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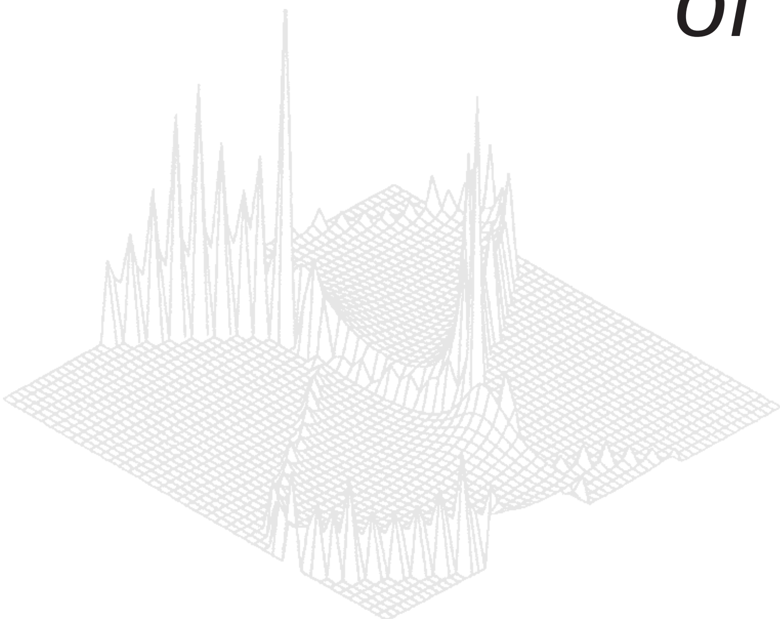

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The Pion Mass Formula

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Abstract

The often used Gell-Mann–Oakes–Renner (GMOR) mass formula for Nambu–Goldstone (NG) bosons in QCD, such as the pions, involves the condensate $\langle\overline{q}q\rangle$, f_π and the quark current masses. Within the context of the Global Colour Model (GCM) for QCD a manifestly different formula was recently found. Remarkably, Langfeld and Kettner have shown the two formulae to be equivalent. Here we note that the above recent analyses refer to the constituent pion and not the exact pion, even within the GCM. Further we generalise the Langfeld–Kettner identity to include the full response of the constituent quark correlators to the presence of a non-zero (and momentum dependent) quark current mass. Results are reported using an effective gluon correlator from meson data fitting.

1. Introduction

The properties of the pion continue to be the subject of considerable theoretical and experimental interest in QCD studies. The pion is an (almost) massless Nambu–Goldstone (NG) boson and its properties are directly associated with dynamical chiral symmetry breaking and the underlying quark–gluon dynamics. Recently there has been renewed interest in the mass formulae for the pion (Cahill and Gunner 1995; Langfeld and Kettner 1996) and the relationship with the well known Gell-Mann–Oakes–Renner (GMOR) (1968) mass formula, as in (1) and (2). Here we extend the study of these relationships and show how one must carefully appreciate the different quantum field theoretic approaches that are actually being employed, often without explicit exposition.

One expects that there should be some perturbative expression for the almost NG boson pion mass in terms of the small quark current masses, and which is built upon the underlying non-perturbative chiral-limit quark–gluon dynamics. While the relation of the low pion mass to the breaking of chiral symmetry dates back to the current algebra era and PCAC (Gell-Mann *et al.* 1968), the often used implementation in QCD has the form

$$m_\pi^2 = \frac{(m_u + m_d)\rho}{f_\pi^2} + ... = \frac{2m\rho}{f_\pi^2} + ... , \tag{1}$$

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where m is the averaged quark current mass, f_π is the usual pion decay constant and the integral $\rho = \langle \bar{q}q \rangle$ is the so-called condensate parameter; for $N_c = 3$

$$\rho = N_c \mathrm{tr}(G(x=0)) = 12 \int \frac{d^4q}{(2\pi)^4} \sigma_s(q^2)\,. \tag{2}$$

(Note: our definition for ρ has an unconventional sign.) In (2) $\sigma_s(s)$ is the chiral limit (i.e. $\sigma_s(s) \equiv \sigma_s(s;0)$ and $\sigma_v(s) \equiv \sigma_v(s;0)$ scalar part of the quark correlator, which utilising only the Lorentz structure, we can write in full generality as

$$G(q;m) = [iA(s;m)q.\gamma + B(s;m) + m]^{-1} = -iq.\gamma\sigma_v(s;m) + \sigma_s(s;m), \tag{3}$$

from which we easily deduce that

$$\sigma_s(s;m) = \frac{B(s;m)+m}{sA(s;m)^2 + [B(s;m)+m]^2}\,, \tag{4}$$

$$\sigma_v(s;m) = \frac{A(s;m)}{sA(s;m)^2 + [B(s;m)+m]^2}\,. \tag{5}$$

However, the expression for ρ in (2) is divergent in QCD, because for large s $B(s)$ decreases like $1/s\ln[s/\mu^2]^{1-\lambda}$, where the anomalous mass dimension is $\lambda = 12/(33-2N_f)$ and μ is the renormalisation scale. When an integration cutoff at $q = \Lambda$ is introduced ρ diverges like $[\ln(\Lambda^2/\mu^2)]^\lambda$. The pion mass m_π is finite and independent of the cutoff when m in (1) is made Λ dependent, defining an $m(\Lambda)$, via $m(\Lambda) = m_R(\mu^2)/[\ln(\Lambda^2/\mu^2)]^\lambda$. This defines the renormalised current mass $m_R(\mu^2)$. The GMOR relation has been considered in various approaches, such as operator product expansions (OPE) (Shifman *et al.* 1979), QCD sum rules (Reinders *et al.* 1985; Narison 1989), and recently finite energy sum rules and Laplace sum rules (Bijnens *et al.* 1995).

In Cahill and Gunner (1995*a*) a new mass formula for the pion mass was derived. That analysis exploited the intricate interplay between the constituent pion Bethe–Salpter equation (BSE) and the non-linear constituent quark equation (CQE) (see Section 8), resulting in the new expression

$$m_\pi^2 = \frac{48}{mf_\pi^2} \int \frac{d^4q}{(2\pi)^4} [\epsilon_s(s)\sigma_s(s) + s\epsilon_v(s)\sigma_v(s)]c(s) + O(m^2)\,, \tag{6}$$

where $c(s)$ is the function

$$c(s) = \frac{B(s;0)^2}{sA(s;0)^2 + B(s;0)^2}\,. \tag{7}$$

Here $\epsilon_s(s)$ and $\epsilon_v(s)$ are functions which specify the response of the constituent quark correlator to the turning on of the quark current mass; see (34) and (35). Note that the GMOR mass formula (1) and (2) appear to be manifestly different from the new expression in (6). However, Langfeld and Kettner (1996) have shown, by further analysis of the CQE, and ignoring for simplicity the $\epsilon_v(s)$ vector response term, that the two mass formulae are equivalent, even though the kernels are indeed different.

