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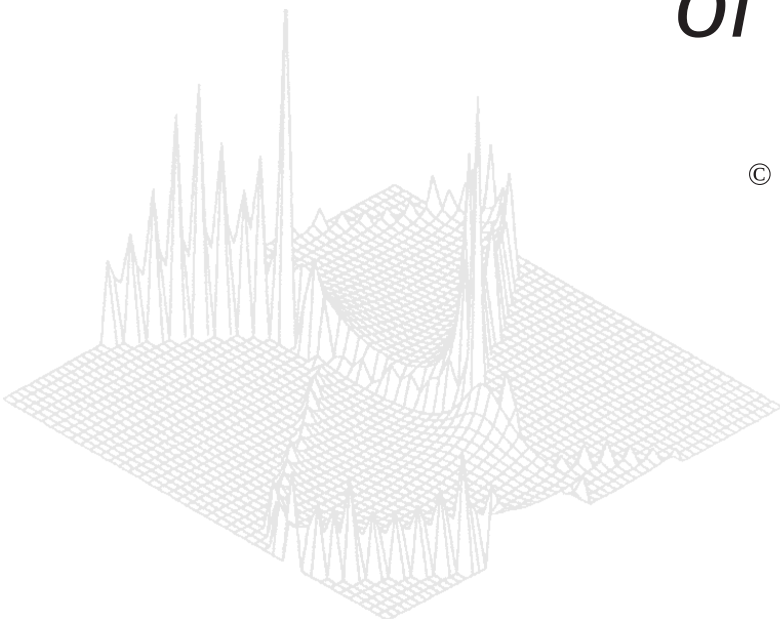

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A Critique of the Gauge Technique

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Abstract

A summary of the successes of and obstacles to the gauge technique (a non-perturbative method of solving Dyson–Schwinger equations in gauge theories) is given, as well as an outline of how progress may be achieved in this field.

1. The Gauge Technique

Nature seems inordinately fond of gauge theories. Chromodynamics, electroweak theory and gravitation are all based on the gauge principle, featuring the groups SU(3), SU(2)×U(1) and SL(2,c), respectively. It is quite likely that any unified gauge model of the fundamental forces will also be a gauge field theory and it is conceivable that a supersymmetric version, in some higher dimension, will be founded on a local supergauge principle, although Nature seems reticent to display her supersymmetric hand in the low energy spectrum of states.

Whenever a field theory is invariant under local group transformations, the resulting Green functions obey a series of 'gauge covariance' relations which specify how the functions change under a variation of gauge.* In QED these relations were originally found by Landau and Khalatnikov (1956, LK for short); later on, Johnson and Zumino (1959) and Zumino (1960) rederived them using functional methods. An immediate consequence of these ideas is the famous divergence properties of Green functions at each vector leg, named Ward–Green–Takahashi (WGT) identities after their early discoverers (Ward 1950; Green 1953; Takahashi 1957).

Heroic efforts have been expended in determining the behaviour of off-shell Green functions. This program has been carried out to high accuracy in perturbation theory for the standard model of particle physics and the results (at least for asymptotic momenta in QCD) are quite reliable because the perturbation series 'converge' to the order in which they have been calculated so far and also because the model is renormalisable.† An alternative approach is to solve the full set of Dyson–Schwinger (DS) equations connecting the various Green functions by

* Purely photonic processes and amplitudes involving on-shell fermions remain *gauge-invariant* of course.

† They will eventually go wrong because the series in e^2 is believed to be only an asymptotic one.

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some suitable truncation procedure. For instance, one simple way is to sum the 'rainbow diagrams' of the self-energy or the 'ladder diagrams' of the vertex, which amounts to simplifying the DS equations for the two- and three-point functions in a particular way. By this means one can already discern the phase structure of a model as the coupling runs through certain critical values.

In gauge theories, thanks to the existence of WGT identities between successive n-point Green functions, a more sophisticated approach suggests itself. The identities allow one to determine (longitudinal) pieces of the Green function in terms of the function with one less vector particle.* Hence if one discards transverse parts of the Green function (orthogonal to the photon momentum) or at least ties them in a particular fashion to the longitudinal parts, then this amounts to a truncation which produces a closed set of equations among the functions—to some level of approximation depending on n. For want of better terminology, we will coin this procedure the 'gauge technique' after Salam (Salam 1963; Salam and Delbourgo 1964; Strathdee 1964)—even though it was originally tied to a specific way of 'solving' the gauge identities and began at level $n = 2$. Anyway, armed with solutions of the DS equations, one may go on to calculate quantities of physical interest, such as decay constants of hadrons (Maris and Roberts 1997; Ivanov *et al*. 1998). This approach (Thompson and Zhang 1987; Roberts and Williams 1994) can be carried out in any number of space–time dimensions D and for any physical fermion mass m and it is quite interesting to see how the results depend on D and m separately, in various background (Cornwall 1986; Maris and Roberts 1998) configurations (temperature, field, etc.).

