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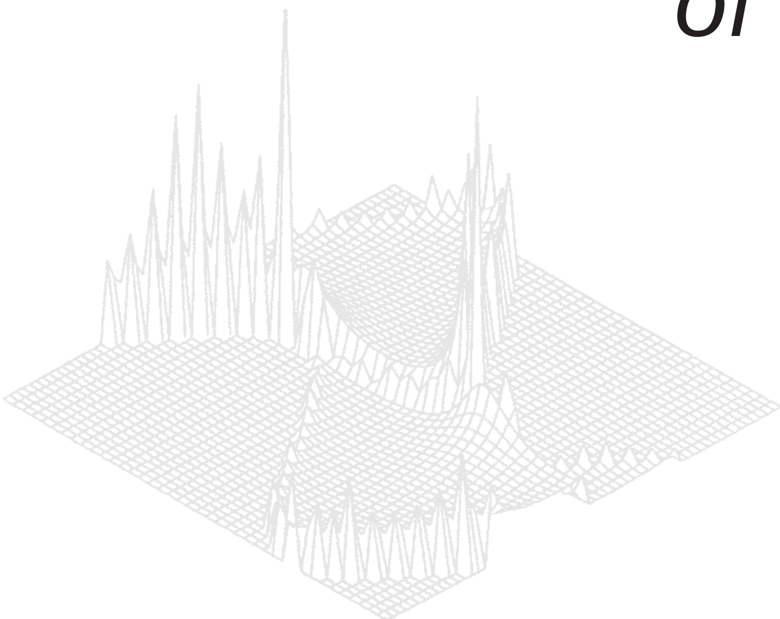
Australian Journal of Physics

Volume 51, 1998



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Aust. J. Phys., 1998, **51**, 959–69

Quasi-dipole Magnetic Fields

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Abstract

This paper describes the development of an analytical technique for including the modification to the poloidal magnetic field that is generated by the toroidal component of motion of the charged particles in the axisymmetric pulsar magnetosphere model of Mestel and co-workers. This 'quasi-dipole' technique, and its 'distorted dipole' case, are developed in such a way as to retain as much as possible of the earlier formalism in which the poloidal magnetic field was taken to be the dipole field of the star. The field is described in terms of a 'magnetic Stokes stream function', which is constant on each poloidal field line. The technique involves using the cylindrical polar radial coordinate and this stream function (in preference to the axial coordinate) as the independent variables, forming a non-orthogonal coordinate set in meridian planes. The field contains an unspecified function, which represents the modification to the dipole field of the star generated by the toroidal magnetospheric currents. Expressions are calculated for the curl of the poloidal magnetic field, which must be matched to an expression obtained for the toroidal electric current density from a magnetosphere model, thus determining the unspecified function.

1. Background

In the 30 years that have elapsed since pulsars were discovered, theorists have not managed to produce a self-consistent model of the pulsar magnetosphere—even for the simplified axisymmetric case of coincident magnetic and rotation axes. Some of the most interesting and promising approaches, particularly for the axisymmetric case, have been introduced by Professor Leon Mestel and his co-workers at the University of Sussex.

In particular, Mestel, Robertson, Wang and Westfold (1985; '$\mathrm{MR}\Omega^2$') proposed an axisymmetric pulsar magnetosphere model in which electrons leave the star with non-negligible speeds and flow with moderate acceleration, and with poloidal motion that is closely tied to poloidal magnetic field lines, before reaching $\mathcal{S}_L$, a limiting surface near which rapid acceleration occurs. As well as these 'Class I' flows, there exist 'Class II flows', which do not encounter a region of rapid acceleration (Burman 1984). The formalism introduced by $\mathrm{MR}\Omega^2$ to describe the moderately accelerated flows can be interpreted in terms of a plasma drift across the magnetic field, following injection along it (Burman 1985*a*).

The $\mathrm{MR}\Omega^2$ formalism for the description of the outflow fully incorporates the toroidal magnetic field generated by the poloidal flow. The general formalism leaves the poloidal magnetic field unspecified, but, in the early detailed development of

 10.1071/PH98003 0004-9506/98/060959$05.00

MRΩ^2 and in my papers (Burman 1985*b*, 1987), that field was taken to be the dipolar field of the star.

