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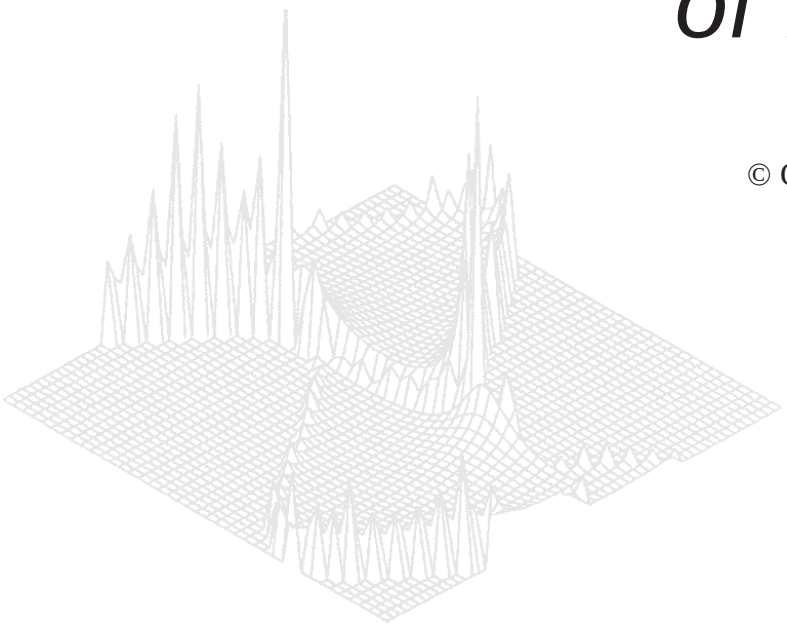

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Lattice Study of the Kink Soliton and the Zero-mode Problem for ϕ^4 in Two Dimensions

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Abstract

Using lattice field theory techniques, we perform an exploratory study of the quantisation of the simplest extended object, the $\lambda\phi^4_{1+1}$ kink soliton, and the associated zero-mode contribution to the kink soliton mass in regions in and beyond the semiclassical regime. The calculations are done in the non-trivial scaling region so that our lattice results can be meaningfully compared with the classical and semiclassical continuum results. We show how to extract the kink from this full quantum field theory treatment and show, as a function of parameter space, where the zero-mode contributions become significant.

1. Introduction

Solitons are non-dispersive localised packets of energy moving uniformly, and resembling extended particles. The elementary particles in nature are also localised packets of energy, being described by some type of quantum field theory. Because of these features, the soliton might appear as the ideal mathematical structure for the description of a particle. When it was realised that many nonlinear field theories used to describe elementary particles, also had soliton solutions and that these solutions might correspond to particle type excitations, the development of methods for soliton quantisation became important. The quantisation of solitons is usually done by performing an expansion in powers of $\hbar$ (loop expansion) such that the classical soliton solution appears as the term of leading order in the expansion and terms of higher order represent the quantum effects. In the mid 1970s there appeared a number of works (Dashen *et al.* 1974*a*, 1974*b*, 1974*c*; Friedberg and Lee 1977) which developed the semiclassical expansion in quantum field theory. In this period, there were schemes being constructed to quantise these solitons. The correspondence between classical soliton solutions and extended-particle states of the quantised theory was established (Dashen *et al.* 1974*a*, 1974*b*; Campbell and Liao 1976) and various methods were used to deal with the so-called 'zero-mode problem' (Gervais *et al.* 1975, 1976*a*, 1976*b*; Faddeev and Korepin 1976; Rajaraman and Weinberg 1975). This problem is a manifestation of the translational invariance of the theory, broken by the introduction of the soliton. Field oscillations around this classical solution contain zero frequency modes, describing displacements of the soliton.