In this paper we shall focus primarily on the archetype gauge theory, QED. This has the advantage of yielding simple abelian identities and of bringing into relief the principal obstacles which the gauge technique must circumvent before it can be considered a fully-fledged method, with results that are above reproach. The purpose of this paper is to highlight some of these problems and to suggest possible ways in which some progress may be achieved.

Since we are dealing with gauge theories, the first issue we face is the gauge covariance of the technique, i.e. how well do the Green functions solutions obey the LK relations. The second question concerns the comparison with perturbation theory (in the fine structure constant α), i.e. how closely do the non-perturbative answers coincide with the perturbation series if we try to expand them in α. The third matter refers to the level of approximation, i.e. to what extent do the results change with the level of truncation n. Since the technique concentrates primarily on the charged particle functions, the fourth issue is what happens to vector propagation, i.e. how will vacuum polarisation pan out in such schemes: this is a serious matter because photon renormalisation, not electron renormalisation, governs the momentum dependence of the running charge. Indeed vertex and charged particle renormalisations are largely irrelevant: they are gauge-dependent and cancel out, leaving little physical trace. It means that going from quenched to unquenched solutions represents a major step physically.

* There is even a 'curly' version of such identities, obtained by Takahashi; they are rather more complicated than the ordinary divergence and we will have little to say about them in this paper. However in two-dimensions, it should be noted that they are equivalent to the axial gauge identities (Delbourgo and Thompson 1982; Thompson 1983; Kondo 1997).

If these issues can be clarified in QED, there are definite lessons (Papavassiliou and Cornwall 1986; Alekseev 1998; Cahill and Gunner 1998) for QCD and gravity.

2. Gauge Identities and SD Equations

First we set up the framework for the ensuing discussion by summarising quickly the identities and the coupled equations for Green functions in QED. If nothing else this will fix our notation. Let $G_{\lambda_1,\lambda_2,.}(p_1,p_2,.;k_1,k_2,.)$ stand for a connected time-ordered Green function, with electron momenta p and photon momenta k, as labelled. It is usual to write $G(p)=S(p)$ and $G_{\mu\nu}(k)=D_{\mu\nu}(k)$ for the two-point electron and photon propagators respectively. It is convenient often to define the one-particle irreducible Green functions $\Gamma(p;k)$ by dropping all the pole parts of $G(p;k)$ and by multiplying out by the inverse propagators of the external lines; for instance, the '3-vertex function' $\Gamma(p',p;k)$ with $k=p'-p$ arises via $G_\lambda(p',p;k)\equiv S(p')\Gamma^\mu(p',p;k)S(p)D_{\mu\lambda}(k)$, while the 'Compton part' $\Gamma(p',p;k',k)$ of electron–photon scattering occurs in

$$\begin{aligned} G_{\lambda\kappa}(p',p;k',k) \equiv D_{\lambda\nu}(k')S(p')[&\Gamma^{\nu\mu}(p',p;k',k) \\ &- \Gamma^\nu(p',p'+k';k')S(p+k)\Gamma^\mu(p+k,p;k) \\ &- \Gamma^\nu(p-k',p;k')S(p'-k)\Gamma^\mu(p',p'-k;k)]S(p)D_{\mu\kappa}(k) \end{aligned} \tag{1}$$

etc. The x-space Fourier transforms are obtained as convolutions; thus

$$G_\lambda(x,y;z)=\int d^4x'd^4y'd^4z'D_{\lambda\mu}(z-z')S(x-x')\Gamma^\mu(x',y';z')S(y'-y),$$

and so on. The reason why we have mentioned the coordinate space version is because the gauge covariance identities, to which we shall presently turn, are best written in x-space.

The Dyson–Schwinger equations connect successive (renormalised) charged Green functions through the series*

$$S(p)(\not{p}-m_0)=Z^{-1}+ie^2\int \bar{d}^4k G_\lambda(p,p-k;k)\gamma^\lambda\,, \tag{2}$$

$$G_\mu(p',p;p'-p)(\not{p}-m_0)=S(p')\gamma_\mu-ie^2\int \bar{d}^4k G_{\mu\lambda}(p',p-k;p'-p,k)\gamma^\lambda\,, \tag{3}$$

etc., plus the uncharged cases starting with,

$$\eta_{\mu\nu}=Z_A\,D^{-1}(k)_{0\mu}{}^\lambda D_{\lambda\nu}(k)-ie^2Z\ \mathrm{Tr}\int \bar{d}^4p G_\mu(p+k,p)\gamma_\nu\,. \tag{4}$$