Numerical work by Fitzpatrick and Mestel (1988*a*, 1988*b*) suggested that the dipole approximation is inadequate. They developed a numerical technique for incorporating the modification to the poloidal magnetic field that is generated by the toroidal motions throughout the magnetosphere. They based their treatment on the hypothesis that those motions are such as to cancel the dipole field of the star, leaving a sextupole poloidal magnetic field at large distances.

An elaboration of the MRΩ^2 model by Mestel and Shibata (1993) incorporates electron–positron pairs, created near and beyond $\mathcal{S}_L$, into the outflowing stream. The gamma rays emitted by the rapidly accelerated electrons in that region result in copious pair production, and the outflowing stream becomes a ternary plasma, consisting of the primary electrons and a dense secondary electron-positron plasma. Mestel and Shibata located $\mathcal{S}_L$ well within the light cylinder—the surface on which the speed of corotation with the star reaches c, the vacuum speed of light.

The purpose of this paper is to present an analytical technique for including the modification to the poloidal magnetic field that is generated by the toroidal motion of the magnetospheric particles. As sketched in a previous interim report (Burman 1996), I am developing the technique in such a way as to retain as much as possible of the earlier formalism in which the poloidal magnetic field was taken to be dipolar. The formalism contains a free function of radial distance from the symmetry axis, which is to be chosen so as to model the distortion of the poloidal magnetic field from the dipole form caused by the toroidal motions of the magnetospheric charges.

2. Electrodynamic Basis

(2a) Steadily Rotating Axisymmetric Systems

The system is steadily rotating at angular frequency Ω. The dimensionless cylindrical radial coordinate X extends from the rotation axis and passes through unity on the light cylinder, which has radius c/Ω about the axis. The dimensionless axial coordinate, also normalised by c/Ω, is denoted by Z. The unit toroidal vector is represented by $\mathbf{t}$ and the toroidal coordinate by ϕ. The steady-rotation condition $\partial/\partial t = \Omega\partial/\partial\phi$ expresses temporal change as arising from rotation of any non-axisymmetric structures at angular frequency Ω in the azimuthal direction. This relation is applicable to scalars and cylindrical polar components of vectors, whereas $\partial/\partial t = \Omega\partial/\partial\phi - \mathbf{\Omega}\times$ for operation on vectors (Westfold 1981).

It follows from Faraday's law and $\nabla.\mathbf{B} = 0$, together with $\partial/\partial t = \Omega\partial/\partial\phi$, that the electric field can be written as the sum of a part $X\mathbf{B}\times\mathbf{t}$, associated with the rotation of the magnetic field structure, and a 'non-corotational' part $-\nabla\psi$ (Mestel 1971, 1973):

$$\mathbf{E} = X\mathbf{B}\times\mathbf{t} - \nabla\psi\,. \tag{1}$$

The gauge-invariant potential ψ is related to the familiar scalar and vector potentials V and $\mathbf{A}$ by $\psi \equiv V - XA_\phi$ (Endean 1972; Westfold 1981).

MRΩ^2 developed their equations in dimensionless form by expressing distances and speeds in units of c/Ω and c, and normalising field and source variables in terms of the equatorial dipolar magnetic field strength at the light cylinder. The last quantity, denoted by B_1, is $\frac{1}{2}(\Omega r_s/c)^3 B_0$ where r_s is the stellar radius and B_0 the polar surface dipolar magnetic field strength. The magnetic field $\mathbf{B}$ and the charge density ρ_e are expressed in units of B_1 and $\Omega B_1/4\pi c$. The system is taken to be axisymmetric, with the magnetic and rotation axes coinciding. Since the system is both steadily rotating and axisymmetric, the operators $\partial/\partial t$ and $\partial/\partial\phi$ are both null when operating on scalars and cylindrical polar components of vectors; $\partial/\partial t = 0$ implies $\partial/\partial\phi = (\mathbf{\Omega}/\Omega)\times$ for operation on vectors.