Here we first demonstrate that in quantum field theoretic analyses different concepts are often being used and confused in the literature. In this respect we carefully distinguish between the constituent pion and the full or exact pion, and its relevant mass expression. Little detailed progress has been made in the exact analysis of QCD, and we use the Global Colour Model (GCM) to illustrate these differences. Further we extend the Langfeld–Kettner identity to include the vector response function $\epsilon_v(s)$. We also generalise the analysis by introducing a momentum dependent quark current mass function $m(s)$, in which case both the new mass formula (6) and the GMOR mass formula (1) and (2) are re-written with the $m(s)$ inside the integration. This merely generalises the previous procedure of using a cutoff at $s = \Lambda^2$ with m constant in the s-interval $\{0, \Lambda^2\}$. Such an $m(s)$ would be expected to arise in a theory of the quark current masses, and hence is beyond QCD.

To be clear we note that this report contains no analysis of the mass formulae for the full pion in QCD, or even in the GCM. However, if the GMOR relation is also the correct QCD result, up to $O(m)$, then that would indicate that the GMOR relation is in fact a generic form that arises whether we are dealing with the full pion or with the constituent pion, and whether we are analysing QCD or some approximation scheme to QCD, such as the GCM, provided we carefully preserve the consequences of the dynamical breaking of chiral symmetry and its activation by the underlying quark–gluon dynamics. We also note that the GMOR formlua has been 'derived' in the past, but such analyses in general appear to have brushed over the various subtleties presented herein.

We exploit the GCM of QCD which has proven to be remarkably successful in modelling low energy QCD (for reviews see Cahill 1992; Cahill and Gunner 1997*a*; Tandy 1996, 1997). The nature of this model is discussed in Section 2. It should be noted that the GCM generates constituent hadrons which necessarily have the form of ladder states. The non-ladder diagrams then arise from hadronic functional integrations over constituent ladder hadrons. In Section 3 we show the difference between the full or exact and the constiutent pion. This involves the use of effective actions and the fact that these effective actions refer to constituent hadrons. In Section 4 the effective action for the chiral limit constituent pion is discussed. In Section 5 the constituent quark correlators are given as solutions of the CQE. The CQE arise as the Euler–Lagrange equations of the hadronised effective action for the GCM. They define the constituent quarks. Fluctuations about the minimum action configurations introduce constituent mesons, and these are described by special BSE. The full (observable) states are produced by dressing each of the constituent states by other states, as is made clear by the functional integral formalism in Section 2. In Section 6 the constituent pion mass formula (6) is derived, but here generalised to include a momentum dependent quark current mass. In Section 7 the Langfeld-Kettner identity is generalised to include the vector response function and the momentum dependent quark current mass. This identity leads from (6) to the GMOR form in (1).

2. The Global Colour Model of QCD

An overview and an insight into the nature of the non-perturbative low energy hadronic regime of QCD is provided by the functional integral hadronisation of QCD. In the functional integral approach correlation functions of QCD are defined by

$$\mathcal{G}(..,x,...) = \int \mathcal{D}\bar{q}\mathcal{D}q\mathcal{D}\mathcal{A}....q(x).....\exp(-S_{QCD}[A,\bar{q},q])\,, \tag{8}$$

in which the kernel includes (but not shown) gluon string structures that render $\mathcal{G}(..,x,...)$ gauge invariant. One example of (8) would be the pion correlation function, which may be defined by the connected part of

$$\mathcal{G}_\pi(x,y,z,w) = \int \mathcal{D}\bar{q}\mathcal{D}q\mathcal{D}A....\bar{q}(x)i\gamma_5\tau_i q(y)..\bar{q}(z)i\gamma_5\tau_i q(w)..\exp(-S_{\rm QCD}[A,\bar{q},q])\,. \tag{9}$$

Even the π–π scattering amplitude is defined by such a functional integral. The pion mass is defined by the position of the pole, with respect to the centre-of-mass (cm) momentum, of the Fourier transform of the translation invariant amplitude $\mathcal{G}_\pi$. Apart from lattice computations a direct computation of these functional integrals is not attempted. Amplitudes, such as (9), when the on-mass-shell conditions are imposed, define the observables of QCD, such as the pion. Theoretical analysis of these amplitudes proceeds by more circumspect techniques, some of which we clarify here.

The correlation functions, as in (9), may be extracted from the generating functional of QCD, $Z_{\rm QCD}[\bar{\eta},\eta,..]$, defined in (10). However, the interactions of low energy hadronic physics, such as π–π scattering, are known to be well described by effective actions which refer only to hadronic states, though the various parameters in these effective actions could only be determined by fitting to experimental data. Hence we expect that the correlations, such as (9), should also be extractable from a hadronic functional integral (as is indeed possible) namely

$$Z_{\rm QCD}[\bar{\eta},\eta,..] = \int \mathcal{D}\bar{q}\mathcal{D}q\mathcal{D}A\exp(-S_{\rm QCD}[A,\bar{q},q] + \bar{\eta}q + \bar{q}\eta) \tag{10}$$

$$\approx \int \mathcal{D}\pi\mathcal{D}\overline{N}\mathcal{D}N...\exp(-S_{\rm had}[\pi,..,\overline{N},N,..] + J_\pi[\bar{\eta},\eta]\pi + ..) \tag{11}$$

$$= Z_{\rm had}[J_\pi[\bar{\eta},\eta],...]\,, \tag{12}$$

which is a hadronic generating functional $Z_{\rm had}[J_\pi[\bar{\eta},\eta],...]$, in which source terms for the various hadrons are naturally induced. A partial derivation of this functional transformation proceeds as follows. First, and not showing source terms for convenience, the gluon integrations are formally performed (ghosts also not shown):

