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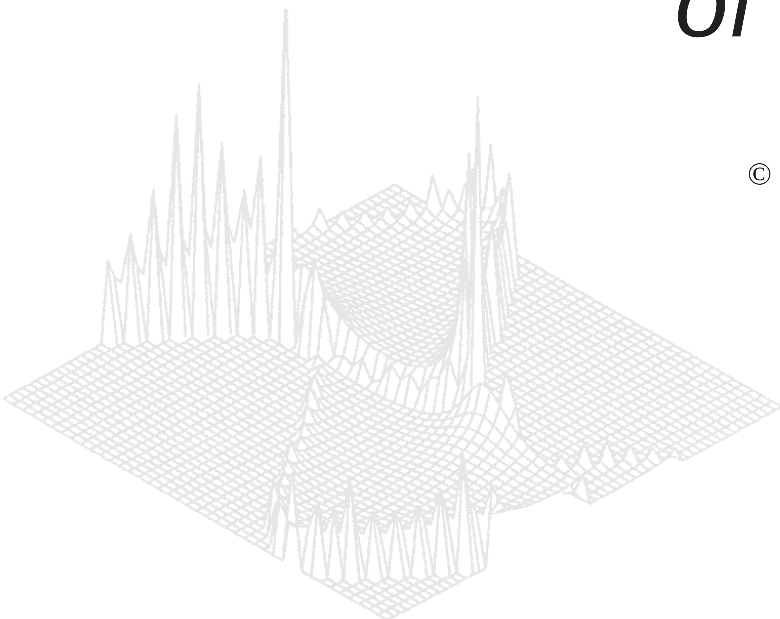

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Magnetic Fields in Spaces with $\mathrm{VII}_0 \times \mathrm{VIII}$ Isometries

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Abstract

The aim of the present paper is to investigate some globally pathological features of a class of static planary symmetric exact solutions with a G_6-group of motion, namely with $g_{44} = -\sinh^2(\alpha z)$, by means of the null oblique geodesics and Penrose diagram. Finally, we derive general expressions for the $A_\mu(x, y, z)_{\mu=\overline{1,3}}$ components of the vector potential, satisfying the source-free Maxwell equations and the Lorentz condition, pointing out the influence of the global pathological properties on the behaviour of magnetostatic fields in such universes.

1. The Geometry of the Model

The properties of globally pathological manifolds and the behaviour of different matter sources in such universes have always been of real interest because of their implications for a better understanding of gravity and spacetime (Brans and Dicke 1961; Hoyle and Narlikar 1964; Guth 1981; Linde 1982; Collins *et al.* 1989). In addition to naked singularities, cosmic strings, Bianchi spacetimes, dynamical isotropisation, hollow cylinders and topological domain walls (Vilenkin and Shellard 1994; Clément and Zouzou 1994; Wang and Letelier 1995 and references therein), black holes in less than four dimensions have been recently investigated. In this respect, the geodesic motion in BTZ black holes, with curvature-regular spacetimes but strongly singular in their causal structure (Banados *et al.* 1993), leads to somewhat unexpected features. For instance, in the $2+1$ anti-de Sitter universe, this new type of black hole can only differ from the background in its global properties through identification of points by means of some *discrete* subgroup of its isometries. The point is that some past-continuations go in closed timelike curves and/or additional Taub–Nut pathologies at the metric singular 'point'.

The aim of this paper is to investigate the global pathology of a class of static planary symmetric exact solutions with a G_6-group of motion and to derive the essential components of a magnetostatic field in this universe.

As investigated previously (Dariescu *et al.* 1997), we deal with the metric

$$ds^2 = \delta_{\mu\nu}\, dx^\mu dx^\nu - e^{2f(x^3)}(dt)^2 \tag{1}$$

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proposed for the uniform and galactic fields (Romain 1963). Introducing the dually-related pseudo-orthonormal tetrads $\{e_a, \omega^a\}_{a=\overline{1,4}}$ as

$$\vec{e}_\mu = \partial_\mu \,, \; e_4 = e^{-f(z)}\partial_t \,; \;\; \omega^\mu = dx^\mu \,, \; \omega^4 = e^{f(z)}dt \tag{2}$$

and employing the Cartan formalism it yields

$$R_{3434} = -R_{33} = R_{44} = -\,\tfrac{1}{2}R = f_{|33} + (f_{|3})^2 \tag{3}$$

and the essential components of the Einstein tensor

$$G_{AB} = \left[f_{|33} + (f_{|3})^2\right]\delta_{AB} \,, \quad A, B = 1, 2 \,. \tag{4}$$