The nonlinearity of electrodynamics becomes evident via the infinite skeleton expansion of the higher-point Green functions, when one expresses the G in terms of their one-particle-irreducible counterpart functions Γ. That is why one is

* Below and later on, we adopt the notation, $\bar{d}^4k\equiv d^4k/(2\pi)^4, \bar{\delta}^4(k)=(2\pi)^4\delta^4(k)$, etc.

obliged to invoke some kind of truncation for solving the equations nonperturbatively, instead of resorting to e^2 expansion—the idea being that a nonperturbative solution may reveal some dependence in $1/e^2$ (usually in an exponent) which is not immediately apparent from the asymptotic e^2 series.

All of the above applies, however one fixes the photon gauge. Now under a change of photon gauge, $D_{\mu\nu}(x) \to D^M_{\mu\nu}(x) = D_{\mu\nu}(x) - \partial_\mu \partial_\nu M(x)$, the renormalised Green functions are modified in a well-defined manner from G to G^M. Thus

$$S^M(x) = \exp[ie^2 M(x)]S(x),$$

$$G^M_\mu(x, y; z) = \exp[ie^2 M(x-y)][G_\mu(x, y; z) + iS(x-y)\partial^z_\mu\{M(x-z) - M(y-z)\}],$$

and so on. Let us call these the 'gauge covariance' or LK relations. It is simple to check that they are consistent with the SD equations in any gauge. A secondary consequence (though historically a primary one) is that the Green functions will satisfy the WGT identities,

$$(p'-p)^\lambda G_\lambda(p', p; p'-p) = S(p) - S(p'),$$

$$k^\mu G_{\mu\nu}(p', p'; k', k) = G_\nu(p', p+k; k') - G_\nu(p'-k, p; k'),$$

whose soft k-limit produces the Ward versions, $G_\mu(p, p; 0) = -\partial S(p)/\partial p^\mu$, etc. The WGT identities also appear straightforward for the 1PI functions Γ, e.g.

$$(p'-p)^\lambda \Gamma_\lambda(p', p; p'-p) = S^{-1}(p') - S^{-1}(p).$$

However, the WGT identities are weaker than the LK relations, which can themselves become quite complicated for the amputated Γ, in contrast to those for the full Green functions G, especially when written in momentum space. It is therefore a vexing business to verify that any Γ, derived somehow in some M-gauge, properly obeys the required covariance identity; by comparison it is easier to investigate the matter for the non-amputated G. In fact if one only amputates the photon legs, the gauge covariance relations simplify a little further and make our task easier. For illustration, take the vertex function $S\Gamma S$; one finds, like $S^M(x-y) = \exp[ie^2 M(x-y)]S(x-y)$, that

$$(S\Gamma S)^M(x, y, ; z) = \exp[ie^2 M(x-y)](S\Gamma S)(x, y; z)\,, \tag{5}$$

$$\partial_z \cdot (S\Gamma S)^M(x, y; z) = iS^M(x-y)[\delta^4(x-z) - \delta^4(y-z)]\,. \tag{6}$$

That completes the quick tour of the basis for the gauge technique.

3. What Gauge Covariance Implies

Before tackling the spinor case, let us start with scalar electrodynamics, where the algebra and arguments are simpler. If Δ connotes the charged scalar propagator and Γ the fully amputated 3-point vertex in some gauge (specified by $M = 0$ say), focus on the three relations:

$$\Delta^M(x,y) = \exp[ie^2 M(x-y)]\Delta(x,y),$$

$$(\Delta\Gamma\Delta)^M(x,y;z) = \exp[ie^2 M(x-y)](\Delta\Gamma\Delta)(x,y;z),$$

$$\partial_z^\mu(\Delta\Gamma_\mu\Delta)^M(x,y;z) = i\Delta^M(x-y)[\delta^4(x-z) - \delta^4(y-z)]\,, \tag{7}$$

associated with the lowest functions in a different gauge M. Now in general, the off-shell 3-point function can be expressed in terms of two invariants, one associated with the longitudinal vertex and the other with a purely transverse vertex ($k = p' - p$):

$$\Gamma_\mu(p',p;k) = (p'+p)_\mu L({p'}^2,p^2,k^2) + [k_\mu({p'}^2 - p^2) - (p'+p)_\mu k^2]T({p'}^2,p^2,k^2)\,, \tag{8}$$

where L and T are symmetric scalar functions under $p^2 \leftrightarrow {p'}^2$. It follows that, when $p^2 = {p'}^2$, the transverse part can be effectively combined with the longitudinal piece; this case applies in particular when one goes on the meson mass shell.

