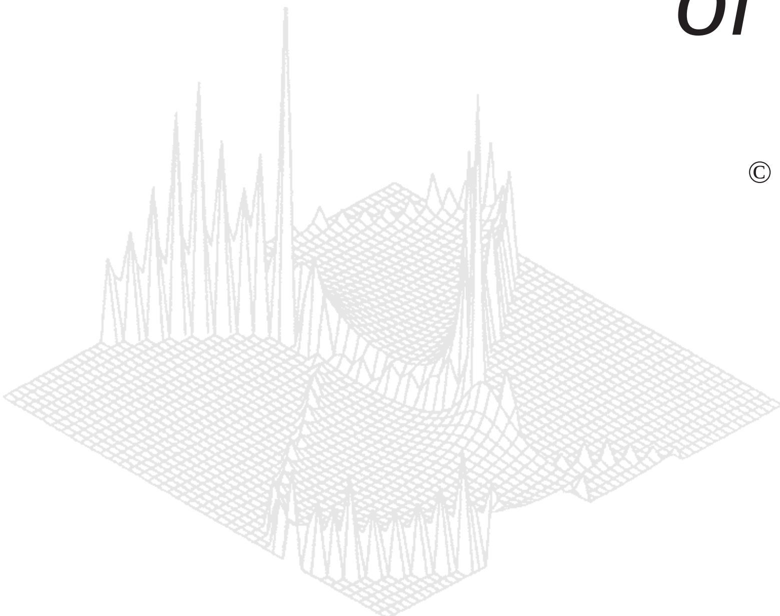
CSIRO PUBLISHING

Australian Journal of Physics



Volume 53, 2000

A journal for the publication of
original research in all branches of physics

www.publish.csiro.au/journals/ajp

All enquiries and manuscripts should be directed to
Australian Journal of Physics
CSIRO PUBLISHING
PO Box 1139 (150 Oxford St)
Collingwood
Vic. 3066
Australia
Telephone: 61 3 9662 7626
Facsimile: 61 3 9662 7611
Email: peter.robertson@publish.csiro.au

Published by **CSIRO** PUBLISHING
for CSIRO Australia and
the Australian Academy of Science

Aust. J. Phys., 2000, **53**, 447–52

The Bose–Einstein Condensation of Anyons

Yong Liu

Laboratory for Numeric Study of Heliospheric Physics (LHP),
Chinese Academy of Sciences, PO Box 8701, Beijing 100 080, P. R. China.
email: yongliu@ns.lhp.ac.cn

Abstract

The probability for the Bose–Einstein condensation of anyons is discussed. It is found that the ideal anyon gas near Bose statistics can display BEC behaviour. In addition, the transition point and the specific heat are determined.

The recent successful experimental observations on Bose–Einstein condensation (BEC) in ultracold trapped atomic gases (Anderson *et al.* 1995; Bradly *et al.* 1995; Davis *et al.* 1995) have created great interest. Many papers on the formation, evolution, interference of the BEC and related topics have been published (see e.g. Grossmann and Holthaus 1995; Castin and Dunn 1996; Kagan *et al.* 1997; Hostin and You 1996).

Although the condensation of an ideal Bose gases has been written about in textbooks for many years and imperfect Bose gases with a hard-sphere interaction were investigated over forty years ago (Huang *et al.* 1957*a*, 1957*b*; Lee and Yang 1958), only an ideal gas within a box or confined by a trap has been studied for the low-dimensional case (de Groot *et al.* 1950; Ketterle and Van Druten 1996; Bagnato and Kleppner 1991). In the two-dimensional case, Hohenberg (1967) has shown that a BEC cannot occur in an ideal system, but Bagnato and Kleppner (1991) argued that, if the system is confined by a spatially varying potential, i.e. the 'trapping' potential, a BEC can in principle occur. It is natural to ask whether a BEC can occur in an interacting two-dimensional gas.