The poloidal part of the electric current density is expressed in terms of a Stokes stream function S:

$$\mathbf{j}_p = X^{-1}\mathbf{t} \times \nabla S, \qquad j_X = X^{-1}\partial S/\partial Z, \qquad j_Z = -X^{-1}\partial S/\partial X\,, \tag{2}$$

with the dimensionless $\mathbf{j}_p$ and S normalised to $\Omega B_1/4\pi$ and $(c/\Omega^2)B_1/4\pi$. The continuity equation $\operatorname{div}\mathbf{j}_p = 0$ is automatically satisfied. The poloidal part of Ampère's law reduces to (MRΩ^2)

$$B_\phi = -S/X\,. \tag{3}$$

It follows from Gauss's law and the toroidal part of Ampère's law that (Mestel *et al.* 1979, equation 2·8)

$$\nabla^2\psi + 2B_Z = -(1 - XV_\phi)\rho_e\,, \tag{4}$$

with ∇^2 dimensionless and ψ expressed in units of cB_1/Ω; the subscripts ϕ and Z denote toroidal and axial components, with V_ϕ the toroidal component of the normalised flow velocity.

Any axisymmetric poloidal magnetic field can be expressed through a 'magnetic Stokes stream function' P by

$$\mathbf{B}_p = X^{-1}\mathbf{t} \times \nabla P, \qquad B_X = X^{-1}\partial P/\partial Z, \qquad B_Z = -X^{-1}\partial P/\partial X\,, \tag{5}$$

with P normalised to $(c/\Omega)^2 B_1$, yielding a dimensionless $\mathbf{B}_p$. The vector potential is $-P\mathbf{t}/X$. The solenoidal condition $\operatorname{div}\mathbf{B} = 0$ is automatically satisfied.

Three auxiliary variables $\bar{P}$, U and Q, defined by

$$\bar{P} \equiv -P, \qquad U \equiv X^{\frac{2}{3}}, \qquad Q \equiv \bar{P}^{\frac{2}{3}}\,, \tag{6}$$

are often convenient. It is U (or X) and Q (or $\bar{P}$), rather than X and Z, that I shall regard as the independent variables, forming a non-orthogonal coordinate set in meridian planes. The poloidal field lines are lines of constant Q (or $\bar{P}$ or P).

(2b) Dipole Fields

A dipole field has the 'stream function'

$$\bar{P} = X^2/R^3 = (\sin^2\theta)/R\,, \tag{7}$$

with $R \equiv (X^2+Z^2)^{\frac{1}{2}}$ the dimensionless spherical polar radial coordinate (normalised by c/Ω) and θ the angle from the Z axis. The cylindrical polar field components and the dipole field magnitude are given in terms of the spherical polar variables R and θ by

$$B_X = (3/2R^3)\sin(2\theta), \qquad B_Z = (3\cos^2\theta - 1)/R^3,$$

$$B_p = (1 + 3\cos^2\theta)^{\frac{1}{2}}/R^3\,. \tag{8}$$

The expressions in (7) for the dipolar form of $\bar{P}$ can be re-arranged as

$$QU = X^2/R^2 = \sin^2\theta\,. \tag{9}$$

Use of these relations in (8) enables the field quantities to be expressed in terms of U (or X) and Q (or $\bar{P}$) as the independent variables (MRΩ^2):

$$B_X = (3\bar{P}/X^2)[QU(1 - QU)]^{\frac{1}{2}}, \qquad B_Z = (2\bar{P}/X^2)(1 - 3QU/2),$$

$$B_p = (2\bar{P}/X^2)(1 - 3QU/4)^{\frac{1}{2}}\,. \tag{10}$$

The toroidal magnetic field generated by the poloidal flow is given by (3) in terms of the stream function S of the poloidal electric current density: $B_\phi = -S/X$. Combining this with B_p from (10) gives

$$B_\phi/B_p = -X(S/2\bar{P}) \div (1 - 3QU/4)^{\frac{1}{2}} \tag{11}$$

for the toroidal-to-poloidal magnetic field ratio when the poloidal currents are accounted for.