It clearly results in the $(2+2)$-decomposition $M_4 = \mathbf{R}^2 \times M_2$, while $G_{44} = 0$ suggests (besides the vacuum as the only *conventional* source) a combined matter-source with the total energy–momentum tensor given by

$$T_{ab} = \lambda[\eta_{a4}\,\eta_{b4} - \eta_{a3}\,\eta_{b3} + \eta_{ab}] \,, \text{ with } \lambda = \text{constant} > 0 \,. \tag{5}$$

This could describe a universal dust, with $\rho = \lambda$, stuck on a z-directed global cosmic string of negative 'tension' $\mu = -\lambda$ imbedded in a static universe of negative cosmological constant, $\Lambda = -\kappa_0\,\lambda$. With (4) and (5), the Einstein equations turn into

$$f_{|33} + (f_{|3})^2 = \alpha^2 \,, \text{ where } \alpha = (\kappa_0\lambda)^{\frac{1}{2}} \,, \tag{6}$$

whose general solution

$$f(z) = \ln\left[c_+ e^{\alpha z} + c_- e^{-\alpha z}\right] \,, \tag{7}$$

with particular choices for the constants $c_\pm$, brings the metric (1) to the 'hyperbolic' cases

$$ds^2 = \delta_{AB}\,dx^A\,dx^B + (dz)^2 - \sinh^2(\alpha z)(dt)^2 \tag{8}$$

and the one with $g_{44} = -\,\cosh^2(\alpha z)$, whose pathological properties have been the subject of previous investigations (Dariescu *et al.* 1997).

In the following we shall focus our attention on the metric (8), defined on $M_4 = \mathbf{R}^2 \times M_2 \subset \mathbf{R}^2 \times \mathbf{R} - \{0\} \times \mathbf{R}$, having $\{z = 0\}$ as a singular point. For the Killing vector fields one gets, besides the usual generators of VII_0 (which correspond to the Euclidian $\mathbf{R}^2$), the following generators:

$$X_{(4,5)} = e^{\pm\alpha t}\left[\partial_z \mp \coth(\alpha z)\partial_t\right]; \;\; X_{(6)} \,=\, \partial_t \tag{9}$$

of G_3' acting on M_2.

According to the Estabrook–Ellis–MacCallum method of enumerating all the G_3 groups (Kramer *et al.* 1980), our G_3', possessing the invariant properties $A_\mu = 0$, $N^{\mu\nu} = \frac{1}{2}\,C^{\mu}_{\cdot\,\alpha\beta}\,\varepsilon^{\alpha\beta\nu} \Rightarrow \text{rank}(N) = 3$ and $|\sigma| = 1$, belongs to the Bianchi type $VIII$ and consequently $G_6 = VII_0 \times VIII$.

2. Null Oblique Geodesics and Penrose Diagram

In order to investigate some of the globally pathological features of the spacetime described by (8), let us analyse the structure of 'oblique' null trajectories. From the 'optical' Lagrangian

$$\Phi = \sinh^{-2}(\alpha z)\,(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = 1\,, \tag{10}$$

one gets by taking a spherically symmetric light source in $(0,\, 0,\, z_0)$ at $t = 0$,

$$\dot{x} = \frac{\sin\chi\,\cos\lambda}{\sinh(\alpha z_0)}\,\sinh^2(\alpha z)\,,\quad \dot{y} = \frac{\sin\chi\,\sin\lambda}{\sinh(\alpha z_0)}\,\sinh^2(\alpha z)\,,$$

$$\dot{z} = \pm\,\sinh(\alpha z)\left[1 - \frac{\sin^2\chi\,\sinh^2(\alpha z)}{\sinh^2(\alpha z_0)}\right]^{\frac{1}{2}}\,, \tag{11}$$

where χ and λ are the usual angular coordinates on S^2,

For the upward (oblique) null trajectories, i.e. $0 \le \chi < \pi/2$, there always exist the turning points

$$z_* = \frac{1}{\alpha}\,\mathrm{arcsinh}\,\frac{\sinh(\alpha z_0)}{\sin\chi}\,, \tag{12}$$

