{2}}[N_3+iN_4]\,. \tag{25}$$

Here we have

$$N_3=(P(\surd(Q_1^2+Q_2^2)+Q_1))^{\frac{1}{2}}\,,$$

$$N_4=(P(\surd(Q_1^2+Q_2^2)-Q_1))^{\frac{1}{2}}\,,$$

but the actual growth rate crucially depends on whether P is positive or negative. In fact we can write

$$\omega=\sigma_0k\mp k(\rho_0)^{\frac{1}{2}}[N_4\mp iN_3] \tag{26}$$

when $P>0$, but

$$\omega=\sigma_0k\mp k(\rho_0)^{\frac{1}{2}}[N_5\mp iN_6] \tag{27}$$

when $P<0$. Here we have

$$N_5=(|P|(\surd(Q_1^2+Q_2^2)+Q_1))^{\frac{1}{2}}\,,$$
$$N_6=(|P|(\surd(Q_1^2+Q_2^2)-Q_1))^{\frac{1}{2}}\,.$$

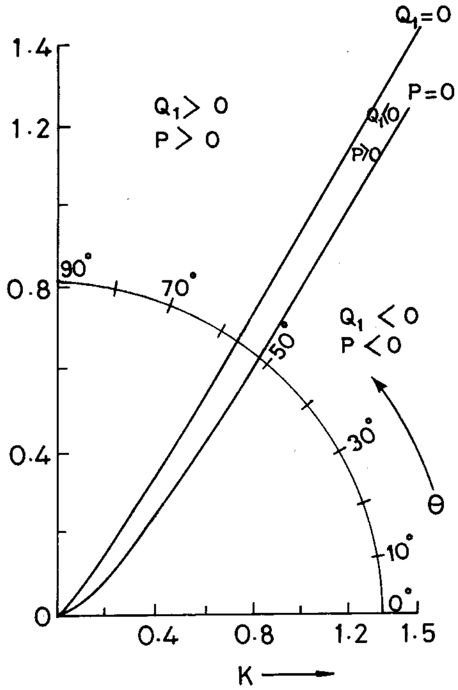

Fig. 1. Polar graph in the (k, θ) plane, where $|k|$ is the radial distance and θ the polar angle (which was originally the angle between the wave vector of the wave and the x-axis). Shown is the zone of stability and its variation with θ.

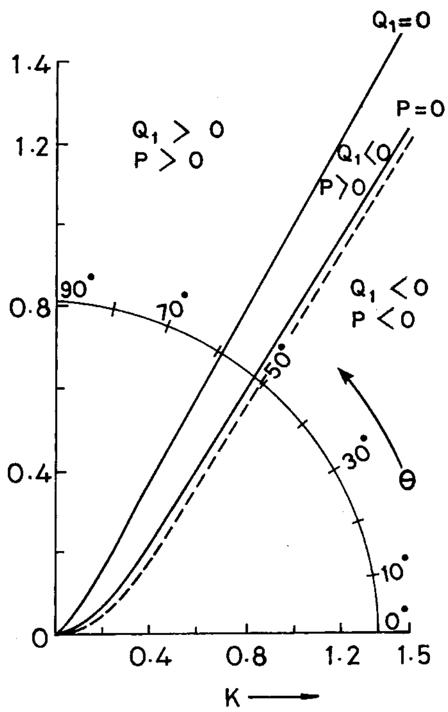

Fig. 2. Same diagram as Fig. 1, but with $\omega_c = 0{\cdot}1$ and 0 (dashed line).

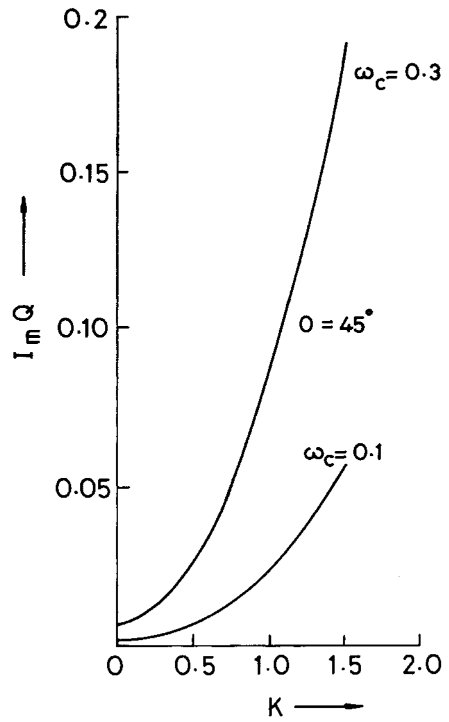

Fig. 3. Variation of ImQ with k.