However, particles obeying fractional statistics (Leinaas and Myrheim 1977; Wilczek 1982; Wilczek and Zee 1983; Wu 1984, 1994; Wu and Zee 1984; Goldhaber and Mackenzie 1984; Haldane 1991; Sen and Bhaduri 1995) can be considered as an interacting system (Sen 1991), and for the two-dimensional anyon gas near Bose statistics, we can expect that a BEC may occur in such a system. In what follows, we will show that this is indeed the case.

To make this paper self-contained, let us repeat the derivation of the equations of state of an anyon gas. All the calculations follow Chaichian *et al.* (1993).

The Hamiltonian of N identical anyons is given by

$$H = \sum_{i=1}^{N} \frac{1}{2m} (\vec{P}_i - \vec{A}_i)^2 \,, \tag{1}$$

where

$$\vec{A}_i = \alpha \sum_{i \neq j} \frac{\vec{k} \times \vec{r_{ij}}}{\vec{r_{ij}}^{\,2}} \qquad (\vec{r_{ij}} \equiv \vec{r_i} - \vec{r_j}) \tag{2}$$

 10.1071/PH99062 0004-9506/00/030447$05.00

is the statistical gauge field. Here $\vec{k}$ is a unit vector perpendicular to the plane and $\alpha = e\phi/\pi$ is the statistical parameter, while e and ϕ are the charge and the flux carried by each anyon.

We define the N-body wave function $\Psi_p(\vec{r_1}, ..., \vec{r_N})$ as

$$\Psi_p(\vec{r_1}, ..., \vec{r_N}) = \prod_{i<j} r_{ij}^{\gamma} \tilde{\Psi}_p(\vec{r_1}, ..., \vec{r_N}) \,. \tag{3}$$

Then for the new function $\tilde{\Psi}_p$, the Hamiltonian reads

$$\tilde{H} = \sum_{i=1}^{N} \left\{ \frac{\vec{P_i}^2}{2m} + \frac{i\alpha}{m} \sum_{j\neq i} \frac{\vec{k} \times \vec{r_{ij}}}{\vec{r_{ij}}^2} \partial_i - \frac{|\alpha|}{m} \sum_{j\neq i} \frac{\vec{r_{ij}}}{\vec{r_{ij}}^2} \partial_i \right\} . \tag{4}$$

Now the singular α^2 interaction terms such as $(1/\vec{r_{ij}}^2)$ in (1) are cancelled in the present form equation (4). Notice that due to the symmetry of the free spectrum under the change $\vec{P_x} \leftrightarrow \vec{P_y}$, the second term in (4) can be shown to vanish. Finally, the $|\alpha|$ interacting term in (4) can be replaced (in a perturbative sense) by a sum of two-body δ potentials (Sen 1991):

$$V = \frac{2\pi|\alpha|}{m} \sum_{i<j} \delta^2(\vec{r_i} - \vec{r_j}) \,. \tag{5}$$

The temperature Green function in momentum space can be derived from the Dyson equation as

$$G(\vec{k}, \omega_n) = G^0(\vec{k}, \omega_n) + G^0(\vec{k}, \omega_n)\Sigma(\vec{k}, \omega_n)G(\vec{k}, \omega_n) \,, \tag{6}$$

where Σ is the proper self-energy and G^0 the free propagator,

$$G^0 = \frac{1}{i\omega_n - (\epsilon_k^0 - \mu)} \,, \tag{7}$$

with $\omega_n = 2n\pi/\beta$, $\epsilon_k^0 = \vec{k}^2/2m$ and μ is the chemical potential.