$$
\begin{aligned}
&\int \mathcal{D}\bar{q}\mathcal{D}q\mathcal{D}A\exp(-S_{\rm QCD}[A,\bar{q},q]) \\
&\qquad\qquad = \int \mathcal{D}\bar{q}\mathcal{D}q\exp\Big(- \int \bar{q}(-\gamma.\partial+\mathcal{M})q \\
&\qquad\qquad\qquad + \frac{g_0^2}{2}\int j_\mu^a(x)j_\nu^a(y)\mathcal{G}_{\mu\nu}(x-y) + \frac{g_0^3}{3!}\int j_\mu^a j_\nu^b j_\rho^c \mathcal{G}_{\mu\nu\rho}^{abc} + ..\Big)\,, \qquad (13)
\end{aligned}
$$

where $j_\mu^a(x) = \bar{q}(x)(\lambda^a/2)\gamma_\mu q(x)$, g_0 is the coupling constant and $\mathcal{G}_{\mu\nu}(x)$ is the exact pure gluon correlator

$$
\mathcal{G}_{\mu\nu}(x-y) = \frac{\int \mathcal{D}AA_\mu^a(x)A_\nu^a(y)\exp(-S_{\rm QCD}[A,0,0])}{\int \mathcal{D}A\exp(-S_{\rm QCD}[A,0,0])}\,. \qquad (14)
$$

A variety of techniques for computing $\mathcal{G}_{\mu\nu}(x)$ exists, such as the gluonic DSE (Brown and Pennington 1989) and lattice simulations (Marenzoni *et al*. 1995). The terms of higher order than the term quartic in the quark fields are beyond our ability to retain in the analysis. However, we can model, in part, the effect of these higher order terms by replacing the coupling constant g_0 by a momentum dependent quark–gluon coupling $g(s)$, and neglecting terms like $\mathcal{G}_{\mu\nu\rho}^{abc}$ and higher order. This $g(s)$ is a restricted form of vertex function. This modification $g_0^2\mathcal{G}_{\mu\nu}(p) \to D_{\mu\nu}(p) = g(p^2)^2\mathcal{G}_{\mu\nu}(p)$ and the truncation in (13) then defines the GCM. We call $D_{\mu\nu}(p)$ the effective gluon correlator.

The GCM is equivalent to using a quark–gluon field theory with the action

$$
S_{\rm GCM}[\bar{q},q,A_\mu^a] = \int \Big(\bar{q}(-\gamma.\partial+\mathcal{M}+iA_\mu^a\frac{\lambda^a}{2}\gamma_\mu)q + \tfrac{1}{2}A_\mu^a D_{\mu\nu}^{-1}(i\partial)A_\nu^a\Big)\,. \qquad (15)
$$

Here $D_{\mu\nu}^{-1}(p)$ is the matrix inverse of $D_{\mu\nu}(p)$, which in turn is the Fourier transform of $D_{\mu\nu}(x)$. This action has only a global colour symmetry, unlike the local colour symmetry that characterises QCD. The gluon self-interactions, that arise as a consequence of the local colour symmetry in (14), lead to $D_{\mu\nu}^{-1}(p)$ being non-quadratic. Its precise form is unknown. In the GCM a general non-quadratic form for $D_{\mu\nu}$ is retained, modelling this significant property of QCD. In Landau gauge we have

$$
D_{\mu\nu}(p) = \Big(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\Big)D(p^2) \text{ and } \mathcal{G}_{\mu\nu}(p) = \Big(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\Big)\mathcal{D}(p^2)\,. \qquad (16)
$$

One can extract the effective $D(s=p^2)$ from meson data as in Cahill and Gunner (1995*b*, 1997*b*), with the most recent results shown in Fig. 1. Also shown are the lattice results for the pure gluon correlator $\mathcal{D}(s)$ from Marenzoni *et al*. (1995). Dividing the effective GCM $D(s)$ by the lattice $\mathcal{D}(s)$ gives $g^2(s)$, and the resultant effective coupling $g(s)$ is shown in Fig. 2. This shows a strong infrared enhancement of the quark–gluon coupling. This $g(s)$ is compared in Fig. 2 with recent lattice calculation results by Skullerud (1998).

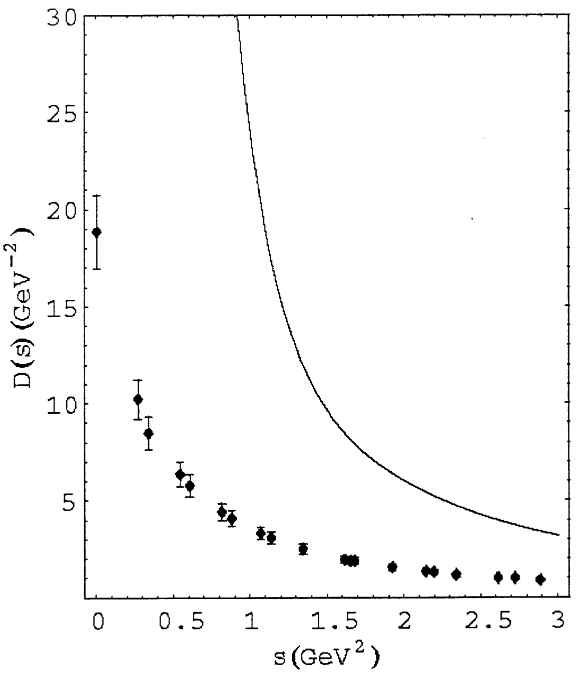

Fig. 1. Effective gluon correlator $D(s)$ (curve) extracted by fitting the GCM to meson data. Also shown are the lattice results for the pure gluon correlator $\mathcal{D}(s)$ from Marenzoni *et al.* (1995). The error bars indicate uncertainties arising from the value of the lattice spacing $a = 0{\cdot}50 \pm 0{\cdot}025$ GeV^{-1}.

The agreement between these completely independent methods of obtaining $g(s)$ demonstrates that the effective gluon correlator, which defines the GCM, is more than a phenomenological treatment, and that it is in fact the actual gluon correlator between quark currents with vertices, and that the higher order gluon correlators between quark currents in hadrons (i.e. over distances $R\ \leq 1$ fm) are not operative. These are presumably relevant to the question of confinement of coloured states. We can summarise this by saying that the GCM is QCD for correlators with $R <$ hadronic size: $\mathrm{QCD}|_{R<\mathrm{had}} \approx \mathrm{GCM}$.

