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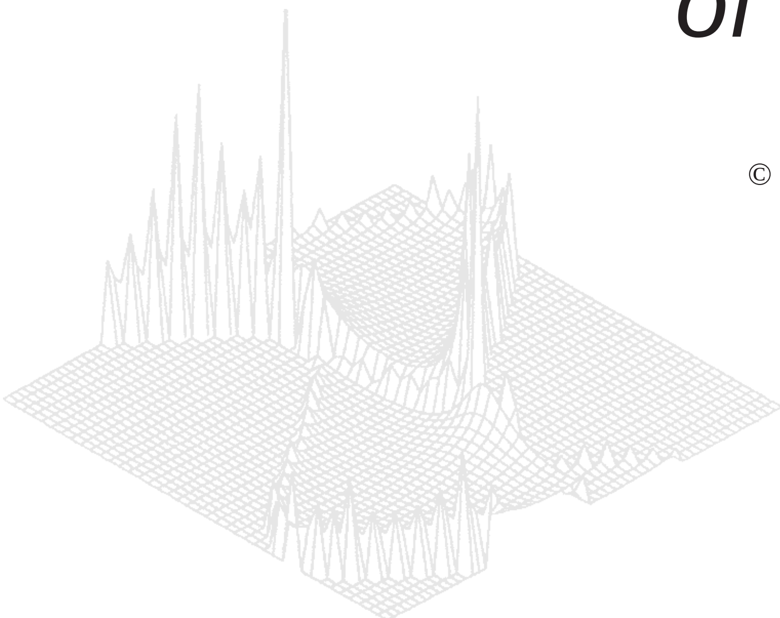

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Effects of Boundary and Electron Inertia on the Ion Acoustic Wave in a Plasma: A Pseudopotential Approach

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Abstract

A pseudopotential approach is used to analyse the propagation of ion-acoustic waves in a plasma bounded by a cylindrical domain. The effect of the finite geometry is displayed both analytically and numerically. The phase velocity of the wave is determined and its variation is studied with respect to the plasma parameters. It is observed that the pseudopotential shows a wide variation of shape due to the imposition of a finite boundary condition. It is shown that if the other parameters are kept within a certain range of values, then the trapping of particles is favoured when the presence of the boundary is taken into account.

1. Introduction

Analysis of nonlinear wave propagation in plasmas is one of the most important aspects of theoretical research in plasma physics. At present there exists more than one method for such a study. The reductive perturbation theory is one of the most common techniques to analyse wave propagation (Washimi and Taniuti 1966), but such an approach always assumes that the amplitude of the propagating wave is small. An independent method is that of the pseudopotential (Schamel 1982; Sagdeev 1966) which has the capability of treating nonlinear waves of arbitrary amplitude. This method was initially used by Sagdeev (1966) and later extensively used by Schamel (1972) and others. An important aspect which is usually overlooked in these theoretical studies of plasma waves is the effect of finite boundaries which is an essential ingredient of all laboratory plasmas (Das and Ghosh 1988; Mukherjee and Roy Chowdhury 1995). These effects have been introduced in the reductive perturbation framework and it was observed that this finite geometry does substantially affect the predictions. So we study here the case of a plasma consisting of electrons and ions confined in a cylindrical wave guide. The ions are assumed to be nonrelativistic due to their relative mass. We also assume that a hydrodynamic description is possible. In the the first part of our work we analyse the effect of electron inertia and finite geometry on the phase velocity of the acoustic wave generated inside. It may be mentioned that some discussion already exists regarding the effect of a boundary, but the result

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is not exhaustive (Sayal and Sarma 1989). In the second part of our paper we consider the shape of the Sagdeev potential and its variation with the boundary and also with electron inertia. It is observed that the dimension of the cylindrical system containing the plasma does have a positive influence on the shape of the Sagdeev potential and, for a particular range of values of the radius, it may help in the trapping of electrons, thereby supporting the formation of a solitary wave.