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Here we study the mass of the simplest topological soliton, that is the $\lambda\phi^4_{1+1}$ kink, using lattice Monte Carlo techniques. The dynamics of this model are governed by a Euclidean Lagrangian density:

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\phi)^2 - \frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4\,, \tag{1}$$

where μ is a bare parameter and λ is the bare coupling constant. For a free scalar field theory $\lambda \to 0$ and $-\mu^2 \to m^2$, where m would then be the bare mass of the ϕ. In the classical theory $\mu^2 > 0$ corresponds to the onset of spontaneous symmetry breaking. There are two types of static solutions, both being static solutions, to the equation of motion; the trivial solutions

$$\phi_0 = \pm\frac{\mu}{\sqrt{\lambda}} \equiv f\,, \tag{2}$$

and the topological solutions

$$\phi_{\mathrm{k,ak}} = \pm\frac{\mu}{\sqrt{\lambda}}\tanh\left(\frac{\mu}{\sqrt{2}}x\right), \tag{3}$$

where 'k' and 'ak' label the topological solutions for the kink and antikink and correspond to the positive and negative sign, respectively. The semiclassical regime corresponds to large f.

The kink provides an example of a particle with an internal structure, distributed over a finite volume rather than concentrated at one point. It possesses a nonzero, conserved quantity called the topological charge Q which is defined as

$$Q = [\phi(x)|_\infty - \phi(x)|_{-\infty}]$$

and which leads to stability of the kink solution. The classical kink mass M_{cl} is defined to be the energy of the static soliton and is given by

$$M_{\mathrm{cl}} = \frac{2\sqrt{2}}{3}\,\frac{\mu^3}{\lambda}\,. \tag{4}$$

The vacuum and kink solutions can be quantised by path integral quantisation or by construction of a tower of approximate harmonic oscillator states around the classical solution $\overline{\phi}$, where either $\overline{\phi} = \phi_0, \phi_{\mathrm{k}}$ or ϕ_{ak}. In a finite box with the length L the soliton mass with a one loop quantum correction becomes (Rajaraman 1982):

$$M_{\mathrm{sol}} = E_{\mathrm{kink}} - E_{\mathrm{vac}} = M_{\mathrm{cl}} + \sum_n \tfrac{1}{2}\omega_n^2 - \sum_n \tfrac{1}{2}\xi_n^2 + \mathcal{O}(\widehat{\lambda})\,, \tag{5}$$

where ω_n and ξ_n are the eigenvalues of the following equation

$$\left[-\frac{\partial^2}{\partial x^2} + \left(3\lambda\phi^2 - -2\mu^2\right)_{\overline{\phi}}\right]\eta_i(x) = \Theta_i^2\eta_i\ , \tag{6}$$

with $\overline{\phi} = \phi_0$ and $\overline{\phi} = \phi_k$ or $\phi_{\rm ak}$ for ω_n and ξ_n respectively. An important point to note is that as $L \to 0$, due to translational invariance, one of the ω vanishes, i.e. $\omega_0 \to 0$.

As L is set to infinity then the sums are replaced by integrals and one ends up with a logarithmically divergent integral and renormalisation is required to render the kink mass finite. Then one arrives at the mass of the continuum kink with one loop corrections (Dashen *et al.* 1974*b*):

$$M_{\rm sol} = \frac{2\sqrt{(2)}}{3\lambda}\mu^3 + \mu\left(\tfrac{1}{6}\sqrt{\tfrac{3}{2}} - \frac{3}{\pi\sqrt{2}}\right) + \mathcal{O}(\lambda)\,. \tag{7}$$

The zero eigenvalue $\omega = 0$ is referred to as the zero mode and has well-known physical consequences. These modes always exist when one quantises a theory with a translationally invariant Lagrangian about a solution that is not translationally invariant. The physical consequence of the existence of the zero-mode is the centre-of-mass motion of the kink. In equation (7) the zero-mode contribution to the kink mass is omitted. That contribution to the energy means that the mass of the quantum kink particle is effectively assumed to be the same as the kink energy. The semiclassical treatment of these zero modes is discussed in a number of works (Dashen *et al.* 1974*a*; Goldstone and Jackiw 1975; Polyakov 1974; Christ and Lee 1975), however, we need not describe these in detail here. It is important to mention that the semiclassical quantisation of the discrete lattice version of the Lagrangian density given by equation (1) is also complicated by the zero-mode problem.