It is the T part which largely governs* the meson 'form factor' because the scalar WGT identity tells us unambiguously that the longitudinal part L cannot depend on the invariant $(p'-p)^2$ in *any* gauge; for it is *always* true that

$$L^M({p'}^2,p^2) = [\Delta^{-1\,M}(p') - \Delta^{-1\,M}(p)]/({p'}^2 - p^2)\,. \tag{9}$$

Like L, the transverse contribution T must also change with gauge function M, but in a subtler way than L and one where the $(p-p')^2$ dependence cannot be so easily forgotten. For suppose that in some gauge, we were to ignore T and wrote at the very least:

$$(\Delta\Gamma_\mu\Delta)(p',p;k) = \Delta(p')(p+p')_\mu\Delta(p)L.$$

Then in another gauge, according to (7), the result would get modified to

$$(\Delta\Gamma_\mu\Delta)(p',p;k) = \int \bar{d}^4k'\,\Delta(p'-k')(p+p'-2k')_\mu\Delta(p-k')LE^M(k'),$$

* It is worth noting that T has no singularities in the triangular variable $k^4 + p^4 + {p'}^4 - 2p^2{p'}^2 - 2p^2k^2 - 2{p'}^2k^2$, but that if we set $p^2 \neq {p'}^2$ there do arise logarithmic divergences (Ball and Chiu 1980) in perturbation theory as $k^2 \to 0$.

where

$$E^M(k) \equiv \int \bar{d}^4x \, \exp[ie^2 M(x) + ik \cdot x] \,. \tag{10}$$

A transverse amplitude T in the off-shell vertex would be ineluctably entrained via the momentum numerator of the integrand. For instance, starting with first order perturbation theory, a transverse Lorentz-covariant lurks within the expression

$$(\Delta\Gamma_\mu\Delta)^M(p',p;k) = \int \bar{d}^4k' \frac{(p+p'-2k')_\mu E^M(k')}{[(p'-k')-m^2][(p-k')^2-m^2]} \,. \tag{11}$$

Only in the limit $e^2M = 0$, when $E^M(k') = \bar{\delta}^4(k')$, will such a transverse term disappear. This argument teaches us two things about ensuring gauge covariance for general M: (i) that it is perilous to neglect the transverse parts of Green functions in non-perturbative treatments, and (ii) that one cannot always disregard the dependence of the amplitude on the momentum of the *vector* leg, i.e. one cannot purely* use functions of $p^2, {p'}^2$.

The discussion increases in substance, richness and delicacy for fermions. Instead of one transverse and one longitudinal part, the vertex contains four independent longitudinal pieces and eight transverse pieces off-shell. In a real tour de force, these pieces have been computed by Ball and Chiu (1980) to first order perturbation theory in the Fermi-Feynman gauge, and by Kizilersu *et al.* (1995) in any gauge. Being off-shell, the answers are extremely involved and we shall content ourselves with making three remarks: (i) there exist transverse covariants now which survive the on-shell spinor limit, such as $i\sigma_{\mu\nu}k^\nu$, that have important physical consequences; (ii) no longer can one combine transverse vertices with longitudinal ones for physical fermions;† (iii) there is a lot more freedom in 'solving' the WGT identity for the vertex, with distinct methods, all deemed free of unphysical singularities and all independent of the square of the photon momentum, yielding vertices differing by specific transverse terms T. (See the Appendix for details.) The various T arise automatically in the spinor version of (8) for arbitrary M, so it is futile to discard them in a general gauge‡ unless one gives up on the idea of satisfying the LK relations—a big disappointment for a gauge theory.

4. Comparison with Perturbation Theory

If one does succeed in obtaining 'acceptable' solutions (presumably with an implicit dependence on e^2) of the DS equations by the gauge technique, then there

* This applies with force to various 'improvements' or corrections to the longitudinal vertex, consistent with multiplicative renormalisability, that have been suggested (Curtis and Pennington 1993; Haeri 1993). In this connection it is worth remembering that the covariant gauge $a = 3$ produces a zero first order in α correction to the scalar propagator or to all orders, in the infrared limit.

† In fact because physical fermions satisfy free equations of motion, the transverse covariants are no longer independent but can be transmogrified into one another.

‡ Nevertheless one should recognise the privileged position of the covariant (Landau) gauge $a = 0$, because the first order in α correction to the wave function renormalisation vanishes identically, as do all rainbow modifications of the spinor propagator—Section 6.

is the obvious question of how well the answers stack up against perturbation theory, when expanded in powers of e^2. At the very least one would hope that they would correct up to first order in α, as they are indubitably correct to zeroth order—but that hardly constitutes progress! It is not much good having them agree with perturbation theory in one gauge (for some L and T) but being wrong in another gauge. However, that is what will likely happen from the transformation property of the G *unless* the various parts carry precisely the correct dependence on M and, as stressed previously, one countenances some k^2 dependence in $S\Gamma S$. And it is no good avouching that the off-shell dependence of the propagator on the momentum looks 'reasonable', with 'suitable behaviour' in the infrared or ultraviolet limits, since one can change the behaviour at will, just by choosing the gauge function M however one likes.