2. Formulation

We consider a plasma consisting of nonrelativistic ions and electrons in a cylindrical wave guide with its axis along the x-axis. With the assumption of a hydrodynamic description we can write the basis equations as (Schamel 1982)

$$
\begin{aligned}
&\frac{\partial n_\mathrm{i}}{\partial t} + \frac{\partial}{\partial x}(n_\mathrm{i}\, u_\mathrm{i}) = 0,\\
&\frac{\partial u_\mathrm{i}}{\partial t} + u_\mathrm{i}\frac{\partial u_\mathrm{i}}{\partial x} = -\frac{\partial \phi}{\partial x},\\
&\frac{\partial n_\mathrm{e}}{\partial t} + \frac{\partial}{\partial x}(n_\mathrm{e}\, u_\mathrm{e}) = 0,\\
&\frac{m_\mathrm{e}}{m_\mathrm{i}}\frac{\partial u_\mathrm{e}}{\partial t} + u_\mathrm{e}\frac{\partial u_\mathrm{e}}{\partial x} + \frac{1}{n_\mathrm{e}}\frac{\partial n_\mathrm{e}}{\partial x} = \frac{\partial \phi}{\partial x},\\
&\frac{\partial \phi^2}{\partial x^2} + \nabla_\perp^2 \phi = n_\mathrm{e} - n_\mathrm{i}\,.
\end{aligned} \tag{1}
$$

In these equations n_e, n_i represent respectively the electron and ion density, u_e, u_i their velocities, and ϕ the electrostatic potential. In the last equation $\nabla_\perp^2 = \partial^2/\partial y^2 + \partial^2/\partial z^2$ is the transverse Laplacian operator. In all equations the velocities are normalised by $\sqrt{kT_\mathrm{e}/m_\mathrm{i}}$, the densities by n_0 (the equilibrium value of the density), the lengths by the Debye length $(kT_\mathrm{e}/4\pi n_0\, e^2)^{1/2}$, and the potential by kT_e/e. To start the analysis we consider the deviation of the quantities n_i, n_e, u_i, u_e, ϕ from their equilibrium value, set

$$
n_\mathrm{i} = n_{0\mathrm{i}} + n_\mathrm{i}', \quad n_\mathrm{e} = n_{0\mathrm{e}} + n_\mathrm{e}', \quad u_\mathrm{i} = u_\mathrm{i}', \quad \text{etc.},
$$

and substitute in equation (1). Here we have neglected the streaming of electrons and ions. We also assume that at equilibrium, $n_{0\mathrm{i}} = n_{0\mathrm{e}} = n_0$, along with the following form of the variations:

$$
\begin{aligned}
n_\mathrm{i}' &= f(r)\exp[\mathrm{i}(kx-\omega t)], \qquad n_\mathrm{e}' = j(r)\exp[\mathrm{i}(kx-\omega t)],\\
u_\mathrm{i}' &= h(r)\exp[\mathrm{i}(kx-\omega t)], \qquad u_\mathrm{e}' = l(r)\exp[\mathrm{i}(kx-\omega t)],\\
\phi' &= m(r)\exp[\mathrm{i}(kx-\omega t)]\,.
\end{aligned} \tag{2}
$$

The linearised form of (1) then reads as

$$\frac{\partial n_{\rm i}'}{\partial t} + n_0 \frac{\partial u_{\rm i}'}{\partial x} = 0,$$

$$\frac{\partial u_{\rm i}}{\partial t} = -\frac{\partial \phi'}{\partial x},$$

$$\frac{\partial n_{\rm e}}{\partial t} + n_0 \frac{\partial u_{\rm e}'}{\partial x} + u_{0\rm e} \frac{\partial n_{\rm e}'}{\partial x} = 0,$$

$$q \frac{\partial u_{\rm e}'}{\partial t} + u_{0\rm e} \frac{\partial u_{\rm e}'}{\partial x} + \frac{1}{n_0} \frac{\partial n_{\rm e}'}{\partial x} = \frac{\partial \phi'}{\partial x},$$

$$\frac{\partial^2 \phi'}{\partial x^2} + \left(\frac{\partial^2 \phi'}{\partial r^2} + \frac{1}{r} \frac{\partial \phi'}{\partial r} \right) = n_{\rm e}' - n_{\rm i}' , \tag{3}$$

where we have used the fact that the equilibrium value of ϕ is zero and

$$\nabla_\perp^2 \phi = \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} .$$