2. Kink on the Lattice

The $\lambda\phi^4$ action on a 2-d lattice can be written as

$$S = -\sum_{n,\mu}\phi_n\phi_{n+\mu} + \sum_n\left(2 - \frac{\widehat{\mu}^2}{2}\right)\phi_n^2 + \frac{\widehat{\lambda}}{4}\phi_n^4\,, \tag{8}$$

where we have defined the dimensionless quantities $\widehat{\mu} \equiv \mu a$ and $\widehat{\lambda} \equiv \lambda a^2$ with a being the lattice spacing. In addition $n \equiv (n_1, n_2)$ is a 2-d vector labeling the lattice sites and μ is a unit vector in the temporal or spatial direction (not to be confused with our parameter $\widehat{\mu}^2 \equiv -\widehat{m}^2$). We have also denoted the field on the neighbouring site of n in the direction of μ by $\phi_{n+\mu}$.

This model exhibits two phases. In some regions of the phase space $\langle\phi\rangle = 0$ and these are the symmetric (unbroken symmetry) regions, whereas in other regions spontaneous symmetry breaking occurs and $\langle\phi\rangle \neq 0$. Classically, for positive values of $\widehat{m}^2 \equiv -\widehat{\mu}^2$ one always has $\langle\phi\rangle = 0$ and for negative $\widehat{m}^2$ (i.e. positive $\widehat{\mu}^2$), where spontaneous symmetry breaking occurs, $\langle\phi\rangle \neq 0$. In this

regime the second order critical line which separates the two phases is the line corresponding to $\widehat{m}^2 = 0$. Beyond the classical limit the phase space structure changes. There are still two phases and there is still a second order phase critical line separating these phases; however, the position of the critical line changes and due to quantum fluctuations washing out 'shallow' spontaneous symmetry breaking it occurs at a finite negative $\widehat{m}^2$ in general.

In order to determine the critical line, we choose several values of $\widehat{m}^2 < 0$ located in the broken symmetry sector where $\langle\phi\rangle \neq 0$. For each value of $\widehat{m}^2$, $\widehat{\lambda}$ can be increased until $\langle\phi\rangle = 0$ and, thus, the critical parameters $(\widehat{m}_c^2, \widehat{\lambda}_c)$ can be found. Of course there is no second order phase transition on a finite lattice; however, by a second order phase transition here we mean that the correlation length is much larger than the lattice dimensions. The critical line is shown in Fig. 1. We have also shown the one loop prediction for the critical line using the light-front formulation (Bender *et al.* 1993).

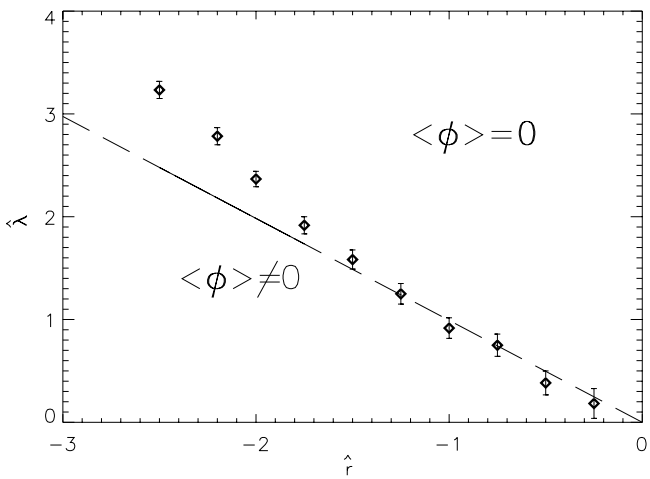

Fig. 1. Plot of the transition line between the broken sector and unbroken sector using Monte Carlo methods (diamonds) and the light-front perturbative predictions (dashed line). We have used the symbol $\widehat{r} \equiv \widehat{m}^2$.