while for the downward ones, with $\pi/2 < \chi = \pi - \gamma \le \pi$, it obviously results in

$$\alpha\,\rho = \arcsin\,\frac{\sin\gamma\,\cosh(\alpha z_0)}{\sqrt{\sinh^2(\alpha z_0) + \sin^2\gamma}} - \arcsin\,\frac{\sin\gamma\,\cosh(\alpha z)}{\sqrt{\sinh^2(\alpha z_0) + \sin^2\gamma}}\,, \tag{13}$$

with $0 \le z \le z_0$. As can be noticed, any of the trajectories intersects the $\{z = 0\} - \mathbf{R}^2$ singular plane within the range

$$0 \le \rho \le b\,,\quad \text{with } b = \frac{1}{\alpha}\left[\frac{\pi}{2} - \arcsin\,\frac{1}{\cosh(\alpha z_0)}\right]\,. \tag{14}$$

For the light rays emitted upward, with χ in between 0 and $\pi/2$, it yields by integrating $d\rho/dz$ from z_0 to $z \le z_*$,

$$\alpha\,\rho = \arcsin\,\frac{\sin\chi\,\cosh(\alpha z)}{\sqrt{\sinh^2(\alpha z_0) + \sin^2\chi}} - \arcsin\,\frac{\sin\chi\,\cosh(\alpha z_0)}{\sqrt{\sinh^2(\alpha z_0) + \sin^2\chi}}\,. \tag{15}$$

Reaching the turning point, each ray goes down toward the plane $\{z = 0\}$, following the equation

$$\rho(z) = \frac{1}{\alpha}\left(\pi - \arcsin \frac{\sin\chi \cosh(\alpha z_0)}{\sqrt{\sinh^2(\alpha z_0) + \sin^2\chi}} - \arcsin \frac{\sin\chi \cosh(\alpha z)}{\sqrt{\sinh^2(\alpha z_0) + \sin^2\chi}}\right). \tag{16}$$

Consequently, at $z = 0$, the disk of radius b flashed by the downward rays is subsequently extended by the circular sector of upper maximal radius π/α flashed by all of the incoming light rays which have already reached their turning points. The corresponding oblique null trajectories are shown in Fig. 1.

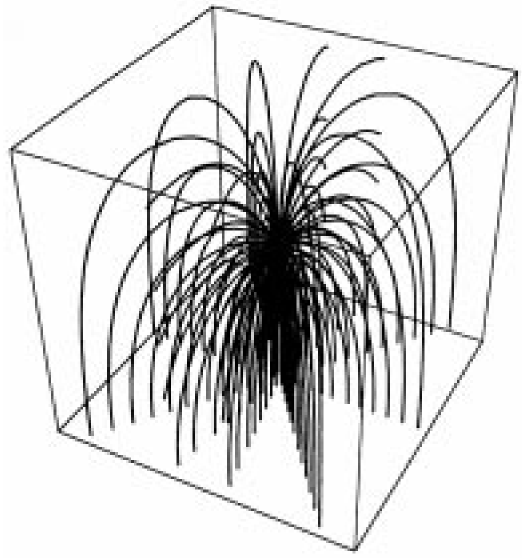

Fig. 1. The oblique null trajectories.

As for the Penrose diagram in Fig. 2, since the first submanifold in the decomposition of M_4 is the usual Euclidean $\mathbf{R}^2$, the global pathology, especially with respect to its conformal structure at infinity, will be mainly revealed by the Lorentzian M_2 submanifold of metric

$$ds_L^2 = (dz)^2 - \sinh^2(\alpha z)\,(dt)^2\,, \tag{17}$$

allowing us to define the compactified Penrose null coordinates

$$\underline{u} = \begin{cases} \pi + \arctan u_- \,, & z \le 0 \\ \arctan u_+ \,, & z \ge 0 \end{cases} \;;\quad \underline{v} = \begin{cases} -\pi + \arctan v_- \,, & z \le 0 \\ \arctan v_+ \,, & z \ge 0 \end{cases}$$

on the whole extension of M_2.