Using now the forms (2) in (3) and eliminating in favour of $m(r)$, we obtain

$$\frac{{\rm d}^2 m(r)}{{\rm d}r^2} + \frac{1}{r} \frac{{\rm d}m(r)}{{\rm d}r} + \alpha^2\, m(r) = 0 , \tag{4}$$

where α is given as

$$\alpha^2 = k^2 \left(\frac{n_0}{\omega^2} + \frac{n_0}{\omega^2 q - k^2} - 1 \right) , \tag{5}$$

and $q = m_{\rm e}/m_{\rm i}$. Now equation (4) is a zero order Bessel equation whose solution can be written as

$$m(r) = J_0(\alpha r) .$$

If R is the radius of the cylindrical wave guide then we must have on the surface of the wave guide

$$J_0(\alpha R) = 0 . \tag{6}$$

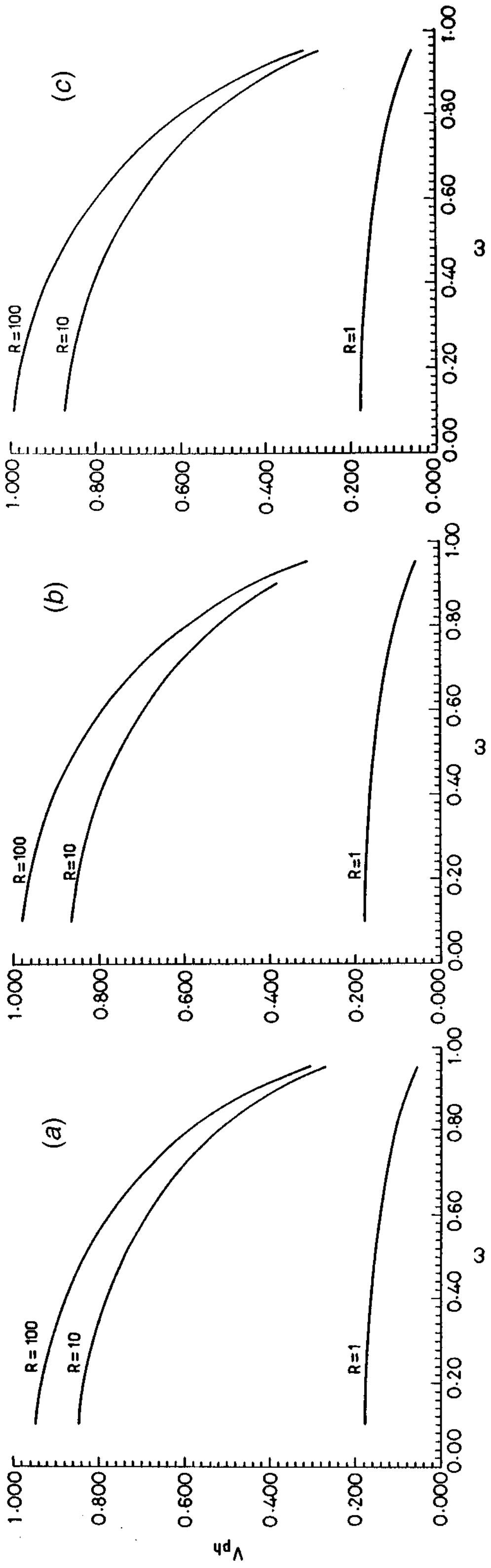

Fig. 1. Variation of phase velocity with ω for three values of the radius when $n_0 = 1$ and $p_{0n} = 5 \cdot 5$: (a) $q = 1/10$, (b) $q = 1/36$ and (c) $q = 1/1836$.