Hadronisation involves a sequence of functional integral calculus changes of variables involving, in part, the transformation to bilocal meson and diquark fields, and then to the usual local meson and baryon fields (sources not shown):

$$Z = \int \mathcal{D}\bar{q}\mathcal{D}q\mathcal{D}A \exp(-S_{\mathrm{QCD}}[A,\bar{q},q] + \bar{\eta}q + \bar{q}\eta)$$

$$\approx \int \mathcal{D}\bar{q}\mathcal{D}q\mathcal{D}A \exp(-S_{\mathrm{GCM}}[A,\bar{q},q] + \bar{\eta}q + \bar{q}\eta) \qquad \text{(GCM truncation)}$$

$$= \int D\mathcal{B}D\mathcal{D}D\mathcal{D}^{\star}\exp(-S[\mathcal{B},\mathcal{D},\mathcal{D}^{\star}]) \qquad \text{(bilocal fields)} \tag{17}$$

$$= \int \mathcal{D}\pi...\mathcal{D}\overline{N}\mathcal{D}N...\exp(-S_{\rm had}[\pi,...,\overline{N},N,..]) \qquad \text{(local fields)}. \tag{18}$$

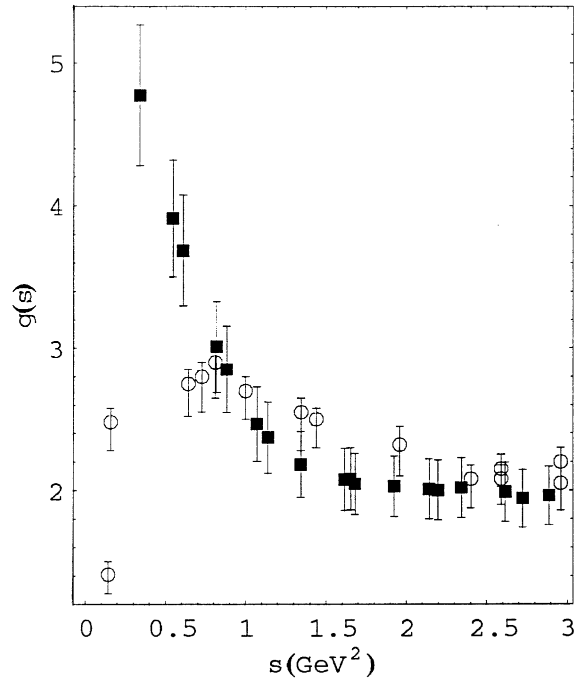

Fig. 2. GCM quark–gluon coupling $g(s)$ (squares). Here $g^2(s)$ was obtained by dividing the GCM effective gluon correlator $D(s)$ (curve in Fig. 1) by the lattice gluon correlator $\mathcal{D}(s)$ (Marenzoni *et al.* 1995). The error bars arise from the lattice spacing uncertainty and from systematic errors in the fitting of the GCM to the meson data. Also shown is $g(s)$ from the lattice calculation of Skullerud (circles) (1998).

The derived hadronic action that finally emerges from this action sequencing, to low order in fields and derivatives, has the form

$S_{\rm had}[\pi,...,\overline{N},N,..] =$

$$\int d^4x{\rm tr}\{\overline{N}(\gamma.\partial + m_N + \Delta m_N - m_N\sqrt{2}i\gamma_5\pi^a\mathcal{T}^a + ..)N\}$$

$$+ \int d^4x\left[\frac{f_\pi^2}{2}[(\partial_\mu\pi)^2 + m_\pi^2\pi^2] + \frac{f_\rho^2}{2}[-\rho_\mu\Box\rho_\mu + (\partial_\mu\rho_\mu)^2 + m_\rho^2\rho_\mu^2]\right.$$

$$+ \frac{f_\omega^2}{2}[\rho \to \omega] - f_\rho f_\pi^2 g_{\rho\pi\pi}\rho_\mu.\pi \times \partial_\mu \pi - i f_\omega f_\pi^3 \epsilon_{\mu\nu\sigma\tau}\omega_\mu \partial_\nu \pi.\partial_\sigma \pi \times \partial_\tau \pi$$
$$- i f_\omega f_\rho f_\pi G_{\omega\rho\pi}\epsilon_{\mu\nu\sigma\tau}\omega_\mu \partial_\nu \rho_\sigma.\partial_\tau \pi$$
$$+ \frac{\lambda i}{80\pi^2}\epsilon_{\mu\nu\sigma\tau}\mathrm{tr}(\pi.F\partial_\mu\pi.F\partial_\nu\pi.F\partial_\sigma\pi.F\partial_\tau\pi.F) + \ldots]\,. \tag{19}$$

The bilocal fields in (17) naturally arise and correspond to the fact that, for instance, mesons are extended states. In (9) we can see that the pion arises as a correlation function for two bilinear quark structures. This bosonisation/hadronisation arises by functional integral calculus changes of variables that are induced by generalised Fierz transformations that emerge from the colour, spin and flavour structure of QCD.

The final functional integration in (18) over the hadrons gives the hadronic observables, and amounts to dressing each hadron by, mainly, lighter mesons. The basic insight is that the quark–gluon dynamics, in (9), is fluctuation dominated, whereas the hadronic functional integrations in (18) are not, and for example the meson dressing of bare hadrons is known to be almost perturbative. In performing the change of variables, essentially normal mode techniques are used. In practice this requires detailed numerical computation of the gluon correlator, quark correlators, and meson and baryon correlators. The mass-shell states of the latter are determined by covariant Bethe–Salpeter and Faddeev equations. The Faddeev computations are made feasible by using the diquark correlation correlators; the diquarks being quark–quark correlations within baryons.

3. Constituent Hadrons

We now come to one of the main points. Using the functional hadronisation we can write $\mathcal{G}_\pi$ in the form (9) or, from (18), in the form:

$$\mathcal{G}_\pi(X,Y) = \int \mathcal{D}\pi..\mathcal{D}\overline{N}\mathcal{D}N...\pi(X)\pi(Y)\exp(-S_{\mathrm{had}}[\pi,...,\overline{N},N,..])\,, \tag{20}$$

in which $X = \frac{1}{2}(x+y)$ and $Y = \frac{1}{2}(z+w)$ are cm coordinates for the pion. We note that now the pion field appears in $S_{\mathrm{had}}[\pi,...,\overline{N},N,..]$ in the exponent of (20), with an effective-action mass parameter m_π. As we now discuss, it is important to clearly distinguish between this mass, and the equations which define its value, from the pion mass that would emerge from the evaluation of the functional integrals in (9) or (20). Equations (9) or (20) define the observable pion mass, whereas the mass in (19) defines the constituent pion mass. There is no reason for these to be equal in magnitude, though they may well both be given by the generic GMOR form.