The first order self-energy is given by (Fetter and Walecka 1971)

$$\Sigma_{(1)}(\vec{k}, \omega_n) \equiv \Sigma_{(1)}(\vec{k}) = V(0) \int \frac{d^2\vec{k}}{(2\pi)^2} n_{\vec{k}}^0 + \int \frac{d^2\vec{k'}}{(2\pi)^2} V(\vec{k} - \vec{k'}) n_{\vec{k'}}^0 \,, \tag{8}$$

where

$$n_{\vec{k}}^0 = \frac{1}{e^{\beta(\epsilon_{\vec{k}}^0 - \mu)} - 1} \tag{9}$$

is the usual Bose distribution and

$$V(\vec{k}) = \frac{2\pi}{m} |\alpha| \tag{10}$$

is the two-particle interacting potential in momentum space.

Now, it is easy to see that

$$\begin{aligned}\Sigma_{(1)} &= \frac{4\pi}{m} \int \frac{d^2\vec{k}}{(2\pi)^2} \frac{1}{z^{-1} e^{\beta(\epsilon^0_{\vec{k}} - \mu)} - 1} \\ &= \frac{2|\alpha|}{\beta} \int_0^\infty dx \, \frac{z}{e^x - z} \\ &= -\frac{2|\alpha|}{\beta} \ln(1-z) \, . \end{aligned} \tag{11}$$

Since $\Sigma_{(1)}$ is independent of ω_n, replacing the free energy $\epsilon^0_{\vec{k}}$ by $\epsilon^0_{\vec{k}} + \Sigma_{(1)}$ we have the particle distribution up to the first order of α. Then the particle density is

$$\begin{aligned} n &= \int \frac{d^2\vec{k}}{(2\pi)^2} \, n_{\vec{k}} \\ &= -\frac{1}{\lambda^2} \ln(1-z) \left\{ 1 - \frac{2|\alpha| z}{1-z} \right\} , \end{aligned} \tag{12}$$

while the pressure p reads

$$\begin{aligned} p\beta &= \int dz \frac{n}{z} \\ &= \frac{1}{\lambda^2} \left\{ g_2(z) - |\alpha| \ln^2(1-z) \right\} , \end{aligned} \tag{13}$$

where $\lambda = \sqrt{h^2/2\pi mkT}$ and

$$g_x(z) = \sum_{l=1}^{\infty} \frac{z^l}{l^x} \, . \tag{14}$$

The same results (12) and (13) were evaluated by Comtet *et al.* (1991). Although these results are the first-order approximation of α, we will take them as the exact ones in the following. In fact, n should be replaced by n_e ($\equiv n - n_0$) in equation (12). Here n_0 represents the particle density staying in the ground state ($\vec{k} = 0$, $\epsilon_{\vec{k}} = 0$). The reason is stated in standard textbooks (Huang 1963; Pathria 1972).

Now, we can show that $(n - n_0)$ given by (12) has a maximum at a certain temperature. We define

$$f(z) = -\ln(1-z) \left\{ 1 - \frac{2|\alpha| z}{1-z} \right\} , \tag{15}$$

and then

$$n - n_0 = \frac{1}{\lambda^2} f(z) \, . \tag{16}$$

To ensure $(n - n_0) \geq 0$, we find that z must satisfy

$$0 \leq z \leq \frac{1}{1 + 2|\alpha|} \, . \tag{17}$$

It is easy to show that $f(z)$ has a maximum in this domain. The point at which $f(z)$ arrives at maximum is determined by

$$2|\alpha|\ln(1-z_0)-(1+2|\alpha|)z_0+1=0\,. \tag{18}$$

Evidently, z_0 is a function of $|\alpha|$.

The physical meaning of the maximum point is that the number of particles that can stay in the excited states is limited. Once the excited states are fully occupied, all leftover particles will be pushed into the ground state, and then a BEC occurs.