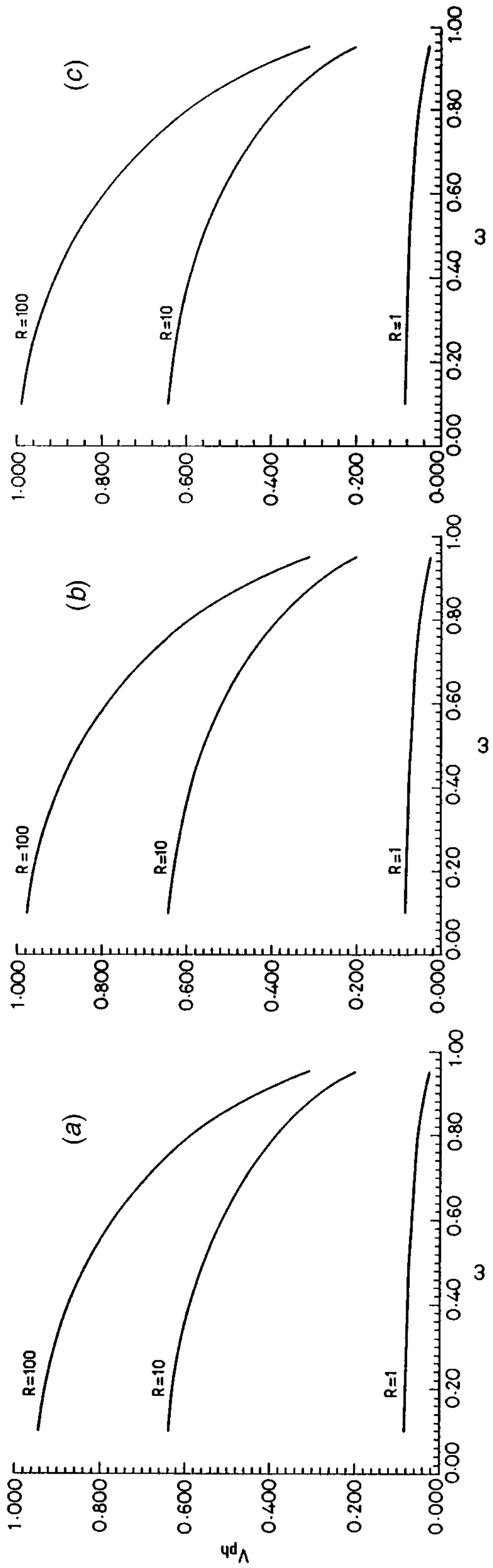

Fig. 2. As for Fig. 1, but with $p_{0n} = 11\cdot 8$.