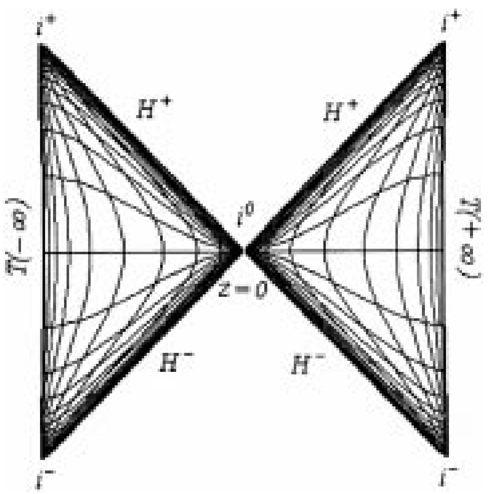

Fig. 2. The Penrose diagram.

There are no spatial infinities at $z = \mp\infty$. Instead, one gets the ultimate universal lines (of the *timelike* fundamental observers) joining i^- to i^+. Obviously, the z =const. timelike (universal) lines are not timelike geodesics. The latter are represented by concave lines, orthogonally joining the two null horizons H^- and H^+, exhibiting past and future event horizons respectively. These (timelike) geodesics practically never reach the $\{z = \mp\infty\}$ 2-planes and quite interestingly, with respect to the (z, t)-parametrisation, the role of the spacial infinity i^0 is actually played by the $\{z = 0\}$-space-like 2-surface.

3. The Magnetic Field

The source-free Maxwell equations

$$\Box A_a = g^{bc} A_{a;bc} = R_{ab} A^b \,, \tag{18}$$

in the case of a magnetostatic field become

$$\Delta A_B + \alpha \coth(\alpha z) \frac{\partial A_B}{\partial z} = 0, \; B = 1, 2 \tag{19}$$

$$\Delta A_3 + \alpha \coth(\alpha z) \frac{\partial A_3}{\partial z} - \frac{\alpha^2}{\sinh^2(\alpha z)} A_3 = 0 \,. \tag{20}$$

In the simplest case $n = 1, 2, 3, \ldots$, introducing the spectral variables

$$k = \alpha \sqrt{n(n+1)} \cos\psi; \quad q = \alpha \sqrt{n(n+1)} \sin\psi \,, \tag{21}$$

the system (19)–(20) possesses the following general solutions:

$$A_B = \sum_{n=0} \int_0^{2\pi} d\psi \left[a_B(n,\psi)\, e^{i(kx+qy)} + \bar{a}_B(n,\psi)\, e^{-i(kx+qy)}\right]$$
$$\times \left\{\mathcal{P}_n(\cosh(\alpha z))\,,\ \mathcal{Q}_n(\cosh(\alpha z))\right\}\,, \tag{22}$$

$$A_3 = \sum_{n=1} \int_0^{2\pi} d\psi \left[a_3(n,\psi)\, e^{i(kx+qy)} + \bar{a}_3(n,\psi)\, e^{-i(kx+qy)}\right]$$
$$\times \left\{\mathcal{P}_n^1(\cosh(\alpha z))\,,\ \mathcal{Q}_n^1(\cosh(\alpha z))\right\}\,, \tag{23}$$

expressed in terms of the linearly-independent Legendre adjoint functions of the second kind (Gradshteyn and Ryzhik 1965)

$$\mathcal{P}_n^m(w) = \frac{1}{\Gamma(1-m)} \left(\frac{w+1}{w-1}\right)^{m/2} F\left(-n\,,\, n+1\,;\, 1-m\,;\, \frac{1-w}{2}\right)\,, \tag{24}$$

$$\mathcal{Q}_n^m(w) = \frac{e^{m\pi i}\,\Gamma(m+n+1)\,\Gamma\left(\frac{1}{2}\right)}{2^{n+1}\,\Gamma\left(n+\frac{3}{2}\right)} (w^2-1)^{m/2}\, w^{-m-n-1}$$
$$\times F\left(\frac{m+n+2}{2}\,,\, \frac{m+n+1}{2}\,;\, n+\tfrac{3}{2}\,;\, \frac{1}{w^2}\right)\,, \tag{25}$$

where $F(\alpha,\, \beta;\, \gamma;\, w)$ are the usual hypergeometric functions and $w = \cosh(\alpha z)$. Using the functional relations