How do the constituent hadrons arise in (18)? In going from (17) to (18) an expansion about the minimum of $S[\mathcal{B},\mathcal{D},\mathcal{D}^\star]$ is performed. First the minimum is defined by Euler–Lagrange equations (ELE):

$$\frac{\delta S}{\delta \mathcal{B}} = 0 \text{ and } \frac{\delta S}{\delta \mathcal{D}} = 0\,. \tag{21}$$

These equations have solutions $\mathcal{B} \neq 0$ and $\mathcal{D} = 0$. Equation (21), with $\mathcal{D} = 0$, after some analysis, is seen to be nothing more than the CQE for the constituent quark correlator [see (29) and (30) in Section 5]. The occurrence of the rainbow form of these equations is *not* an approximation within the GCM. The non-rainbow diagrams, corresponding to various more complicated gluon dressing of the quarks, are generated by the additional functional integrals in (18). The generation of a minimum with $\mathcal{B} \neq 0$ is called the formation of a condensate, here a $\bar{q}q$ condensate. In the GCM $\mathcal{D} = 0$ means that no diquark or anti-diquark type condensates are formed.

Next in going from (17) to (18) we must consider the fluctuations or curvatures of the action for the bilocal fields. One finds that the curvature $\delta^2 S/\delta\mathcal{B}\delta\mathcal{B}$ when inverted gives the constituent meson correlators, but with only ladder gluon exchanges. Again non-ladder diagrams are generated by the functional hadronic integrations in (18). Similarly inversion of the curvatures in the diquark sector $\delta^2 S/\delta\mathcal{D}\delta\mathcal{D}^*$ leads to diquark correlators, but with only ladder gluon exchanges between the constituent quarks.

We note that the generalised bosonisation with meson and diquark fields leads to some additional complications that we shall consider elsewhere, but which do not impinge on the basic point being made here. This meson–baryon hadronisation is based upon a generalised Fierz transformation that induces the appropriate colour singlet anti-quark–quark correlations, and colour anti-triplet quark–quark correlations that are in the correct colour state for quark–quark correlations within a colour singlet baryon.

Hence we see that in the exponent in (18) there arise special correlators for quarks, mesons, diquarks and even baryons. These particular correlators and their associated fields will be defined to be the *constituent* states. They could also be described as core states. The observables are generated by the hadronic functional integrals in (18), and correspond to dressing each constituent or core state with other such states. Hence the parameters in the hadronic effective action in (19) refer to the constituent states.

Nevertheless, one often compares these parameters with the parameter values for the fully dressed constituent states, that is the observable hadronic states. This appears to be valid because in general the dressing produces only a small shift in the parameter values. However, one known exception is the nucleon where pion dressing of the constituent nucleon state reduces its mass by some 200–300 MeV. This mass shift emerges from consideration of the functional integral

$$\mathcal{G}_N(X,Y) = \int \mathcal{D}\pi..\overline{N}(X)N(Y)\exp(-S_{\rm had}[\pi,...,\overline{N},N,..])\,, \tag{22}$$

where mainly the pions, but as well other mesons are used to dress the nucleon. Of course one usually casts this into the form of a non-linear integral equation for the meson dressed nucleon correlation function.

4. Chiral Limit Constituent Pions

When the quark current masses $\mathcal{M} \to 0$ $S[\bar{q}, q, A^a_\mu]$ has an additional global $U_L(N_F) \otimes U_R(N_F)$ chiral symmetry: writing $\bar{q}\gamma_\mu q = \bar{q}_R\gamma_\mu q_R + \bar{q}_L\gamma_\mu q_L$ where $q_{R,L} = P_{R,L}q$ and $\bar{q}_{R,L} = \bar{q}P_{L,R}$ we see that these two parts are separately invariant under $q_R \to U_R q_R, \bar{q}_R \to \bar{q}_R U_R^\dagger$ and $q_L \to U_L q_L, \bar{q}_L \to \bar{q}_L U_L^\dagger$. Its consequences may be explicitly traced through the GCM hadronisation. First the ELE $\delta S/\delta\mathcal{B} = 0$ have degenerate solutions. In terms of the constituent quark correlator this degeneracy manifests itself in the form

$$G(q;V) = [iA(q)q.\gamma + VB(q)]^{-1} = \zeta^\dagger G(q;\mathbf{1})\zeta^\dagger \,, \tag{23}$$

where

$$\zeta = \sqrt{V}, \quad V = \exp(i\sqrt{2}\gamma_5\pi^a F^a)\,, \tag{24}$$

in which the $\{\pi^a\}$ are arbitrary real constants. The degeneracy of the minimum implies that some fluctuations in $\delta^2 S/\delta\mathcal{B}\delta\mathcal{B}$ have zero mass; these are the NG BSE states, and this indicates the realisation of the Goldstone theorem.

In the hadronisation, in going from (17) to (18), new variables are forced upon us to describe the degenerate minima (vacuum manifold). This is accomplished by a coordinatisation of the angle variables $\{\pi^a\}$:

$$U(x) = \exp(i\sqrt{2}\pi^a(x)F^a)\,, \tag{25}$$

$$V(x) = P_L U(x)^\dagger + P_R U(x) = \exp(i\sqrt{2}\gamma_5\pi^a(x)F^a)\,. \tag{26}$$

The NG part of the hadronisation then gives rise to the constituent pion effective action

$$\int d^4x \Bigg(\frac{f_\pi^2}{4}\mathrm{tr}(\partial_\mu U\partial_\mu U^\dagger) + \kappa_1 \mathrm{tr}(\partial^2 U\partial^2 U^\dagger) + \frac{\rho}{2}\mathrm{tr}\Bigg([\mathbf{1} - \frac{U+U^\dagger}{2}]\mathcal{M}\Bigg)$$

$$+\kappa_2\mathrm{tr}([\partial_\mu U\partial_\mu U^\dagger]^2) + \kappa_3\mathrm{tr}(\partial_\mu U\partial_\nu U^\dagger\partial_\mu U\partial_\nu U^\dagger) + \ldots\Bigg)\,. \tag{27}$$

This is the ChPT effective action (Gasser and Leutwyler 1985), but with the added insight that all coefficients are given by explicit and convergent integrals in terms of A and B, which are in turn determined by $D_{\mu\nu}$. The higher order terms contribute to $\pi\pi$ scattering. The dependence of the ChPT coefficients upon $D_{\mu\nu}$ has been studied in Cahill and Gunner (1995*b*) and Frank and Meissner (1996), in which the GCM constituent pion expressions for the various parameters were used. However, in view of the apparent generic role of the GMOR relation one should keep in mind the possibility that the functional form of the dependences of the parameters $\kappa_1, ..$ on the quark correlation functions A and B might also be generic. At present the final functional integral dressing to obtain the pion observables has not been carried out; this amounts to the assumption that the constituent pion forms are sufficiently accurate. The hadronisation procedure also gives a full account of NG-meson–nucleon coupling.