To clarify these points, let us consider covariant gauges, parametrised by a real number a:

$$D^a_{\mu\nu}(k) = -\eta_{\mu\nu}/k^2 + (1-a)k_\mu k_\nu/k^4.$$

(The values $a = 0, 1, 3$ define the Landau, Fermi–Feynman and Fried–Yennie gauges respectively.) Upon introducing a regularisation scale, like the electron mass m, the Fourier transform of $M(k) \equiv -a/k^4$ is obtained as $iM(x) = -a\ln(-m^2x^2)/16\pi^2$. Therefore the coordinate space Green functions G of two charged fields separated by x will be multiplied by the gauge factor

$$\exp[ie^2M(x)] = (-m^2x^2)^{-a\epsilon}; \qquad \epsilon \equiv e^2/16\pi^2 = \alpha/4\pi,$$

with transform,

$$E^M(k) = \int d^4x\, \exp(ip\cdot x)(-m^2x^2)^{-a\epsilon}$$

$$= -\frac{16\pi^2 i\Gamma(2-a\epsilon)}{k^4\Gamma(a\epsilon)}(-k^2m^2/4)^{a\epsilon}.$$

Expanding in powers of e^2, one can check that

$$E^M(k) = \bar{\delta}^4(k) - ie^2a/k^4 + O(e^4),$$

with the second term on the right corresponding to the change of covariant gauge parameter a to first order perturbation theory.

The transverse term has a complicated analytic form which only simplifies in various asymptotic limits; the behaviour in the ultraviolet regime is one such limit and King (1983) and Haeri (1988) put it to good use in correcting multiplicative renormalisability that is jeopardised in simple gauge technique ansatze; however, their procedure fails in the soft photon limit and does not do justice to the analytic behaviour in momentum transfer. To do better, it is useful to look at the form of T in scalar electrodynamics say, to first order in α or M as above.

An examination of the Feynman graph integral shows that it can be expressed in the parametric form

$$(p'^2 - p^2)T = \frac{e^2}{16\pi^2}\int d\alpha d\beta d\gamma\ \delta(1-\alpha-\beta-\gamma)$$

$$\times\frac{N(\alpha,\beta,\gamma)}{m^2(1-\alpha)-(p'^2\beta+p^2\gamma)\alpha-(p-p')^2\beta\gamma}\,, \tag{12}$$

where the parametric numerator (for $a=1$ say) is given by

$$N = 2(p'^2-p^2)\alpha + (\beta-\gamma)[p^2(1-\beta)+p'^2(1-\gamma)+m^2(3-\alpha)]$$

$$-3(\beta-\gamma)\beta\gamma[(p'^2-p^2)\alpha+(p-p')^2(\beta-\gamma)].$$

The important thing is that the integral produces a result which vanishes for $p^2 = p'^2$.

It is interesting to see whether the answer can be rewritten as some kind of dispersion relation. To that end, consider the self-energy first. The standard Feynman parametric form leads to the scalar integral

$$I = \int_0^1 d\alpha\ \chi(p,\alpha)/(p^2\alpha - m^2).$$

By changing variable to $W^2 = m^2/\alpha$, the expression above can be easily converted into the more familiar form

$$I = \int_{m^2}^{\infty} \rho(W^2)/(p^2 - W^2).$$

Turning to the proper vertex part, if we make the change of variables in the Feynman parametric integral, $\beta+\gamma=\sigma, \beta-\gamma=\sigma u$ and $\sigma = m^2/W^2$, we obtain an expression like

$$\int_{m^2}^{\infty} dW^2 \int_{-1}^{1} du\ N(p',p,u,W)/D(p',p,u,W),$$

for the transverse part. The denominator D takes the form

$$p'^2(1+u)/2 + p^2(1-u)/2 + (p'-p)^2(1-u^2)(1-m^2/W^2)(1-u^2)/4 - W^2,$$

and we see that only near the fermion mass shell ($W^2 = m^2$) and in the soft photon limit can one disregard the dependence on the momentum transfer. (In the ultraviolet limit where the spinor momentum is a long way from the $p^2 = m^2$ one can make other sorts of approximations.)