$$\frac{d}{dz}\{\mathcal{P}_n^1\,,\, \mathcal{Q}_n^1\} + \alpha\, \coth(\alpha z)\, \{\mathcal{P}_n^1\,,\, \mathcal{Q}_n^1\}$$
$$= \frac{\alpha}{\sqrt{w^2-1}} \left[(w^2-1)\, \frac{d}{dw}\, \{\mathcal{P}_n^1\,,\, \mathcal{Q}_n^1\} + w\, \{\mathcal{P}_n^1\,,\, \mathcal{Q}_n^1\}\right], \tag{26}$$

$$(w^2-1)\, \frac{d}{dw}\, \{\mathcal{P}_n^1\,,\, \mathcal{Q}_n^1\} + w\, \{\mathcal{P}_n^1\,,\, \mathcal{Q}_n^1\}$$
$$= n\, \left[\left\{\mathcal{P}_{n+1}^1\,,\, \mathcal{Q}_{n+1}^1\right\} - w\, \{\mathcal{P}_n^1\,,\, \mathcal{Q}_n^1\}\right]\,, \tag{27}$$

$$\left\{\mathcal{P}_{n+1}^1\,,\, \mathcal{Q}_{n+1}^1\right\} - w\, \left\{\mathcal{P}_n^1\,,\, \mathcal{Q}_n^1\right\} (n+1)\sqrt{w^2-1}\, \{\mathcal{P}_n\,,\, \mathcal{Q}_n\}\,, \tag{28}$$

the general Lorentz condition

$$\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} + \alpha\, A_3\, \coth(\alpha z) = 0 \tag{29}$$

becomes explicitly

$$\sum_{n=1} \sqrt{n(n+1)}\ \{\mathcal{P}_n(\cosh(\alpha z))\,,\ \mathcal{Q}_n(\cosh(\alpha z))\} \int_0^{2\pi} d\psi$$
$$\times \left\{ \left[\cos\psi\, a_1(n,\psi) + \sin\psi\, a_2(n,\psi) - i\sqrt{n(n+1)}\, a_3(n,\psi)\right] e^{i(kx+qy)} \right.$$
$$\left. - \left[\cos\psi\, \bar{a}_1(n,\psi) + \sin\psi\, \bar{a}_2(n,\psi) + i\sqrt{n(n+1)}\, \bar{a}_3(n,\psi)\right] e^{-i(kx+qy)} \right\} = 0\,.$$

As it can be seen, the following polar-type structure of the spectral amplitudes $\{a_\mu(n,\psi)\}_{\mu=\overline{1,3}}$,

$$a_1(n,\psi) = i\,c(n)\cos\psi\,,\ \ a_2(n,\psi) = i\,c(n)\sin\psi\ \Rightarrow\ \ a_3 = \frac{c(n)}{\sqrt{n(n+1)}}\,, \tag{30}$$

gives all of the vacuum 'longitudinal' modes (of magnetic type), since

$$B_A \sim \left\{ \frac{\partial \mathcal{P}_n}{\partial z} - \alpha\, \mathcal{P}_n^1\,,\ \frac{\partial \mathcal{Q}_n}{\partial z} - \alpha\, \mathcal{Q}_n^1 \right\} \equiv 0,\ \ B_3 \equiv 0\,. \tag{31}$$

Working with the most general algebraic relation between the spectral coefficients that satisfy the Lorentz condition

$$a_3(n,\psi)\ =\ -\frac{i}{\sqrt{n(n+1)}}\ \left[a_1(n,\psi)\cos\psi + a_2(n,\psi)\sin\psi\right]\,, \tag{32}$$

the solutions (22) and (23) turn into

$$A_B = \sum_{n=0} \{\mathcal{P}_n\,,\ \mathcal{Q}_n\} \int_0^{2\pi} d\psi \left[a_B(n,\psi)\, e^{i(kx+qy)} + \bar{a}_B(n,\psi)\, e^{-i(kx+qy)}\right]\,,$$
$$A_3 = \sum_{n=1} \{\mathcal{P}_n^1\,,\ \mathcal{Q}_n^1\}\, \frac{i}{\sqrt{n(n+1)}} \tag{33}$$
$$\times \int_0^{2\pi} d\psi \left\{ -\ [a_1(n,\psi)\ \cos\psi + a_2(n,\psi)\ \sin\psi]\, e^{i(kx+qy)} \right.$$
$$\left. + [\bar{a}_1(n,\psi)\ \cos\psi + \bar{a}_2(n,\psi)\ \sin\psi]\, e^{-i(kx+qy)} \right\}\,. \tag{34}$$

