Transient Electromagnetic Fields for a Polarized Conductive Sheet

James R. Wait
2210 East Waverly
Tucson AZ 85719 USA

Abstract
We formulate the problem of a transient magnetic dipole located over a thin conductive sheet which may exhibit frequency dispersion or I.P. (induced polarization). A first order analysis is carried through for the case when the source current is a step function. It is shown that the I.P. slow tail can be of opposite polarity to the principal electromagnetic response.

Introduction
In geophysical exploration and non-destructive testing, the thin sheet conductor is a useful idealization of the target. In such cases, the conductivity-thickness product, $\sigma d$, or conductance is a key parameter. When the sheet, located in free space, is excited by an external pulsed magnetic dipole, it has been shown that the image of the source is a like dipole but its distance from the sheet recedes in time (Wait, 1956). Such a concept was originally put forth by Maxwell (1891) a century ago. The normalized time parameter, for such a problem, is $t(\mu_0\sigma d)\mu_0 = 4\pi \times 10^{-7}$ is the permeability of the surrounding free space, and $I$ is some scale length such as the distance of the source to the sheet.

An interesting extension of the thin sheet induction problem is to allow the conductivity of the sheet to be a function of frequency which we designate as $\sigma(j\omega)$ for a time factor $\exp(j\omega t)$. We then inquire if the image concept is still applicable. And, if so, what modifications of the time domain formulation are needed? For purposes of illustration, we choose a simple Debye type model for the frequency dispersion of the sheet conductivity. A closely related problem was discussed recently in this journal by Smith and West (1988). We wish to complement their development with an approximate analysis; and it is hoped that the new insight provided is not entirely trivial.

Basic Transform Representation
The formal solution of the quasi-static time harmonic problem is well known (e.g., Wait 1951, Grant and West 1965) so we may summarize the solution very briefly here. To cast the needed result into Laplace transform notation, we merely replace $j\omega$ by $s$, the transform variable. Also, we define a normalized sheet admittance $y(s)$ by the relation

$$y(s) = \sigma(s) d\mu_0/2$$

(1)

The conductive sheet, itself, is to be located in the plane $z = 0$ of a cylindrical coordinate system ($r, \phi, z$). The source, located at $z = h$, is a small loop of area $dA$ carrying a current $I(t)$ which is zero for $t < 0$. The corresponding transform is obtained from

$$I(s) = \mathcal{L}\{I(t) e^{-st} dt = L I(s)$$

(2)

where $L$ is the Laplace transform operator.

Now, within our quasi-static assumption, the magnetic field outside the sheet can be obtained from a time-varying magnetic potential $U(t)$ according to

$$\hat{h}(t) = -\mathbf{\nabla} U(t)$$

(3)

where

$$U(t) = L^{-1} U(s)$$

(4)

where $L^{-1}$ is the inverse Laplace transform operator. For $z > 0$, we may write

$$U(s) = -\frac{j(s) dA}{4\pi} \left[ \frac{\partial}{\partial z} \frac{1}{r} - \frac{\partial}{\partial z} \int_0^\infty f(s) e^{-k(z-n)} J_0(kr) dk \right]$$

(5)

where

$$f(s) = \left[ s + (k/y(s)) \right]^{-1}$$

(6)

and

$$n = \left[ z^2 + (z-n)^2 \right]^{1/2}$$

(7)

Explicit Inversion of Transform
To specialize further, we will now adopt a turn-on or step function current excitation. Thus

$$I(t) = I_0 u(t)$$

(8)

where $u(t) = 0$ for $t < 0$ and $u(t) = 1$ for $t > 0$. Then we deduce that the time dependent potential is

$$U(t) = -\frac{j(s) dA}{4\pi} \left[ \frac{\partial}{\partial z} \frac{1}{r} - \frac{\partial}{\partial z} \int_0^\infty L^{-1} f(k,s) e^{-k(z-n)} J_0(kr) dk \right] u(t)$$

(9)
Now we restrict attention to weakly dispersive conductivity such that the fractional change of \( \sigma(w) \), over all significant frequencies, is small compared with one. Thus we may write

\[
\frac{f(k,s)}{f_0(k,s)} \approx \frac{y(s)}{y_\infty} \frac{f_0(k,s)}{y_\infty} = y_\infty \frac{f_0(k,s)}{y_\infty} \quad (10)
\]