Two methods were proposed by Ciria and Tarancon (1994) for calculating the kink mass on the lattice (it should be noted that in this reference there was an error in the presentation of the semiclassical results, which were shown to be smaller than they actually are). Here we use one of these methods which uses a local parameter and is much less susceptible to finite size effects. In this method one imposes an anti-periodic spatial boundary condition, giving rise to a topological excitation with non-zero topological charge. Since the kink has the lowest energy in the topological sector this topological excitation corresponds to the kink. It is shown that for a a fixed $\widehat{m}$, one has (Ciria and Tarancon 1994):

$$M_{\rm sol}(\widehat{\beta}) = \int_{\beta_c}^{\widehat{\beta}} d\beta' \Omega(\widehat{\beta}') \equiv \frac{1}{T} \int_{\widehat{\beta}_c}^{\widehat{\beta}} d\beta' \frac{1}{\beta'} \left[\langle S_a\rangle - \langle S_p\rangle\right] , \tag{9}$$

where $M_{\rm sol}$ is the soliton mass, $\widehat{\beta} = 1/\widehat{\lambda}$, $\widehat{\beta}_c$ is the inverse of the dimensionless critical bare coupling $\widehat{\lambda}_c$, T is the length of the lattice in the temporal direction, and $\langle S_p \rangle$ and $\langle S_a \rangle$ are the mean action of the system with a periodic and anti-periodic spatial boundary condition, respectively.

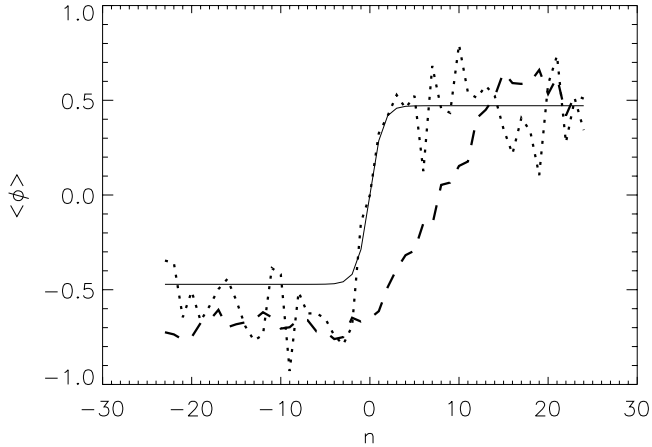

Fig. 2. Comparative plot of averages of fields on a time slice with $f = 0\cdot5$ and $\widehat{m}^2 = -1$ for an unconstrained lattice (dashed line) and a constrained lattice (dotted line). The constraint fixes the centre of the kink to the centre of the lattice for all time slices. The solid line is the classical solution.

As we mentioned earlier, in the semiclassical quantisation one encounters the zero-mode problem with its physical consequences being the energy contribution associated with the centre-of-mass motion of the kink. An interesting question is whether this problem persists beyond the semiclassical regime. To answer whether the zero-mode problem persists beyond the semiclassical regime, one can examine one of the consequences of the existence of a zero mode, that is, the kink displacements. On a lattice with anti-periodic boundary condition in the special direction, we set $\widehat{m}^2 \equiv -\mu^2 = -1$ and $\widehat{\lambda} = 4$ giving $f = 0\cdot5$. These parameters were chosen because they which corresponds to a region beyond the semiclassical regime. Then, for an arbitrary time slice, we took the value of the field at each site (ϕ_n) for a number of configurations and an average over these configurations was calculated in order to improve the signal-to-noise ratio. We have shown the results in Fig. 2. The movement of the kink due to the translational mode clearly persists beyond the semiclassical regime, as is evidenced by the blurring of the kink shape (cf. the classical shape). This occurs because when averaging over different (unconstrained) kink configurations, we are averaging over fluctuations of the kink centre which tends to flatten and blur out the otherwise relatively sharp kink shape. We repeated the same procedure for different time slices and different number of sampled configurations and these results showed the same behaviour.