In our NSE the coefficients P and Q depend on the two quantities (k, θ). Our motivation is to analyse their behaviour as a function of these variables. So we have displayed their variation in a plane where k represents the radial distance and θ stands for the polar angle. In such a diagram the regions of stability can be displayed very elegantly.

In the above derivation we have assumed that Q_1 and Q_2 are comparable in magnitude because only then does a complex form of Q make sense. But in the present case we have observed (see Fig. 3) that $|Q_2| < 1$, and one may safely neglect Q_2^2. Hence the above condition reduces to the usual modulational stability criterion that $PQ_1 < 0$ or $PQ_1 > 0$. At this point we may note that our equation (17) reduces to that given by Mishra *et al.* (1990) when $Q_2 = 0$ for the nonmagnetised case.

In the diagrams we depict the polar plots of $P = 0$ and $Q_1 = 0$, which clearly exhibit the dependence of stability on the angle θ. Here Q_1 denotes the real part of Q. The two cases for $\omega_c = 0 \cdot 3$ and $\omega_c = 0 \cdot 1$ are given in Figs 1 and 2. The variation of the imaginary part Q_2 is shown in Fig. 3, also for $\omega_c = 0 \cdot 1$ and $\omega_c = 0 \cdot 3$. It is important to notice that, whatever its variation with respect to the wave vector, the magnitude of Q_2 is always $\ll 1$. This fact may be responsible for the observation that the region of stability does not change appreciably with ω_c. It is important to note that $\theta = 60^0$ still remains a critical angle which divides the region of stability and instability. A different type of situation (due to second harmonic generation) was considered by Kako and Hasegawa (1976), but also in the unmagnetised case. Lastly, we may note that the NSE does possess an envelope solitary wave solution which can be written as

$$\phi^h = \alpha \mathrm{sech}(\gamma(x - Qt))\exp(i(kx - \omega t))\,,$$

which is the required envelope soliton solution generated due to the interaction mechanism discussed following equation (15).

3. Another Type of Modulation

Since we are considering oblique propagation of the nonlinear wave it is possible to consider another type of modulation, observed through a different form of stretched variables:

$$\phi^h = \epsilon^{\frac{1}{2}}\phi^h(\xi, \eta, \tau)\exp(i(k_x x + k_y y - \omega t))\,,$$
$$\xi = \epsilon^{\frac{1}{2}}(x - v_{gx}t), \qquad \eta = \epsilon^{\frac{1}{2}}(y - v_{gy}t), \qquad \tau = \epsilon t\,, \tag{28}$$

and hence we get a three-dimensional NSE written as

$$i\frac{\partial \phi^h}{\partial \tau} + a\frac{\partial^2 \phi^h}{\partial \xi^2} - 2b\frac{\partial^2 \phi^h}{\partial \xi \partial \eta} + c\frac{\partial^2 \phi^h}{\partial \eta^2} + d|\phi^h|^2\phi^h = 0\,, \tag{29}$$

where

$$a = \frac{\omega^3}{2k^4}\left(1 - \frac{\omega_c^2}{\omega^2}\right)^2\left[1 - (1 + 3\omega^2 + \omega_c^2)\left(1 - \frac{\omega_c^2}{\omega^2}\right)\cos^2\theta - (1 + k^2)\omega_c^2\right],$$

$$c = \frac{\omega^3}{2k^4}\left(1 - \frac{\omega_c^2}{\omega^2}\right)^2\left[1 - (1 + 3\omega^2 + \omega_c^2)\left(1 - \frac{\omega_c^2}{\omega^2}\right)\sin^2\theta - (1 + k^2)\omega_c^2\right],$$

$$b = \frac{\omega^3}{2k^4}\left(1 - \frac{\omega_c^2}{\omega^2}\right)^3(1 + 3\omega^2 + \omega_c^2)\sin\theta\cos\theta\,,$$

$$d = \frac{\omega k^2}{4}\frac{\cos^2\theta}{1 - \omega_c^2/\omega^2}\left|1 + i\frac{\omega_c}{\omega}\tan\theta\right|^2.$$