Here \( y_\infty = \lim_{s \to \infty} y(s) \) is the high frequency limit of the normalized admittance. Furthermore, in our example,

\[
f_0(k,s) = \frac{1}{s + (k/y_\infty)} \quad (11)
\]

and

\[
f_0(k,s) = \frac{\partial f(k,s)}{\partial y(s)} \quad \frac{y(s)}{y_\infty} = \frac{\partial f_0(k,s)}{\partial y_\infty} \quad (12)
\]

Thus, it follows that

\[
f_0(k,s) = \left( \frac{k}{y_\infty} \right)^2 \left[ \frac{1}{s + (k/y_\infty)} \right]^2 \quad (13)
\]

Equation (9) can now be decomposed as follows

\[
U(t) = u^{EM}(t) + u^{IP}(t) \quad (14)
\]

The first part is

\[
u^{EM}(t) = - \frac{I_0 da}{4\pi} \left[ \frac{\partial}{\partial z} \left( \frac{1}{R(t)} \right) - \frac{1}{R(t)} \right] \quad (15)
\]

where

\[
1/R(t) = \int_0^\infty \left[ \frac{1}{s + (k/y_\infty)} \right] \left[ e^{-k(\sigma + i\omega)} \right] \quad (16)
\]

\[
= \int_0^\infty \frac{e^{-k(y_\infty^2)}}{e^{-k(y_\infty^2)}} \left( \frac{1}{s + (k/y_\infty)} \right) \quad (17)
\]

\[
= \frac{1}{R} \left[ \frac{1}{s + (k/y_\infty)^2} \right] \quad (18)
\]

Carrying out the differentiation, we obtain

\[
u^{EM}(t) = \frac{I_0 da}{4\pi} \left[ \frac{z + i\omega}{z + i\omega} \right]^{3/2} \quad (19)
\]

\[
= \left( z + i\omega \right) \left( \frac{z + i\omega}{z + i\omega} \right) \quad (20)
\]

Clearly \( u^{EM}(t) \) is the electromagnetic response computed as if the conductivity of the sheet was \( \sigma(\infty) \) or \( \sigma_0 \) for all frequencies. Then we see that \( R'(t) \) is the time dependent distance from the image at \( z = -(h + t\gamma_\infty) \) to the observer at \( (r, z) \). Of course, the contribution from the image vanishes as \( t \) becomes sufficiently large.

Now the second part of (14) is given by

\[
u^{IP}(t) = \frac{I_0 da}{4\pi} \left[ \frac{y(s)}{y_\infty} \right] \quad (21)
\]

\[
\approx \frac{1}{R} \left[ \frac{z + i\omega}{z + i\omega} \right]^{3/2} \quad (22)
\]

\[
\approx \left[ \frac{z + i\omega}{z + i\omega} \right] \quad (23)
\]

\[
\approx \left[ \frac{1}{s + (k/y_\infty)^2} \right] \quad (24)
\]

\[
\approx \left[ \frac{1}{s + (k/y_\infty)^2} \right] \quad (25)
\]

where the latter approximate form (25) is valid if \( g >> \gamma_\infty /k \) for all significant values of \( k \). Here \( g \) can be dabbled the I.P. time constant while \( \gamma_\infty \) is the E.M. time constant bearing in mind that \( 1/R \) is a characteristic scale length. Then, using (20) we see that

\[
u^{IP}(t) = \frac{I_0 da}{4\pi} \left[ \frac{y(s)}{y_\infty} \right] \quad (26)
\]

\[
\approx \left[ \frac{1}{s + (k/y_\infty)^2} \right] \quad (27)
\]

where \( R \) is given by (7) being the distance from the geometrical image point, at \( z = -h, \) to the observer at \( (r, z) \).

**Concluding Remarks**

It is remarkable that the I.P. slow tail, characterized by the time constant \( g \), appears to be equivalent to a time varying magnetic pole at the image point \( z = -h \) but it decays with time exponentially. Also it is proportional to the conductivity of the sheet and varies inversely with the I.P. time constant. An alternative interpretation of the \( 1/R \) dependence for the potential is to view the pole contribution as a semi-infinite magnetic current extending downwards from \( z = -h \) to \( -\infty \).

But, of course, we should bear in mind that we have made a number of physical approximations to simplify the analysis. Essentially we have assumed that the dispersion or polarizability is weak (e.g. \( m < 0.2 \)) and that the I.P. time constant is somewhat greater than the E.M. time constant. It would not be difficult to relax such assumptions, as Smith and West (1988) have done in a closely related problem, but then numerical effort is required and some insight is lost.
References


(Paper received: 19/9/89)