In order to verify our interpretation of these results we imposed the constraint $\phi(M) = 0$ with $M = (x_0, N/2)$ for all times x_0 and then repeated our calculations,

i.e. we imposed a constraint which fixed the kink centre to the centre of the lattice. For the same time slices as in the previous case and for a similar number of constrained configurations, the average value of the field for each site was again calculated. These results are also included in Fig. 2. The constrained lattice configuration indeed more closely resembles the classical kink solution with its centre located at the centre of the lattice.

Both in numerical MC studies and the analytical calculations it is important to find the renormalisation group trajectories (RGT). Along these curves and close to an infrared (IR) fixed point the physics described by the lattice regularised quantum field is constant and only the value of the cut-off (lattice spacing) is changing. This region is called the scaling region. The $\lambda\phi^4$ theory in 2-d is an interacting theory. That is, in addition to having a Gaussian fixed point at $\widehat{m}^2 = \widehat{\lambda} = 0$, where the renormalised coupling $\widehat{\lambda}_r$ vanishes, it has other infrared fixed points at which $\widehat{\lambda}_r$ is non-zero. The best candidates for these fixed points are along the critical line where a second order phase transition occurs. In this model the only non-trivial critical region is along the transition line shown in Fig. 1. In the vicinity of the fixed point the vertex functions strongly scale (Zinn-Justin 1993) and one expects that close to the critical line, there should be segments of phase space where the ratios of dimensionless vertex functions remain essentially constant, indicating the scaling region.

In our calculations it is important to find the scaling region corresponding to a non-trivial IR fixed point. That is, one should try to find trajectories away from the Gaussian fixed point. On trivial fixed point trajectories, even though spontaneous symmetry breaking can still occur and hence a kink solution can exist, the vacuum is governed by a free field. We used $R(\widehat{m}_r, \widehat{\lambda}_r) \equiv \widehat{m}_r^2/\widehat{\lambda}_r$ as a dimensionless parameter for probing the scaling region. This parameter R can be calculated accurately using the effective potential method (Ardekani and Williams 1998). The scaling region corresponds to regions where R is approximately constant. Our entire calculations are performed within the scaling region so that they can be legitimately compared with the continuum-semiclassical predictions for the soliton mass, given by equation (7).

For a fixed value of $\widehat{m}^2 = -1$ we found a range of values of $\widehat{\lambda}$ in the broken symmetry sector, that is $0{\cdot}2 < \widehat{\lambda} < 0{\cdot}8$, for which the values of $R(\widehat{m}_r, \widehat{\lambda}_r)$ were approximately constant, determining a segment of the scaling region corresponding to $\widehat{m}^2 = -1$. Then, for some values of $\widehat{\lambda}$ within this region we calculated $\Delta S = \langle S_a \rangle - \langle S_p \rangle$. We have plotted $\Delta S/T$ versus $\widehat{\beta}$ along with its classical value in Fig. 3. Note that having obtained $\Delta S/T$, equation (9) can then be used to calculate the soliton mass. We calculated the soliton mass with and without imposing a constraint on the centre of the kink. The comparison of these results with each other and with the classical and semiclassical values is shown in Fig. 4. As is evident, for these parameters choices the zero-mode contribution cannot be resolved within statistical uncertainties. The statistical uncertainties, as one expects, increase as one approaches the critical line which complicates the detection of the zero-mode contribution to the soliton mass. It is interesting to note that the Monte Carlo results for the soliton mass are less than the classical mass but larger than one loop semiclassical predictions.