Equations (7) and (9) show that the dipole field lines, $P =$ constant or $Q =$ constant, have equations $\bar{P}R = \sin^2\theta = QU$. Expressions for their slope follow on taking B_X and B_Z from (8) or (10):

$$\begin{aligned} dZ/dX = B_Z/B_X &= \tfrac{2}{3}(3\cos^2\theta - 1)/\sin(2\theta) \\ &= \tfrac{2}{3}(1 - 3QU/2) \div [QU(1 - QU)]^{\frac{1}{2}}\,, \end{aligned} \tag{12}$$

in terms of either θ alone or the U-Q coordinates. The rate of change of the slope is given by

$$\begin{aligned} d^2Z/dX^2 &= -(1/9R)(1 + \cos^2\theta)/(\sin^2\theta.\cos\theta) \\ &= -(2/9U^{\frac{3}{2}})(1 - QU/2) \div [QU(1 - QU)]^{\frac{1}{2}} \end{aligned} \tag{13}$$

(see Section 5c below) in terms of either spherical polar or U-Q coordinates.

The standard formula for the radius of curvature of a plane curve (in the X-Z plane) is

$$\rho \equiv [1 + (dZ/dX)^2]^{\frac{3}{2}} \div |d^2Z/dX^2|\,. \tag{14}$$

Substituting the derivatives from (12) and (13) gives

$$\begin{aligned}\rho &= (R/3)(1+3\cos^2\theta)^{\frac{3}{2}} \div [(1+\cos^2\theta).\cos^2\theta.\sin\theta] \\ &= (4U^{\frac{1}{2}}/3Q)(1-3QU/4)^{\frac{3}{2}} \div [(1-QU/2)(1-QU)] \qquad (15)\end{aligned}$$

for the dimensionless radius of curvature of dipole field lines in terms of either spherical polar or U-Q coordinates. The field-line radius of curvature is particularly important in the pulsar context, because of its bearing on curvature radiation.

3. Quasi-dipole Formalism

(3a) The Quasi-dipole Field

In order to retain as much as possible of the detailed MRΩ^2 formalism that has been developed for the dipolar case, I shall write the poloidal magnetic field in the form

$$B_X = (3\bar{P}/X^2)[QU\alpha(U,Q)]^{\frac{1}{2}}, \qquad B_Z = (2\bar{P}/X^2)\beta(U,Q),$$

$$B_p = (2\bar{P}/X^2)\Delta^{\frac{1}{2}}, \qquad \Delta(U,Q) \equiv \beta^2 + 9QU\alpha/4\,, \qquad (16)$$

where $\alpha(U,Q)$ and $\beta(U,Q)$ are as-yet-unspecified (but not independent) functions of position. Use of $B_\phi = -S/X$ and B_p from (16) gives

$$B_\phi/B_p = -X(S/2\bar{P}) \div \Delta^{\frac{1}{2}} \qquad (17)$$

for the toroidal–poloidal field ratio.

The forms for B_X and B_Z in (16) define the 'quasi-dipole' field. They are suggested by the MRΩ^2 forms, in (10) above, for a dipole field, and express the field components in terms of U (or X) and Q (or $\bar{P}$), rather than X and Z, as the independent variables. In the dipole case (MRΩ^2):

$$\alpha = 1-QU, \qquad \beta = 1-3QU/2, \qquad \Delta = 1-3QU/4; \qquad (18)$$

since $QU = \sin^2\theta$ for a dipole field, these can be expressed as functions of θ alone:

$$\alpha = \cos^2\theta, \qquad \beta = (3\cos^2\theta - 1)/2, \qquad \Delta = (1+3\cos^2\theta)/4\,. \qquad (19)$$

So long as α and β are left unspecified, there is no loss of generality in using the quasi-dipole forms. The functions α and β are not independent, but are related by the solenoidal condition on $\mathbf{B}$, so there is, in effect, a single free function of U and Q.