Now, putting everything together and employing the $U(1)$-gauge covariant definition of the Maxwell tensor $\mathbf{F} = d\mathrm{A}$, we are in the position to write the observable $\vec{B}$ components as

$$
\begin{aligned}
B_1 = \alpha \sum_{n=1} \int_0^{2\pi} d\psi \, \cos\psi \Big\{ & [a_1(n,\psi)\,\sin\psi - a_2(n,\psi)\,\cos\psi]\, e^{i(kx+qy)} \\
& + [\bar{a}_1(n,\psi)\,\sin\psi - \bar{a}_2(n,\psi)\,\cos\psi]\, e^{-i(kx+qy)} \Big\} \left\{ \mathcal{P}_n^1 \,,\; \mathcal{Q}_n^1 \right\} \,,
\end{aligned} \tag{35}
$$

$$
\begin{aligned}
B_2 = \alpha \sum_{n=1} \int_0^{2\pi} d\psi \, \sin\psi \Big\{ & [a_1(n,\psi)\,\sin\psi - a_2(n,\psi)\,\cos\psi]\, e^{i(kx+qy)} \\
& + [\bar{a}_1(n,\psi)\,\sin\psi - \bar{a}_2(n,\psi)\,\cos\psi]\, e^{-i(kx+qy)} \Big\} \left\{ \mathcal{P}_n^1 \,,\; \mathcal{Q}_n^1 \right\} \,,
\end{aligned} \tag{36}
$$

$$
\begin{aligned}
B_3 = \; -i\alpha \sum_{n=1} \sqrt{n(n+1)} \int_0^{2\pi} d\psi \Big\{ & [a_1(n,\psi)\sin\psi - a_2(n,\psi)\cos\psi]\, e^{i(kx+qy)} \\
& - [\bar{a}_1(n,\psi)\,\sin\psi - \bar{a}_2(n,\psi)\,\cos\psi]\, e^{-i(kx+qy)} \Big\} \left\{ \mathcal{P}_n ,\; \mathcal{Q}_n \right\} \,.
\end{aligned} \tag{37}
$$

Finally, assuming $a_1 = \sin\psi$ and $a_2 = -\cos\psi$, the magnetostatic field components (35)–(37) are generically represented in Fig. 3 as functions of x and z, for the $\mathcal{P}$ and $\mathcal{Q}$ modes corresponding to $n = 2$.

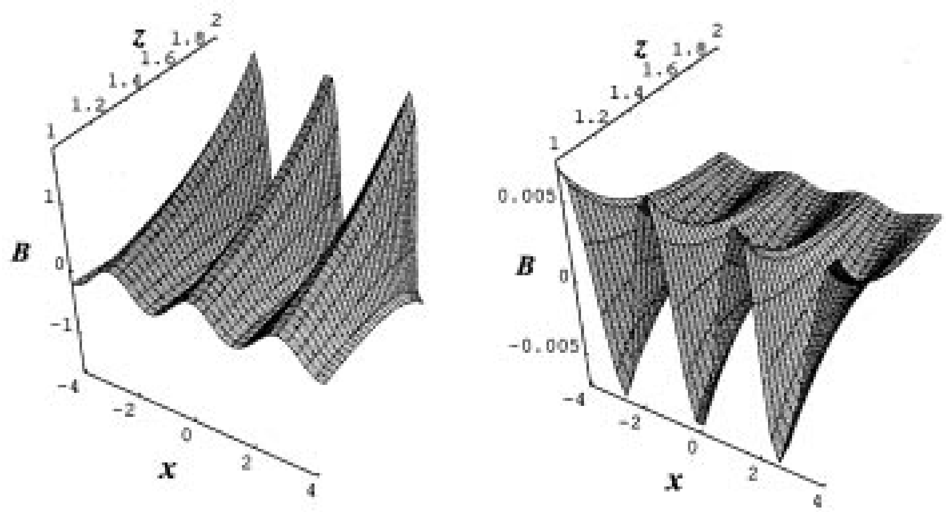

Fig. 3. Generic representation of the magnetostatic field components given by equations (35)–(37), for $a_1 = \sin\psi$, $a_2 = -\cos\psi$ and $y = 0$. The left and right surfaces represent the $\mathcal{P}$ and $\mathcal{Q}$ parts for $n = 2$ respectively.

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