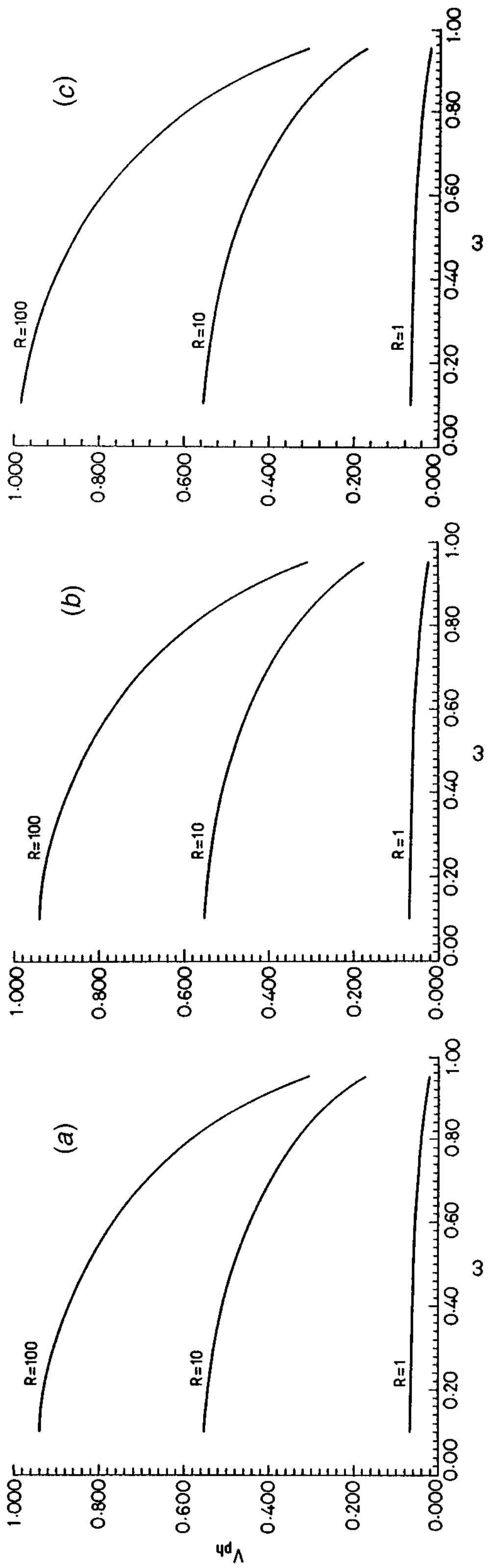

Fig. 3. As for Fig. 1, but with $p_{0n} = 14\cdot9$.

Thus, if p_{0n} is a root of equation (6), then $\alpha R = p_{0n}$, or $\alpha = P_{0n}/R$. Substituting in equation (5) we get

$$p_{0n}^2 = (kR)^2\left(\frac{n_0}{\omega^2} + \frac{n_0}{\omega^2 q - k^2} - 1\right), \tag{7}$$

which is the required equation for the phase velocity. This equation is fourth order in k and can be solved analytically. The detailed variation of this phase velocity with frequency, the ratio of masses q, and the radius R are shown in Figs 1–5. In fact $J_0(r)$ has many zeros, but we have chosen only the nearest ones, $p_{0n} = 5 \cdot 5,\ 11 \cdot 8,\ 14 \cdot 9$ and hence considered different values of R, the radius of the wave guide. These values are to be used in solving the dispersion relation (7).

In Fig. 1 we depict the variation of the phase velocity with respect to ω for $R = 1$, 10 and 100 and for different q values, corresponding to the zero $p_{0n} = 5 \cdot 5$ of the Bessel function in equation (7). In Fig. 2 the same variation is displayed for $p_{0n} = 11 \cdot 8$, another mode of the Bessel function. The third zero of the Bessel function, $p_{0n} = 14 \cdot 9$, is shown in Fig. 3. The cylindrical geometry actually gives rise to these different modes due to the presence of the Bessel function. To show that such modes make a difference in actual physical values of the phase velocity, we plot the values of the phase velocity for these three p_{0n} or or q values for a fixed value of q or p_{0n} in Figs 4 and 5 respectively. So we may conclude that the phase velocity increases with an increase in R and also it has larger values for the smaller zeros of the Bessel function, which is the dispersion relation in the present situation.

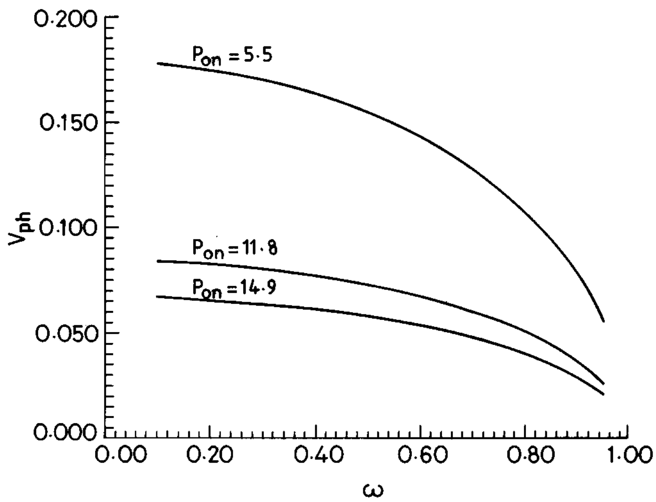

Fig. 4. Variation of phase velocity with ω for three values of p_{0n} when $n_0 = 1$, $q = 0 \cdot 1$ and $R = 1$.