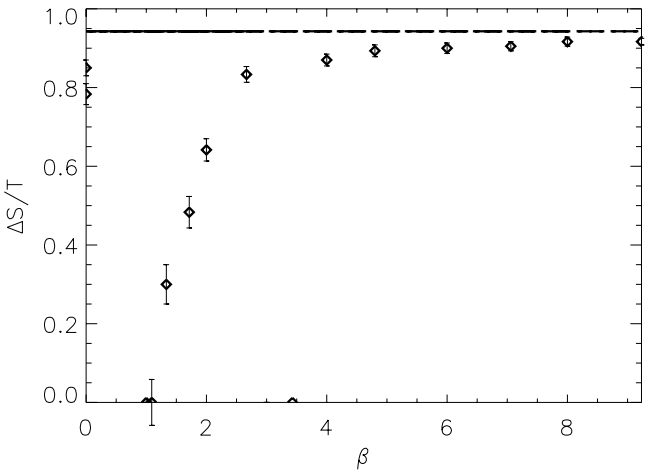

Fig. 3. Plot of $\Delta S/T$ versus $\beta = 1/\widehat{\lambda}$ with $\widehat{m}^2 = -1\cdot 0$. The straight horizontal line is the classical value for $\Delta S/T$.

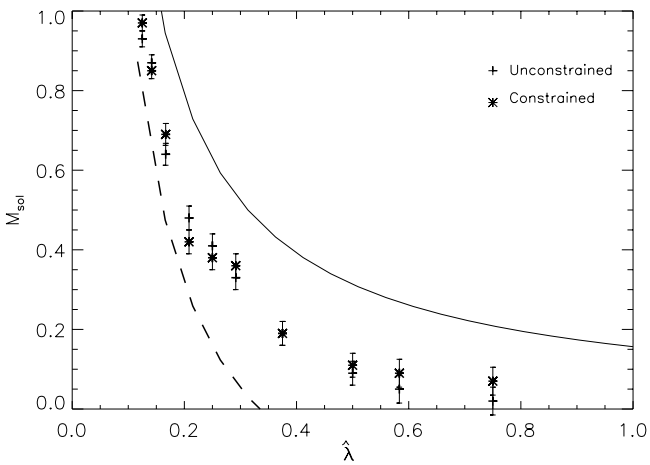

Fig. 4. Plot of soliton mass versus $\widehat{\lambda} = 1/\beta$ with $\widehat{m}^2 = -1$. The Monte Carlo results are compared with the classical (solid line) and semiclassical results (dashed line).

Next, for $\widehat{m}^2 = -2\cdot 2$, we repeated the same procedure and calculated the soliton mass for a number of bare couplings within the scaling region. The calculations were again done with and without a constraint on the centre of the kink and the results and their comparison with the classical and semiclassical predictions are shown in Fig. 5. Our results are consistent with Ciria and Tarancon (1994). There are two important observations. First, unlike the previous case the MC calculated soliton masses are lower than the semiclassical values which must be due to the higher order corrections. The other important observation is that, close to the critical line, it appears to be clear that there is a positive contribution to the soliton mass due to the zero-mode.

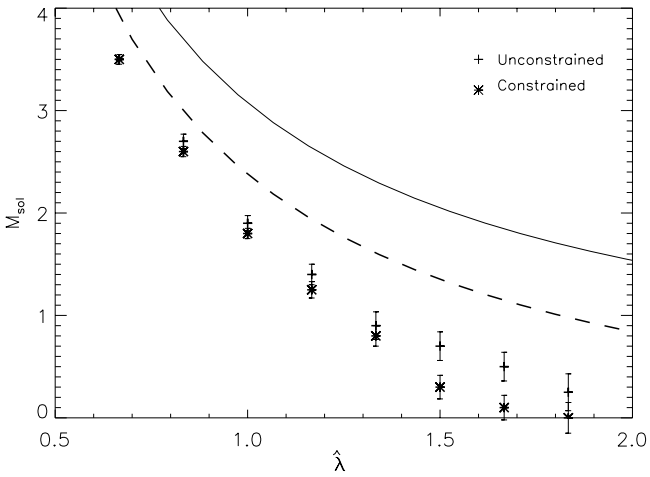

Fig. 5. Plot of soliton mass versus $\widehat{\lambda} = 1/\beta$ with $\widehat{m}^2 = -2\cdot 2$. The Monte Carlo results are compared with the classical (solid line) and semiclassical results (dashed line).