(3b) First Derivatives of Quasi-dipole Fields

The field components will now be expressed in terms of the stream function by

$$B_X = -X^{-1}\bar{P}_{,Z} \quad \text{and} \quad B_Z = X^{-1}\bar{P}_{,X} \qquad (20)$$

with a comma denoting partial differentiation with respect to its following suffix. Differentiating the definitions $Q \equiv \bar{P}^{\frac{2}{3}}$ and $U \equiv X^{\frac{2}{3}}$ gives, after using (20) to substitute for $\bar{P},_X$ and $\bar{P},_Z$,

$$Q^{-1}Q,_X = 2XB_Z/3\bar{P}, \qquad Q^{-1}Q,_Z = -2XB_X/3\bar{P}, \qquad dU/dX = 2U/3X\,. \tag{21}$$

These will now be used in calculating the first partial derivatives of the expressions in (16) for B_X and B_Z defining a quasi-dipole magnetic field, hence enabling the divergence and curl of the quasi-dipole field to be formed.

Divergence. Differentiating B_X from (16) at constant Z, using B_Z from (20) in reverse for $\bar{P},_X$ and B_Z from (16), together with $Q,_X$ and dU/dX from (21), gives

$$(X/B_X)B_X,_X = (8\beta - 5)/3 + (U\alpha,_U + 2Q\beta\alpha,_Q\,)/3\alpha\,. \tag{22}$$

Differentiating B_Z from (16) at constant X, using B_X from (20) in reverse for $\bar{P},_Z$ and $Q,_Z$ from (21) gives

$$(X/B_X)B_Z,_Z = -2\beta - (4Q/3)\beta,_Q\;. \tag{23}$$

Since $\nabla.\mathbf{B} = B_X,_X + B_X/X + B_Z,_Z$ in cylindrical polar coordinates (X, ϕ, Z) with axisymmetry, use of (22) and (23) for the derivatives, and B_X from (16), yields

$$3\alpha(X/B_X)\nabla.\mathbf{B} = 2\alpha(\beta - 1) + U\alpha,_U + 2Q\beta\alpha,_Q - 4Q\alpha\beta,_Q\;. \tag{24}$$

Note from B_X in (16) and the definitions of Q and U, that $3X/B_XQU = X^2/P^2\alpha^{\frac{1}{2}}$; hence (24) can be written as

$$(\alpha^{\frac{1}{2}}X^2/P^2)\nabla.\mathbf{B} = 2\alpha(\beta-1)/QU + Q^{-1}\alpha,_U + 2U^{-1}\beta\alpha,_Q - 4U^{-1}\alpha\beta,_Q\;. \tag{25}$$

The functions α and β are required to be such that this is identically zero.

Curl. Differentiating B_X from (16) at constant X, using B_X from (20) in reverse for $\bar{P},_Z$ and $Q,_Z$ from (21), and then using B_X from (16) again, gives

$$(X^3/2\bar{P})B_X,_Z = -(3QU/2)(4\alpha + Q\alpha,_Q\,)\,. \tag{26}$$

Differentiating B_Z from (16) at constant Z, using B_Z from (20) in reverse for $\bar{P},_X$ and B_Z from (16) again, and also $Q,_X$ and dU/dX from (21), gives

$$(X^3/2\bar{P})B_Z,_X = 2\beta(\beta - 1) + (2U/3)\beta,_U + (4Q\beta/3)\beta,_Q\;. \tag{27}$$

Subtracting (27) from (26), dividing through by QU and using $U \equiv X^{\frac{2}{3}}$ in the coefficient on the left, results in

$$\begin{aligned} -(U^2X/12\bar{P}Q)(\nabla\times\mathbf{B})_\phi &= \alpha + \beta(\beta-1)/3QU \\ &\quad + (Q/4)\alpha,_Q + (1/9Q)\beta,_U + (2\beta/9U)\beta,_Q\;. \end{aligned} \tag{28}$$

In application, this result for $(\nabla \times \mathbf{B})_\phi$ must be matched to j_ϕ, the normalised toroidal component of the electric current density.

4. Distorted Dipole Formalism

(4a) The Distorted Dipole Field

The dipolar form X^2/R^3 for $\bar{P}$ can be regarded as an equation for Z and re-arranged (MRΩ^2) to give $Z^2 = (1-QU)U^2/Q$; this enables Z to be eliminated, leaving U and Q as the coordinates. The idea behind the distorted dipole approximation is to take a form for $\bar{P}$ which is sufficiently similar to the dipolar one that it readily allows the elimination of Z as a coordinate, but is more general in that it contains a free function of distance from the axis.