The GCM is in turn easily related to a number of the more phenomenological models of QCD. They include the Nambu–Jona-Lasinio model (NJL) (Reinhardt 1990), chiral perturbation theory (ChPT) (Gasser and Leutwyler 1985), the MIT and cloudy bag model (Thomas *et al.* 1981), soliton models (Frank and Tandy 1992), quantum hadrodynamics (QHD) (Serot 1992) and the quantum meson coupling model (QMC) (Guichon 1988). We also indicate that the pure gluon correlation function, in (14), may be obtained from lattice computations, and used in the GCM.

5. Action Minimum and Pionic Fluctuations

The GCM involves the solution of various integral equations for the constituent correlation functions. As we saw in Section 3, the first equation involves the determination of the constituent quark correlator from the CQE in, necessarily, the rainbow form (the so-called vacuum equation of the GCM in Landau gauge):

$$G^{-1}(p) = i\not{p} + m(p^2) + \int \frac{d^4q}{(2\pi)^4} D_{\mu\nu}(p-q)\gamma_\mu \frac{\lambda^a}{2} G(q) \frac{\lambda^a}{2}\gamma_\nu \,. \tag{28}$$

Separating this CQE into its scalar and vector parts (see equation 3), we obtain

$$B(p^2;m) = 4\int \frac{d^4q}{(2\pi)^4} D(p-q). \frac{B(q^2;m)+m(s)}{q^2A(q^2;m)^2 + [B(q^2;m)+m(s)]^2}\,, \tag{29}$$

$$[A(p^2;m)-1]p^2 = \tfrac{4}{3}\int \frac{d^4q}{(2\pi)^4} D(p-q)\left(p.q + 2\frac{q.(p-q)p.(p-q)}{(p-q)^2}\right)$$
$$\times \frac{A(q^2;m)}{q^2A(q^2;m)^2 + [B(q^2;m)+m(s)]^2}\,. \tag{30}$$

We have included, for generality, a phenomenological momentum dependent current mass $m(s)$ for the quarks. The usual procedure, as in Langfeld and Kettner, is to introduce a cutoff Λ, and to choose an s independent $m(s)$, but with the value of that constant m now Λ dependent. This is in fact our final choice, and the results are shown in Fig. 3. However, our analysis supports a more general result with $m(s)$ dependent on s, in which case the GMOR relation in (2) and (3) needs to be re-written with $m(s)$ included inside the integration. A colour factor arises from $\sum_a \frac{\lambda^a}{2}\frac{\lambda^a}{2} = \frac{4}{3}\mathbf{1}$, and in (29) a factor of 3 from the Dirac algebra. Using Fourier transforms (29) may be written in the form, here for $m=0$,

$$D(x) = \tfrac{1}{4}\frac{B(x)}{\sigma_s(x)}\,, \tag{31}$$

which implies that knowledge of the constituent quark correlator determines the effective GCM gluon correlator. Multiplying (31) by $B(x)/D(x)$, and using Parseval's identity for the RHS, we obtain the identity

$$\int d^4x \frac{B(x)^2}{D(x)} = 4\int \frac{d^4q}{(2\pi)^4} B(q)\sigma_s(q)\,. \tag{32}$$

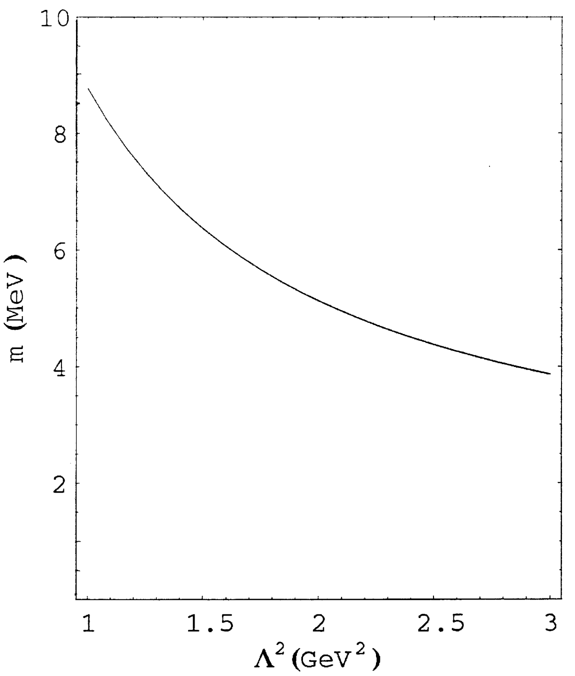

Fig. 3. Averaged $u-d$ quark current mass from (1) for various cutoffs Λ in the condensate integral in (2), when using the $\sigma_s(s)$ corresponding to the effective gluon correlator shown in Fig. 1, and using $m_\pi = 138{\cdot}5$ MeV and $f_\pi = 93{\cdot}0$ MeV.

The second basic equation is the ladder form BSE for the constituent pion mass-shell state, which arises from the mesonic fluctuations about the minimum determined by (29) and (30). Again this ladder form cannot be generalised without causing double counting of some classes of diagrams at a later stage, and without also damaging the intricate interplay between (29), (30) and the BSE:

$$\Gamma^f(p,P) = 2\int \frac{d^4q}{(2\pi)^4} D(p-q)\mathrm{tr}_{\mathrm{SF}}(G_+T^gG_-T^f)\Gamma^g(q,P)\,, \tag{33}$$

where $G_\pm = G(q \pm \frac{1}{2}P)$. This BSE is for isovector NG bosons, and only the dominant $\Gamma = \Gamma^f T^f i\gamma_5$ amplitude is retained. The spin trace arises from projecting onto this dominant amplitude. Here $\{T^b, b = 1,..,N_F^2-1\}$ are the generators of SU(N_F), with $\mathrm{tr}(T^fT^g) = \frac{1}{2}\delta_{fg}$.

The BSE (33) is an implicit equation for the mass shell $P^2 = -M^2$. It has solutions *only* in the time-like region $P^2 \leq 0$. Fundamentally this is ensured by (29) and (30) being the specification of an absolute minimum of an effective action after a bosonisation. Nevertheless, the loop momentum is kept in the space-like region $q^2 \geq 0$; this mixed metric device ensures that the quark and gluon correlators remain close to the real space-like region where they have been most thoroughly studied. Very little is known about these correlators in the time-like region $q^2 < 0$.