What this exercise demonstrates is that under no stretch of the imagination can one invoke a transverse vertex which is purely a function of p^2 and ${p'}^2$. Even an invokation like

$$(S\Gamma^T_\mu S)(p',p) = \int dW \frac{1}{\not{p}' - W} i\sigma_{\mu\nu}(p'-p)^\nu \frac{1}{\not{p} - W},$$

misses the point altogether, since it fails to include the correct analytic structure for a magnetic *form factor*. Remember also that the covariance LK relation under change of gauge will inevitably create such structure, if none were initially present. Therefore we insist that any substantive improvements to the gauge technique ought to include these kinds of effects and agree with first order perturbation theory at the very least. Ansatze of the type,

$$\Gamma^T_\mu(p',p) = \int_m^\infty dW \int_{-1}^1 du\ \rho^T(W) K^T_\mu / D(p',p,u,W)$$

stand some chance of capturing the main feature of transverse corrections.

5. Level of Truncation

In an effort to improve upon the three-point Green function it is good to proceed to the next level of the DS equations. This will relate the three and four-point Green functions, via a Bethe–Salpeter like equation—see Parker (1984) and Delbourgo and Zhang (1984). Note that in order to make use of the higher-point WGT identities, it is sensible to write this equation with one of the spinor legs preferred rather than the photon leg as is normally done; for with that choice one can use the four-point WGT identity to relate that amplitude to the three-point Green function and thereby arrive at a self-consistent equation for the three-point function. (This is in complete analogy to the favourite way of handling the propagator and vertex function together and we can think of it as the $n = 3$ level improvement of the gauge technique, in contrast to the conventional $n = 2$ level.)

What is more to the point is that, at this level, the longitudinal and transverse vertices are treated *on an equal footing* while the two-point function (propagator) is obtained secondarily through the divergence of the three-point function. In practice, though, the equation for the vertex is *very* difficult to solve, for one is not entirely sure of its *full* analytic representation, except through the perturbation expansion. The best one can do in such circumstances is to use a double dispersion relation (in a Feynman parametric form say) and try to determine the spectral function self-consistently, but even that is not straightforward. The only progress to date has been the determination (Delbourgo and Zhang 1984) of the propagator spectral function in a manner which coincides with perturbation theory to order α^2. Hopefully this problem will receive due attention in future.

6. Unquenching—Vacuum Polarisation

In most of the self-consistent calculations of the charged field propagator, the photon is taken as bare: this represents the so-called 'quenched approximation'. Including the effect of vacuum polarisation leads to a highly nonlinear equation for the propagator even in the best method of solving for the propagator, since vacuum polarisation is determined by charged loop effects; that is one of the main reasons why the problem is normally avoided. The best that one can do in that situation is to make an inspired guess at the behaviour of the dressed vector propagator $D(k)$ and (i) either include this in the computation of $S(p)$, or (ii) introduce a running coupling in the interaction between the vector and the charged field. Now in QED the photon receives *small* logarithmic corrections in the ultraviolet regime, but its low energy properties are largely unaffected; for that reason the quenched approximation is not too drastic a procedure for electrodynamics, at least in four dimensions.* But in QCD, the gluon exhibits asymptotic freedom in the ultraviolet—again a logarithmic correction—which hints at infrared slavery (confinement of colour?) at low energies and a propagator which is possibly more singular than the undressed form (Alekseev 1998; Cahill and Gunner 1998). Thus a variety of models have been proposed and the corresponding quark $S(p)$ found; the results depend critically on the assumed form of $D(k)$ which is itself influenced by the gluon self-interactions and the ghost field effects. There is still some dispute about what is the correct form of $D(k)$ and whether the quark propagator is an entire function of the momentum. We do not wish to get involved in these arguments; suffice it to say that the method with the best chance of being correct is the one that handles the gauge covariance properties correctly, in tandem with the gauge-invariant effective coupling $g = g_0 Z_1^{-1} Z_2 Z_3^{\frac{1}{2}}$.

In this connection it is worth recalling (Larin and Vermaseren 1993) the perturbative expansions of the beta functions for QED and QCD, which are of course gauge independent. Let $\epsilon \equiv g^2/16\pi^2 \to \alpha/4\pi$ for QED; then

$$\beta(\epsilon) = \epsilon[\gamma_3(\epsilon) + 2\gamma_2(\epsilon) - 2\gamma_1(\epsilon)] \equiv \sum_{n=1}^{\infty} \beta_n \epsilon^{n+1}\,.$$

(In QED the anomalous dimensions are equal, $\gamma_2 = \gamma_1$, so $\beta = \epsilon\gamma_3$ is determined purely by the anomalous dimension of the photon field.) Up to three loops, we have

$$\beta_{\rm QED} = \tfrac{4}{3}\epsilon^2\left[1 + 3\epsilon - \left(\tfrac{3}{2} + \frac{11N}{3}\right)\epsilon^2 + \dots\right]$$

$$\beta_{\rm QCD} = \epsilon^2\left[\left(\frac{2N}{3} - 11\right) + \left(\frac{38N}{3} - 102\right)\epsilon - \left(\frac{325N^2}{54} - \frac{5033N}{18} + \frac{2857}{2}\right)\epsilon^2 + \dots\right], \tag{13}$$

where N stands for the number of charged fermions or flavours. The dependence on the gauge parameter a in QED arises through the anomalous scaling function