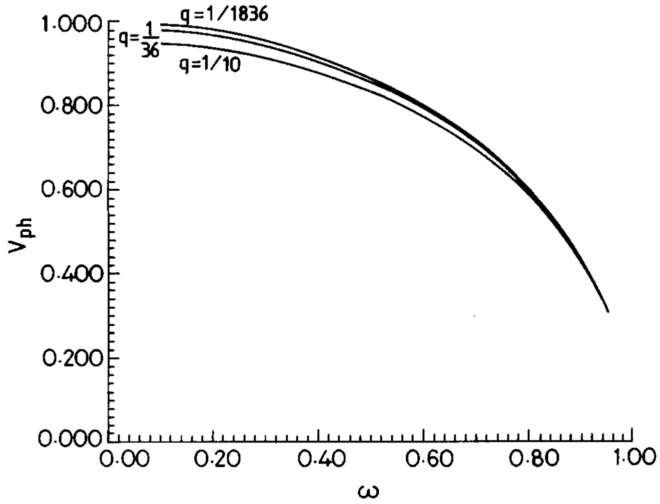

Fig. 5. Variation of phase velocity with ω for three values of q when $n_0 = 1,\ q = 0\cdot 1,\ R = 100$ and $p_{0n} = 5\cdot 5$.

3. Pseudopotential and Solitary Wave

We now consider the problem of arbitrary amplitude nonlinear wave propagation with the help of a pseudopotential formalism taking care of the boundary condition on the surface of the cylindrical wave guide. From our previous analysis we now know that the radical variation of all the perturbed quantities is given by $J_0(kr)$. Using this fact we deduce the following equations from (1). It may be pointed out that the radial dependence through $J_0(kr)$ for the perturbed quantities is implied by the geometric symmetry of the situation. That is, we assume that the disturbances produced by the perturbation proceed only in the x direction.

Actually we substitute

$$n_{\rm i} = 1 + J_0(kr)\, N_{\rm i}(xt), \qquad u_{\rm i} = u_{\rm i0} + J_0(kr)\, U_{\rm i}(xt),$$

$$n_{\rm e} = 1 + J_0(kr)\, N_{\rm e}(xt), \qquad u_{\rm e} = u_{\rm e0} + J_0(kr)\, U_{\rm e}(xt),$$

$$\phi = J_0(kr)\, \phi(xt),$$

and obtain

$$J_0\, \frac{\partial N_{\rm i}}{\partial t} + J_0\, \frac{\partial U_{\rm i}}{\partial x} + J_0 U_{\rm i0}\, \frac{\partial N_{\rm i}}{\partial x} + J_0^2\, \frac{\partial}{\partial x}(N_{\rm i}\, U_{\rm i}) = 0\,, \tag{8}$$

$$J_0\, \frac{\partial U_{\rm i}}{\partial t} + J_0\, U_{\rm i0}\, \frac{\partial U_{\rm i}}{\partial x} + J_0^2\, U_{\rm i}\, \frac{\partial U_{\rm i}}{\partial x} = -J_0\, \frac{\partial \phi}{\partial x}\,, \tag{9}$$

$$J_0 q(1+J_0\,N_{\rm e})\frac{\partial U_{\rm e}}{\partial t} + J_0(1+J_0\,N_{\rm e})(U_{\rm e0}+J_0\,U_{\rm e})\frac{\partial U_{\rm e}}{\partial x}$$

$$+J_0\,\frac{\partial N_{\rm e}}{\partial x} = J_0(1+J_0\,N_{\rm e})\frac{\partial \phi}{\partial x}\,. \qquad (10)$$