The non-perturbative quark–gluon dynamics are expressed here in (29) and (30). Even when $m = 0$ equation (29) can have non-perturbative solutions with $B \neq 0$. This is the dynamical breaking of chiral symmetry.

When $m = 0$ equation (33) has a solution for $P^2 = 0$; the Goldstone theorem effect. For the zero linear momentum state $\{P_0 = 0, \vec{P} = \vec{0}\}$ it is easily seen that (33) reduces to (29) with $\Gamma^f(q,0) = B(q^2)$. When $\vec{P} \neq \vec{0}$, then $\Gamma^f(q,P) \neq B(q)$, and (33) must be solved for $\Gamma^f(q,P)$.

6. Constituent Pion Mass

We shall now determine an accurate expression for the mass of the constituent pion when $m(s)$ is small but non-zero. This amounts to finding an analytic solution to the BSE (33), when the constituent quark correlators are determined by (29) and (30). The result will be accurate to order m:

For small $m \neq 0$ we can introduce the Taylor expansions in $m(s)$:

$$B(s;m) + m(s) = B(s) + m(s)\epsilon_s(s) + O(m^2)\,, \tag{34}$$

$$A(s,m) = A(s) + m(s)\epsilon_v(s) + O(m^2)\,. \tag{35}$$

For large space-like s we find that $\epsilon_s \to 1$, but for small s we find that $\epsilon_s(s)$ can be significantly larger than 1. This is an infrared region dynamical enhancement of the quark current mass by gluon dressing, and indicates the strong response of the chiral limit constituent quark correlator to the turning on of the current mass. A plot of $\epsilon_s(s)$ is shown in Cahill and Gunner (1995*a*). Higashijima (1984) and Elias (1993) have also reported similar enhancements of the current quark masses in the infrared region.

Even in the chiral limit the constituent quark running mass $M(s) = B(s)/A(s)$ is essential for understanding any non-perturbative QCD quark effects. The integrand of a BSE contains the gluon correlation function, constituent quark correlation functions and the form factor for the state (see for example equation 33). This integrand shows strong peaking at typically $s \approx 0{\cdot}3\,\mathrm{GeV}^2$. At this value we find that $M(s) \approx 270$ MeV. This is a property of the constituent hadrons. It does not include any effects from the dressing of these hadrons via (18). This mass is called the constituent quark mass. Because of the infrared region enhancement of the quark current mass we find that this constituent mass rises quickly with quark current mass.

Because the pion mass m_π is small when m is small, we can perform an expansion of the P_μ dependence in the kernel of (33). Since the analysis is Lorentz covariant we can, without loss of validity, choose to work in the rest frame with $P = (im_\pi, \vec{0})$, giving

$$\Gamma(p) = \tfrac{1}{6}m_\pi^2 \int \frac{d^4q}{(2\pi)^4} D(p-q) I(s) \Gamma(q) \tag{36}$$

$$+4\int \frac{d^4q}{(2\pi)^4} D(p-q) \frac{1}{s[A(s)+m(s).\epsilon_v(s)]^2 + [B(s)+m(s).\epsilon_s(s)]^2} \Gamma(q) + ...,$$

where

$$I(s) = 6(\sigma_v^2 - 2(\sigma_s{\sigma_s}' + s\sigma_v{\sigma_v}') - s[\sigma_s{\sigma_s}'' - ({\sigma_s}')^2] - s^2(\sigma_v{\sigma_v}'' - ({\sigma_v}')^2)\,. \tag{37}$$

By using Fourier transforms the integral equation (36), now with explicit dependence on m_π, can be expressed in the form of a variational mass functional,

$$\begin{aligned} m_\pi[\Gamma]^2 = &-\frac{24}{f_\pi[\Gamma]^2}\int \frac{d^4q}{(2\pi)^4}\frac{\Gamma(q)^2}{s[A(s)+m(s)\epsilon_v(s)]^2+[B(s)+m(s)\epsilon_s(s)]^2} \\ &+\frac{6}{f_\pi[\Gamma]^2}\int d^4x\frac{\Gamma(x)^2}{D(x)}\,, \end{aligned} \tag{38}$$

in which

$$f_\pi[\Gamma]^2 = \int \frac{d^4q}{(2\pi)^4} I(s)\Gamma(q)^2\,. \tag{39}$$

The functional derivative $\delta m_\pi[\Gamma]^2/\delta\Gamma(q) = 0$ reproduces (36). The mass functional (38) and its minimisation is equivalent to the constituent pion BSE in the near chiral limit. To find an estimate for the minimum we need only note that the change in m_π^2 from its chiral limit value of zero will be of first order in m, while the change in the zero linear momentum frame $\Gamma(q)$ from its chiral limit value $B(q^2)$ will be of second order in m.

Hence to obtain m_π^2 to lowest order in m, we may replace $\Gamma(q)$ by $B(q^2)$ in (38), and we have that the constituent pion mass is given by

$$\begin{aligned} m_\pi^2 = &\frac{48}{f_\pi[B]^2}\int \frac{d^4q}{(2\pi)^4}m(s)\frac{\epsilon_s(s)B(s)+s\epsilon_v(s)A(s)}{sA(s)^2+B(s)^2}\frac{B(s)^2}{sA(s)^2+B(s)^2} \\ &-\frac{24}{f_\pi[B]^2}\int \frac{d^4q}{(2\pi)^4}\frac{B(s)^2}{sA(s)^2+B(s)^2}+\frac{6}{f_\pi[B]^2}\int d^4x\frac{B(x)^2}{D(x)}+O(m^2)\,. \end{aligned} \tag{40}$$

However, the pion mass has been shown to be zero in the chiral limit. This is confirmed as the two $O(m^0)$ terms in (40) cancel because of the identity (32). Note that it might appear that f_π would contribute an extra m dependence from its kernel in (37). However, because the numerator in (38) is already of order m, this extra contribution must be of higher order in m.