* In three dimensions charged loops alter substantially the low energy behaviour from $D(k) = 1/k^2$ to $D(k) \propto 1\sqrt{-k^2}$, while in two dimensions the vector becomes massive through the Schwinger mechanism.

for the spinor*

$$\gamma_2(\epsilon) = \epsilon[a - \tfrac{3}{2}\epsilon + \tfrac{3}{2}\epsilon^2 + \ldots].$$

Notice that the gauge dependence arises only at one-loop level and that it vanishes in the Landau gauge $a = 0$. More significantly there are higher order in α contributions which cannot be ignored; it is therefore fatuous to suppose that one can simply set the coefficient of $\not{p}$ in the spinor propagator equal to one in the Landau gauge—this is simply incorrect in higher orders.

Anyhow it is reasonable to enquire what repercussions, if any, does the gauge technique have on vacuum polarisation. There are two important points to check in this connection: whether the polarisation tensor $\Pi_{\mu\nu}(k)$ is transverse and whether it is gauge-independent. Since

$$\Pi_{\mu\nu}(k) = ie^2 \,\mathrm{Tr} \int \bar{d}^4 p (S\Gamma_\mu S)(p + k, p)\gamma_\nu,$$

it is fairly clear that a non-perturbative calculation of Π which dresses the fermions (S) but leaves the full vertex as bare ($\Gamma \to \gamma$), will not produce a transverse tensor; therefore this strategy is unacceptable. It is also easy to concoct a longitudinal approximation to the three-point Green function that *does* lead to transverse polarisation, e.g. a mass-weighted spectral representation, as described in the Appendix. Carrying out the computations in the Landau gauge say, where

$$S(x - y) = \int dW\, \rho_0(W) S_F(x - y|W),$$

$$(S\Gamma^L S)(x, y; z) = \int dW\, \rho_0(W) S_F(x - z|W)\gamma S_F(z - y|W)\,, \tag{14}$$

with $S_F(x|W) \equiv (i\gamma\cdot\partial + W)\Delta_F(x|W)$, we will obey gauge covariance by stipulating that, in any other gauge M, the expressions above are to be multiplied by $\exp[ie^2 M(x - y)]$. This then leads to the gratifying result that the resulting vacuum polarisation is transverse and gauge-independent, since

$$\begin{aligned}\Pi_{\mu\nu}(z) &\propto \lim_{x,y\to 0} \mathrm{Tr}(S\Gamma_\mu S)(x, y; z)\gamma_\nu \\ &= \lim_{x,y\to 0} \mathrm{Tr} \int dW\, \rho_0(W) S_F(x - z|W)\gamma_\mu S_F(z - y|W)\gamma_\nu \mathrm{e}^{ie^2 M(x-y)} \\ &= \mathrm{Tr} \int dW\, \rho_0(W) S_F(x - z|W)\gamma_\mu S_F(z - y|W)\gamma_\nu\,, \end{aligned} \tag{15}$$

* See Larin and Vermaseren (1993) for a complete determination of the anomalous scaling functions in QCD, which are too long to reproduce for the purposes of this paper.

if we approach the equal location limit $x = y$ along a certain direction. This would imply that vacuum polarisation is gauge independent and is given by the Landau gauge result, $\Pi_{\mu\nu}(z) = \int dW\, \rho_0(W)\Pi_{\mu\nu}(z|W)$, corresponding to a weighted mass integral. Actually this result is still not correct: the Green function (14) is insufficient to account for all higher order quantum corrections, because we must supplement it by a transverse (Landau gauge) contribution, which certainly does not spoil the transversality property. Unless this is added, β_{QED}, will almost certainly be wrong.

Thus, from all of this discussion, we see that the most pressing problem in patching up the gauge technique is to incorporate transverse Green functions which have correct analytic and gauge-covariance properties. Until this is done, any *physical* results that are claimed to be a consequence of the technique are not to be fully trusted.