If we let $z=z_0$ for a certain number of particles, the transition point T_c can be determined by

$$T_c=\frac{h^2}{2\pi mk}\,\frac{n}{f(z_0)}\,. \tag{19}$$

Finally, let us investigate the behaviour of the specific heat. According to the well-known formula of thermodynamics, the energy of the system is given by

$$\begin{aligned} U &\equiv -\left(\frac{\partial}{\partial\beta}\ln\mathcal{Z}\right)_{z,V} \\ &= kT^2\left\{\frac{\partial}{\partial T}\left(\frac{PV}{kT}\right)\right\}_{z,V}\,, \end{aligned} \tag{20}$$

and the specific heat by

$$\frac{C_v}{Nk}=\frac{1}{Nk}\left(\frac{\partial U}{\partial T}\right)_{N,V}\,. \tag{21}$$

When $T\le T_c$, replacing z by z_0, we have

$$p\beta=\frac{2\pi mkT}{h^2}\left\{g_2(z_0)-|\alpha|\ln^2(1-z_0)\right\}, \tag{22}$$

$$\frac{U}{V}=\frac{2\pi mk^2T^2}{h^2}\left\{g_2(z_0)-|\alpha|\ln^2(1-z_0)\right\}, \tag{23}$$

$$\frac{C_v}{Nk}=2\,\frac{T}{T_c}\,\frac{1}{f(z_0)}\left\{g_2(z_0)-|\alpha|\ln^2(1-z_0)\right\}. \tag{24}$$

When $T\ge T_c$, the evaluation is a little more complicated. From

$$p\beta=\frac{2\pi mkT}{h^2}\left\{g_2(z)-|\alpha|\ln^2(1-z)\right\} \tag{25}$$

we obtain

$$\frac{U}{V}=\frac{2\pi mk^2T^2}{h^2}\left\{g_2(z)-|\alpha|\ln^2(1-z)\right\}, \tag{26}$$

$$\frac{C_v}{Nk} = 2\frac{T}{T_c}\,\frac{1}{f(z_0)}\left\{g_2(z) - |\alpha|\ln^2(1-z)\right\}$$
$$+\frac{T^2}{T_c}\,\frac{1}{f(z_0)}\left\{\frac{\partial}{\partial z}\left[g_2(z) - |\alpha|\ln^2(1-z)\right]\right\}\left(\frac{\partial}{\partial T}\,z\right)_n\,, \tag{27}$$

where $((\partial/\partial T)z)_n$ can be deduced from (12),

$$\left(\frac{\partial}{\partial T}\,z\right)_n = \ln(1-z)\{1-(1+2|\alpha|)z\}\Big/T\left\{1-\frac{2|\alpha|z}{1-z}+\frac{2|\alpha|}{1-z}\,\ln(1-z)\right\}. \tag{28}$$

Substituting this result into equation (27), we get

$$\frac{C_v}{Nk} = 2\,\frac{T}{T_c}\,\frac{1}{f(z_0)}\left\{g_2(z) - |\alpha|\ln^2(1-z)\right.$$
$$\left.-\frac{\ln^2(1-z)}{2z}\,\frac{[1-(1+2|\alpha|)z]^2}{1-(1+2|\alpha|)z+2|\alpha|\ln(1-z)}\right\}. \tag{29}$$

Here z is given by equation (12), i.e.

$$n = -\frac{2\pi mkT}{h^2}\,\ln(1-z)\left\{1-\frac{2|\alpha|z}{1-z}\right\}. \tag{30}$$

From (30), it is easy to find that for each T there are two values of z in the domains $(0, z_0)$ and $(z_0, 1/(1+2|\alpha|))$ corresponding to certain n. If z takes values in the domain $(z_0, 1/(1+2|\alpha|))$, then nothing will happen. However, the specific heat C_v may become negative for certain z in $[0, z_0]$, which can be seen from (29). When $T \to T_c$, z may approach z_0 from two directions in which case $z > z_0$ and $z < z_0$. When $z \to z_0$ from the direction of $z > z_0$, then $C_v \to +\infty$. But if $z \to z_0$ from the direction of $z < z_0$, then $C_v \to -\infty$. So, we find that the domain $(0, z_0)$ has no physical meaning and should be excluded. The discontinuity of the specific heat at $T = T_c$ is evident. But the presence of the singularity of the specific heat when $T \to T_c^+$ is difficult to understand. Perhaps the reason is that the approximation taken here is not precise enough. This needs to be studied further.