Hence, we finally arrive at the analytic expression, to $O(m)$, for the constituent NG boson (mass)^2 from the solution of the BSE,

$$m_\pi^2 = \frac{48}{f_\pi[B]^2}\int \frac{d^4q}{(2\pi)^4}m(s)\frac{\epsilon_s(s)B(s)+s\epsilon_v(s)A(s)}{sA(s)^2+B(s)^2}\frac{B(s)^2}{sA(s)^2+B(s)^2}+O(m^2)\,. \tag{41}$$

Equation (6), or (41) with the $m(s)$, is the new form of the NG mass formula derived in Cahill and Gunner (1995*a*). It would appear that expression (6) is manifestly different to the conventional GMOR form in (1) and (2). However, here we generalise the identity found by Langfeld and Kettner (1996) that shows these forms to be equivalent.

7. Relating the Mass Formulae

Inserting (34) and (35) into (29), and expanding in powers of $m(s)$, we obtain up to terms linear in m, and after using (29) with $m = 0$ to eliminate the $O(m^0)$ terms, we get

$$m(p^2)\epsilon_s(p^2) = m(p^2) + 4\int \frac{d^4q}{(2\pi)^4} D(p-q) \frac{m(q^2)\epsilon_s(q^2)}{q^2A(q^2)^2 + B(q^2)^2}$$

$$-4\int \frac{d^4q}{(2\pi)^4} D(p-q) \frac{B(q^2)^2 2m(q^2)\epsilon_s(q^2)}{[q^2A(q^2)^2 + B(q^2)^2]^2}$$

$$-4\int \frac{d^4q}{(2\pi)^4} D(p-q) \frac{B(q^2)A(q^2)2m(q^2)q^2\epsilon_v(q^2)}{[q^2A(q^2)^2 + B(q^2]^2}\,. \tag{42}$$

We now multiply (42) throughout by $B(p^2)/[p^2A(p^2)^2 + B(p^2)^2]$, and integrate with respect to p. Using again the chiral limit of (29) there is some cancellation of terms, and we are left with a generalised Langfeld–Kettner identity

$$2\int d^4p \frac{B(p^2)^2}{p^2A(p^2)^2 + B(p^2)^2}\left(\frac{B(p^2)m(p^2)\epsilon_s(p^2)}{p^2A(p^2)^2 + B(p^2)^2} + \frac{p^2A(p^2)m(p^2)\epsilon_v(p^2)}{p^2A(p^2)^2 + B(p^2)^2}\right)$$

$$= \int d^4p \frac{m(p^2)B(p^2)}{p^2A(p^2)^2 + B(p^2)^2}\,. \tag{43}$$

Remarkably, noting (4) and (5), we see that using this identity in (6) or (41) finally completes the derivation of a *generalised* GMOR expression for the mass of the constituent pion,

$$m_\pi^2 = \frac{12}{f_\pi^2}\int \frac{d^4q}{(2\pi)^4}[m_u(s) + m_d(s)]\sigma_s(q^2)\,. \tag{44}$$

The usual GMOR relation in (1) and (2) follows when the momentum dependent mass $m(s)$ is restricted to a constant up to some momentum Λ and zero beyond that, i.e. if one introduces the usual cutoff prescription. Then we must use a Λ dependent, but s independent current mass $m_{u,d}(\Lambda)$, as discussed in Langfeld and Kettner (1996).

We see that despite its apparently simple form the GMOR expression actually depends on two identities that follow from the *non-linear* constituent quark equation (CQE), and on the subtle interplay between this CQE and the BSE for the constituent pion. These in turn arise from the careful self-consistency rendered by the functional integral prescription that ensures that the fluctuation spectrum for the bilocal action is precisely related to the Euler–Lagrange equations. *Ad hoc* alterations to this connection will invalidate the above derivation of the GMOR expression.

Finally, having identified various subtleties in relating the pion mass to the quark current masses, we show in Fig. 3 the averaged $u-d$ quark current mass from (1) for various cutoffs Λ(GeV) in the condensate integral in (2), when using the $\sigma_s(s)$ corresponding to the effective gluon correlator shown in Fig. 1, and using $m_\pi = 138\cdot 5\,\mathrm{MeV}$ and $f_\pi = 93\cdot 0\,\mathrm{MeV}$. We show results for the cutoff varying from $1\,\mathrm{GeV}^2$ to $3\,\mathrm{GeV}^2$. Fitting to the form $m(\Lambda) = m_R(\mu^2)/[\ln(\Lambda^2/\mu^2)]^\lambda$, for sufficiently large Λ, we obtain $m_R(\mu^2 = 1\,\mathrm{GeV}^2) = 4\cdot 8\,\mathrm{MeV}$, giving $\rho(\mu^2 = 1\,\mathrm{GeV}^2) = (258\,\mathrm{MeV})^3$. As shown herein the use of the GMOR relation is exactly equivalent, to $O(m)$, to the much more difficult task of solving the pionic BSE. The current mass results in Fig. 3 can be understood to arise from the lattice gluon correlator in Fig. 1 together with the quark–gluon coupling of Fig. 2, which we have extracted from the meson data (Cahill and Gunner 1997*b*). These results indicate the efficacy of combining lattice and continuum modellings in QCD studies—an approach which is just coming to fruition.

8. Conclusion

We have indicated the careful considerations that must be given to modelling QCD via the GCM and the manner in which this leads to hadronic effective actions. We have defined constituent quarks, mesons, diquarks and baryons as those states that appear in the effective action, i.e. in the exponent, as in (18). These constituent states are then further dressed by the functional integrations in (18). Remarkably this GCM structuring of the quantum field theoretic analysis implies, at least in the simplest version of the GCM, that the constituent states are described by sums of rainbow or ladder diagrams, and that the functional integrations then build up all the remaining diagrams, amounting to the vast array of crossed diagrams and vertex dressings etc. The fact that in most cases these extra dressings do not cause large changes in the values of the constituent masses, coupling constants, etc., would appear to indicate that the effect of the inclusion of these extra diagrams is not manifestly large. Of course this is not surprising because the GCM hadronisation allows us to assess the significance of a constituent state through its mass; low mass states should be more important than very massive states. This implies that the pion dressing is the largest such effect. Inclusion of this dressing for the constituent nucleon is known to be significant, and amounts to the inclusion of various non-ladder diagrams in the observable nucleon.

We have also carefully indicated that it is the mass of the constituent pion that is analysed here, and by using various identities that follow from the non-linear equation for the constituent quark one can show that the mass of this constituent pion is indeed given by the GMOR formula, with the scalar part of the constituent quark correlation function appearing. This does not preclude the fact that presumably the observable pion also has its mass obeying a GMOR formula, but in which the full quark scalar correlation function appears. That is, the GMOR relation is generic. Discussions of the pion mass need to carefully indicate which pion it is that is being discussed.

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