Note that here we have considered both the streaming of electrons and ions. In each of these equations the dependence on the radial distance is in the Bessel function $J_0(kr)$. To remove this we multiply each equation by $J_0(kr)$ and integrate over r from 0 to R, whence we get

$$N_{\rm i} = \frac{U_{\rm i}}{M-(\alpha U_{\rm i}+U_{\rm i0})}\,, \qquad (11)$$

where

$$\alpha = \int_0^R J_0^3(r)\,r\,{\rm d}r/I,$$

$U_{\rm i0}$ is the streaming velocity of the ion and we have used a wave front $\xi = x - Mt$ in integrating these equations. The electron equation similarly yields

$$N_{\rm e} = \frac{U_{\rm e}}{M-(\alpha U_{\rm e}+U_{\rm e0})}\,, \qquad (12)$$

whereas equation (9) yields

$$-MU_{\rm i} + U_{\rm i0}\,U_{\rm i} + \tfrac{1}{2}\alpha\,U_{\rm i}^2 = -\phi\,. \qquad (13)$$

Finally from equation (10) we get

$$\left(\frac{B}{3} - \frac{\alpha^2}{3}\right)U_{\rm e}^3 + \left(\frac{\alpha M}{2} - \frac{\alpha U_{\rm e0}}{2}\right)U_{\rm e}^2 + \gamma U_{\rm e} + \delta = 0\,, \qquad (14)$$

$$\gamma = MU_{\rm e0} - qM^2 - U_{\rm e0}^2 - \frac{M-U_{\rm e0}}{M} + qMU_{\rm e0}\,, \qquad (15a)$$

$$\delta = (U_{\rm e0}-M)\phi - \left(\frac{M-U_{\rm e0}}{\alpha}\right)\log M - \frac{U_{\rm e0}(M-U_{\rm e0})}{M\alpha}\,, \qquad (15b)$$

$$\beta = \int_0^R J_0^4(r)\,r\,{\rm d}r/I; \qquad I = \int_0^R J_0^2(r)\,{\rm d}r\,. \qquad (15c)$$

Finally, Poisson's equation yields

$$\begin{aligned}\frac{\partial^2\phi}{\partial\xi^2} &= \phi + N_\mathrm{e} - N_\mathrm{i}\\ &= \phi + \frac{U_\mathrm{e}}{M - \alpha U_\mathrm{e} - U_\mathrm{e0}} + \left[\frac{1}{\alpha} - \frac{1}{\alpha}\left(1 - \frac{2\alpha\phi}{(M - U_\mathrm{i0})^2}\right)^{-\frac{1}{2}}\right]\\ &= -\,\frac{\partial V}{\partial\phi}\,, \qquad (16)\end{aligned}$$

where U_e is to be solved from equation (14) and substituted. This cubic equation for U_e can be analytically solved by Cardan's method and we only consider real values of U_e for physical considerations. The real values of U_e are written as

$$U_\mathrm{e} = \frac{z - a_1}{a_0}; \qquad Z = U - \frac{H}{U};$$

$$U^3 = \frac{-G + \sqrt{G^2 + 4H^3}}{2}, \qquad G^2 + 4H^3 > 0\,,$$

with

$$G = a_0^2\,\delta - a_0\,a_1\,\gamma + 2a_1^3\,,$$

$$H = \tfrac{1}{3}a_0\,\gamma - a_1^2\,,$$
$$a_0 = \tfrac{1}{3}(\beta - \alpha^2); \qquad a_1 = \tfrac{1}{3}\left(\frac{\alpha M}{2} - \frac{\alpha U_\mathrm{e0}}{2}\right).$$

Equation (16) in conjunction with (14) gives us the pseudopotential equation in the presence of the boundary and of electron inertia. Due to the complicated dependence of U_e on ϕ, equation (16) can be solved only numerically (Baboolal *et al*. 1988, 1989). The results of such an analysis are given in Figs 6 and 7. Fig. 6*a* corresponds to the situation with no boundary and Fig. 6*b* to the case when the cylindrical boundary is present, where we have taken $p_{0n} = 5\cdot 5$. It may be observed that the presence of the boundary favours the trapping of particles and hence the formation of solitons. To understand the soliton formation more clearly we can consider the special situation when ϕ is small. Then we can expand the algebraic expressions (16) and (15) in powers of ϕ and keep only lowest powers. This immediately leads to