In conclusion, we have investigated the related properties of a two-dimensional ideal anyon gas. We find that, when we approximate to first order in the statistical parameter α, the ideal anyon gas near Bose statistics can display BEC behaviour. In addition, the transition point and the specific heat are determined.

Because the effect of the statistical gauge potential is equivalent to a two-body δ interaction potential when we approximate to first order, we can expect that a δ interacting system may display BEC behaviour in the sense of perturbation. Further work on this will be reported elsewhere.

References

Anderson, M. H., Ensher, J. R., Matthews, M. R., Wieman, C. E., and Cornell, E. A. (1995). *Science* **269**, 198.
Bagnato, V., and Kleppner, D. (1991). *Phys. Rev.* A **44**, 7439.
Bradly, C. C., Sackett, C. A., Tollet, J. J., and Hulet, R. G. (1995). *Phys. Rev. Lett.* **75**, 1687.
Castin, Y., and Dum, R. (1996). *Phys. Rev. Lett.* **77**, 5315.
Chaichian, M., Felipe, R. G., and Montonen, C. (1993). *J. Phys.* A **26**, 4017.
Comtet, A., McCabe, J., and Duvry, S. (1991). *Phys. Lett.* B **260**, 372.
Davis, K. B., Mewes, M. O., Andrews, M. R., Van Druten, N. J., Durfee, D. S., Kurn, D. M., and Ketterle, W. (1995). *Phys. Rev. Lett.* **75**, 3969.
de Groot, S. R., Hooyman, G. J., and ten Seldam, C. A. (1950). *Proc. R. Soc. London* A **203**, 266.
Fetter, A. L., and Walecka, J. D. (1971). 'Quantum Theory of Many-particle Systems' (McGraw–Hill: New York).
Goldhaber, A. S., and Mackenzie, R. (1984). *Phys. Lett.* B **214**, 271.
Grossmann, S., and Holthaus, M. (1995). *Phys. Lett.* A **208**, 188.
Haldane, F. D. M. (1991). *Phys. Rev. Lett.* **67**, 937.
Honhenberg, P. C. (1967). *Phys. Rev.* **158**, 383.
Hoston, W., and You, L. (1996). *Phys. Rev.* A **53**, 4254.
Huang, K. (1963). 'Statistical Mechanics' (Wiley: New York).
Huang, K., Yang, C. N., and Luttinger, J. M. (1957*a*). *Phys. Rev.* **105**, 776.
Huang, K., Yang, C. N., and Luttinger, J. M. (1957*b*). *Phys. Rev.* **105**, 767.
Kagan, Y., Surkov, E. L., and Shlyapnikov, G. V. (1997). *Phys. Rev.* A **55**, R18.
Ketterle, W., and Van Druten, N. J. (1996). *Phys. Rev.* A **54**, 656.
Lee, T. D., and Yang, C. N. (1958). *Phys. Rev.* **112**, 1419.
Leinaas, J., and Myrheim, J. (1977). *Nuovo Cimento* B **37**, 1.
Pathria, R. K. (1972). 'Statistical Mechanics' (Pergamon: London).
Sen, D. (1991). *Nucl. Phys.* B **360**, 397.
Sen, D., and Bhaduri, R. K. (1995). *Phys. Rev. Lett.* **74**, 3912.
Wilczek, F. (1982). *Phys. Rev. Lett.* **48**, 1144; **49**, 957.
Wilczek, F., and Zee, A. (1983). *Phys. Rev. Lett.* **51**, 2250.
Wu, Y.-S. (1984). *Phys. Rev. Lett.* **52**, 2103.
Wu, Y.-S. (1994). *Phys. Rev. Lett.* **73**, 922.
Wu, Y.-S., and Zee, A. (1984). *Phys. Lett.* B **147**, 325.

Manuscript received 2 July, accepted 23 November 1999