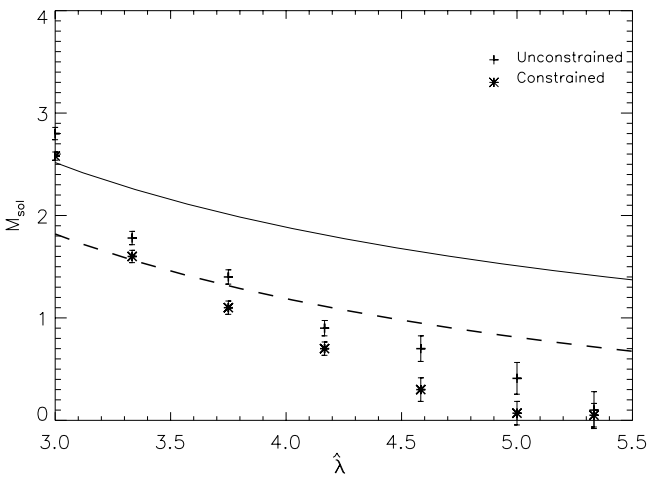

Fig. 6. Plot of soliton mass versus $\widehat{\lambda} = 1/\beta$ with $\widehat{m}^2 = -4$. The Monte Carlo results are compared with the classical (solid line) and semiclassical results (dashed line).

Finally, for $\widehat{m}^2 = -4$ we again calculated the soliton mass, with and without a constraint on the configurations. The results are shown in Fig. 6 and again the zero-mode contribution to the soliton mass appears to be positive.

All our calculations were done on a 48×48 lattice. As one approaches the critical line the correlation length increases and the finite size effects become more significant. However, since the Monte Carlo calculation of masses are based on a local parameter $\Omega(\beta)$ the finite size effects are smaller than one might, in general, expect (Groeneveld *et al.* 1981). In our calculations we kept the correlation length below a half of the lattice length.

We note that in addition to the straightforward elimination of the zero-mode degree of freedom, the imposition of a constraint on the centre of the kink resulted also in more stable configurations and a reduction of the statistical uncertainties on $\langle S_a \rangle$ and consequently on $M_{\rm sol}$. These instabilities are more significant close to the critical line where the field fluctuations are larger. In general we found that the statistical uncertainties on $\langle S_a \rangle$ were much larger than on $\langle S_p \rangle$.

3. Summary and Conclusions

We have made an exploratory study of the quantisation of extended objects using the tools provided by Monte Carlo lattice techniques. In particular, we have studied the simplest case, the kink soliton in $1+1$ dimensions. Where our lattice results could be compared with previous studies (Ciria and Tarancon 1994), they were found to be in agreement. Careful attention was paid to the issue of the zero-mode or centre-of-mass motion problem by comparing results for kinks with fixed centres and for kinks whose centres were left free to fluctuate. We found that for the cases studied here, the energy associated with the centre-of-mass motion appears to be positive. We also ensured that our lattice results were obtained in the non-trivial scaling region and with sufficiently large lattice volumes. Hence meaningful comparisons with the continuum classical and semiclassical results could be made. Having established the feasibility of this approach, future detailed studies should explore a larger range of parameter space with increased statistics and should attempt to fully characterise the differences between the full and semiclassical quantisation of the kink. Once that is done, it would be extremely interesting to turn our attention to the more physically interesting question of the quantisation of solitons in higher dimensions, such as those which arise in theories of elementary particle physics.

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