$$U_\mathrm{e} \approx \alpha_0 + \alpha_1\,\phi + \alpha_2\,\phi^2 + \dots,$$

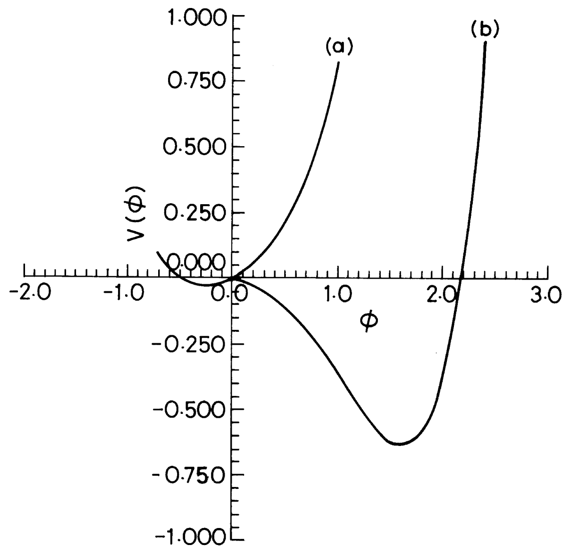

Fig. 6. Variation of $V(\phi)$ with ϕ when $M = 1 \cdot 5$, $q = 0 \cdot 1$ and $U_{\rm i0} = U_{\rm e0} = 0$, with $\alpha = -4 \cdot 9$ and $\beta = 25 \cdot 75$ for $p_{0n} = 5 \cdot 5$: (*a*) no boundary case and (*b*) cylindrical boundary case.

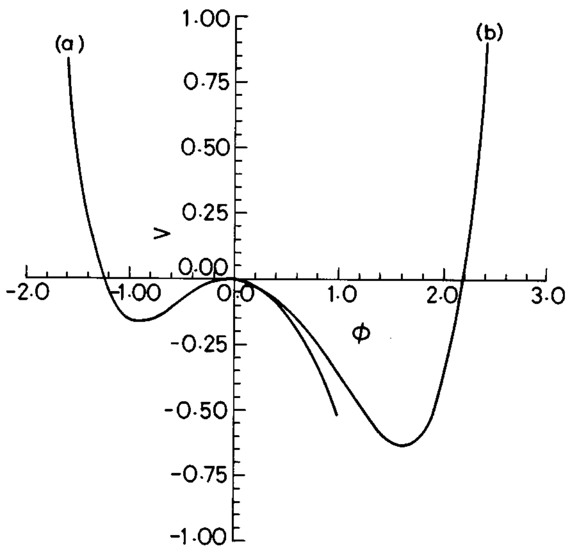

Fig. 7. As for Fig. 6, but with $\alpha = -25 \cdot 35$ and $\beta = 685 \cdot 8$, for $p_{0n} = 11 \cdot 8$.

which when used in (16) leads to

$$\frac{\partial^2 \phi}{\partial \xi^2} = \beta_0 - \beta_1 \, \phi + \beta_2 \, \phi^2 + \ldots .$$

An easy integration leads to $\phi = \eta\,\mathrm{sech}^2(k\xi)$, the usual profile of solitons. But in the general case when the amplitude is not small, the analytic approach is not possible. In this computation only the case $q = 0{\cdot}1$ is considered. On the other hand, an interesting situation develops if we consider a different zero of the Bessel function, say $p_{0n} = 11{\cdot}8$, as shown in Fig. 7. In Fig. 7*b* we have again drawn the curve shown in Fig. 6*b* to compare with the situation which emerges in the case $p_{0n} = 11{\cdot}8$ given in Fig. 7*a*. There is a drastic change in the behaviour of the curve and the trapping of particles is now less favoured compared to the case $p_{0n} = 5{\cdot}5$.

4. Discussion

In our analysis we have discussed the behaviour of the phase velocity and the pseudopotential (Sagdeev) in the presence of a cylindrical boundary and electron inertia. It is observed that due to the existence of multiple modes of solution of the dispersion equation (involving a Bessel function), the usual conclusions of the free space analysis can be changed completely. The pseudo-potential undergoes a dramatic change and a whole new set of phenomena can occur.

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