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Appendix

In scalar electrodynamics, if $\Delta(p^2) = \int dW^2\rho(W^2)/(p^2 - W^2)$ stands for the meson propagator, the longitudinal expressions for the vertex,

$$\Gamma^L_\mu(p', p) = (p + p')_\mu \frac{\Delta^{-1}(p'^2) - \Delta^{-1}(p^2)}{p'^2 - p^2},$$

$$(\Delta\Gamma^L_\mu\Delta)(p', p) = (p + p')_\mu \frac{\Delta(p^2) - \Delta(p'^2)}{p'^2 - p^2}$$

$$= \int dW^2\rho(W^2)\left[\frac{1}{p'^2 - W^2}(p + p')_\mu \frac{1}{p^2 - W^2}\right],$$

are entirely equivalent. Although one may add any amount of transverse amplitude without affecting the WGT identity, it would be perverse to introduce such an additional piece *unless there are good reasons to do so*, like reaching agreement with perturbation theory, eliminating subdivergences or trying to patch up the gauge covariance relation. This is precisely what has motivated King (in QED) and Haeri (in QCD) to incorporate particular transverse terms in the ultraviolet regime (King 1983; Haeri 1988). No such corrections are needed in the infrared regime, since all Green functions are effectively governed by the nonperturbative behaviour of the charged particle propagator (Delbourgo and West 1977*a*, 1977*b*; Delbourgo 1979; Atkinson and Slim 1979).

In spinor electrodynamics the situation is much less clearcut because of the plasticity of the gamma-matrix algebra. Begin with the spinor propagator, written in the equivalent forms,

$$S(p) = \int dW\rho(W)/(\not{p} - W) \quad \text{or} \quad S^{-1}(p) \equiv \not{p}A(p^2) + B(p^2).$$

For short, write $A \equiv A(p^2), B \equiv B(p^2), A' \equiv A(p'^2), B' \equiv B(p'^2)$ and $F \equiv p^2A^2 - B^2$, etc. Then there are at least three 'obvious' ways of 'solving the gauge identities', all of which are singularity free. From the proper vertex identity, one may factor out the momentum transfer and arrive at the first version (Ball and Chiu 1980),

$$\Gamma_\mu^L(p', p) = \tfrac{1}{2}\gamma_\mu(A' + A) + \frac{(p' + p)_\mu}{p'^2 - p^2}[(B' - B) + \tfrac{1}{2}(\not{p}' + \not{p})(A' - A)].$$

A second way is to carry out the factorisation for the full Green function:

$$-(S\Gamma_\mu^L S)(p', p) = \frac{\gamma_\mu}{2}\left(\frac{A'}{F'} + \frac{A}{F}\right) + \frac{(p' + p)_\mu}{p'^2 - p^2}\left[\frac{\not{p}' + \not{p}}{2}\left(\frac{A'}{F'} - \frac{A}{F}\right) - \left(\frac{B'}{F'} - \frac{B}{F}\right)\right].$$

A third way is to take advantage of the dispersive representation as a weighted mass integral and thereby arrive at the form

$$(S\Gamma_\mu^L S)(p', p) = \int dW \rho(W) \frac{1}{\not{p}' - W}\gamma_\mu \frac{1}{\not{p} - W}.$$

These three versions are not identical to one another, in contrast to the scalar case. They differ from one another by particular transverse components (which of course have no effect on the WGT identity) and no version is more natural than any other at this level, unless other considerations intervene; thus they all behave smoothly as $p^2 \to p^2$ and they agree with lowest order perturbation theory. For instance, the difference between the first and second versions of the proper vertices can be expressed as

$$\frac{A'}{(\not{p}'A' - B')}T_\mu(\not{p}A + B) + A'T_\mu + (\not{p}'A' + B')T_\mu \frac{A}{(\not{p}A - B)} + T_\mu A,$$

where $2T_\mu = \gamma_\mu - (\not{p}' - \not{p})(p' + p)_\mu/(p'^2 - p^2)$ is a transverse Lorentz-covariant. Similarly, the third version can be rewritten in a more revealing form:

$$(S\Gamma_\mu^L S)(p', p) = \int dW \rho(W) \frac{1}{\not{p}' - W} 2T_\mu \frac{1}{\not{p} - W} + \frac{(p' + p)_\mu}{p'^2 - p^2}[S(p) - S(p')].$$

Haeri and Haeri (1992) have shown that this particular spectral form of the longitudinal vertex can be converted into the equivalent but more elegant form

$$(S\Gamma_\mu^L S)(p', p) = \frac{(\not{p}'\gamma_\mu + \gamma_\mu \not{p})S(p) - S(p')(\not{p}'\gamma_\mu + \gamma_\mu \not{p})}{p'^2 - p^2},$$

which corresponds to the proper vertex solution

$$\Gamma_\mu^L(p',p) = \frac{\not{p}'\gamma_\mu \not{p}(A'-A) + (\not{p}'\gamma_\mu + \gamma_\mu \not{p})(B'-B) + \gamma_\mu({p'}^2 A' - p^2 A)}{{p'}^2 - p^2},$$

featuring the inverse propagator functions A and B.

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