To reduce equation (29) to a simpler form we make a further change of variable:

$$X = p\xi + q\eta, \qquad Y = r\xi + s\eta\,, \tag{30}$$

so that we get

$$i\frac{\partial \phi^h}{\partial \tau} + p_1\frac{\partial^2 \phi^h}{\partial X^2} + p_2\frac{\partial^2 \phi^h}{\partial Y^2} + d\left|\phi^h\right|^2\phi^h = 0\,, \tag{31}$$

with

$$p_1 = (a - 2b + c)(ac - b^2)/(b - c)^2\,,$$

$$p_2 = (a - 2b + c)\,,$$

Whereas a one-dimensional NLS equation (i.e. one space dimension) is always known to sustain an envelope soliton, that is not the case with the two-dimensional one. We may again analyse the region of stability by plotting the polar curve of $p_1 = p_2 = 0$. Note that $p_1 = p_2 = 0$ leads to the condition

$$a - 2b + c = 0\,,$$

which reduces to the fact that

$$\theta = \tfrac{1}{2}\arcsin(K/L)\,,$$

where

$$K = 2(1 - (1+k^2)\omega_c^2) - \left(1 + 4\omega_c^2 + \frac{3k^2}{1+k^2}\right)\frac{k^2}{k^2 + \omega_c^2(1+k^2)}\,,$$

$$L = \left(1 + 4\omega_c^2 + \frac{3k^2}{1+k^2}\right)\frac{k^2}{k^2 + \omega_c^2(1+k^2)}\,.$$

This was used to draw the polar plot in Fig. 4 for the three cases $\omega_c = 0$, $\omega_c = 0\cdot 1$ and $\omega_c = 0\cdot 3$.

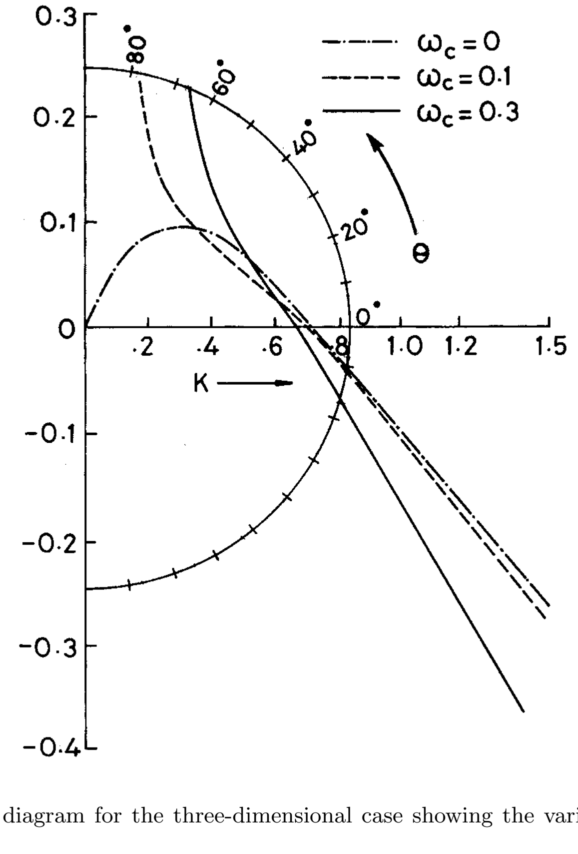

Fig. 4. Polar diagram for the three-dimensional case showing the variation of p_1, p_2.

We observe that the situation changes drastically so far as the dispersive coefficients p_1, p_2 are concerned, but the variation of the nonlinear term remains almost the same (not shown in Fig. 4). So it can be inferred that the effect of the magnetic field changes the stability scenario in the multidimensional propagation, but not in the case of the one dimensional NLS. In equation (30) the choice of the constants p, q, r and s is found to be given by $p/q = (b-c)/(a-b)$ and $r/s = 1$.

4. Conclusion and Discussion

In our analysis we have shown how the interaction of the ion-acoustic wave with the plasma slow response behaves under the influence of a constant magnetic field. The stability condition for the obliquely propagating wave is discussed for two kinds of modulation. The emergence of the NLS equation with a complex coefficient is a new feature. It is also observed that in the limit $\omega_c \to 0$, our equation reduces to that derived earlier in the unmagnetised case. It is also interesting to observe that, depending on the method of modulation, we can have a different form of nonlinear wave equation for the description. Lastly, the stability criterion is depicted in a plot which clearly shows the dependence on the angle θ.

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