I define 'distorted dipole fields' by expressing the magnetic stream function by (Burman 1996)

$$\bar{P} = X^2/D^3, \qquad QU = X^2/D^2, \qquad D^2 \equiv X^2 g(U) + Z^2\,, \tag{29}$$

where $g(U)$ is a free function of distance from the axis. In the dipole case, $g(U) \equiv 1$ and D reduces to the dimensionless distance from the star.

Substituting $\bar{P}$ from (29) into (20) for the poloidal field components yields the quasi-dipole forms (16) with

$$\alpha(U,Q) \equiv 1 - QUg(U), \qquad \beta(U,Q) \equiv 1 - 3QUf(U)/2,$$

$$\Delta = 1 - (3QU/4)(4f-3) - (3QU/2)^2(g-f^2)\,, \tag{30}$$

where

$$f(U) \equiv d(U^3 g)/d(U^3) \equiv (U/3)dg/dU + g\,. \tag{31}$$

With the forms in (30) for α and β, the forms in (16) for B_X and B_Z become

$$B_X = (3\bar{P}/X^2)[QU(1-QUg)]^{\frac{1}{2}}, \qquad B_Z = (2\bar{P}/X^2)(1-3QUf/2)\,. \tag{32}$$

Using $\bar{P}$, QU and D from (29), first with α from (30) in B_X from (16), then with β from (30) in B_Z from (16), yields

$$B_X = 3XZ/D^5, \qquad B_Z = [2Z^2 - (3f-2g)X^2]/D^5\,, \tag{33}$$

expressing the distorted dipole magnetic field components as functions of X and Z. These can also be written, along with D from (29), in terms of the spherical polar coordinates as

$$B_X = (R^2/D^2)(3/2D^3)\sin(2\theta),$$

$$B_Z = (R^2/D^2)[2\cos^2\theta - (3f-2g)\sin^2\theta]/D^3,$$

with

$$D^2/R^2 = 1 + [g(U)-1]\sin^2\theta, \qquad U \equiv (R\sin\theta)^{\frac{2}{3}}\,. \tag{34}$$

The forms in (29) for QU and D^2, with α from (30), show that

$$X/D = (QU)^{\frac{1}{2}}, \qquad Z/D = \alpha^{\frac{1}{2}}. \tag{35}$$

So the following equations give the spherical polar angle θ in terms of the U-Q coordinates:

$$\tan\theta = X/Z = (QU/\alpha)^{\frac{1}{2}}, \qquad \sin\theta = X/R = (1+\alpha/QU)^{-1/2}. \tag{36}$$

The form for B_X in (33) demonstrates satisfaction of the necessary requirements on B_X of vanishing on the symmetry axis $X = 0$ and on the equatorial plane $Z = 0$. The distorted dipole component B_Z, given in (32), (33) and (34), vanishes on quasi-conical surfaces given by

$$QU = 2/[3f(U)]; \quad \text{i.e.} \quad \tan^2\theta = 2/(3f - 2g). \tag{37}$$

In the dipole case, the $B_Z = 0$ surface is the (double) cone $QU = \frac{2}{3}$ or $\tan^2\theta = 2$, corresponding to $\theta = 54^\circ \cdot 7$.

The forms in (29) for QU and D^2 show, after using α from (30) and the definition of U, that

$$Z^2 = (U^2/Q)\alpha(U,Q). \tag{38}$$

So the form of magnetic stream function defined in (29) is algebraically attractive in that—just as occurs in the dipole case—it enables Z to be eliminated analytically, leaving U and Q as the independent spatial variables: they seem to form a natural (though non-orthogonal) coordinate set.

The idea behind the distorted dipole approximation is to select a form of magnetic stream function which is sufficiently similar to the dipolar one that it readily allows the elimination of the axial coordinate—thus enabling the mathematics to be developed along similar lines to the treatment of the dipole case—but is more general in leaving some scope to model the distortion of the poloidal magnetic field from the dipole form. The aim is to have a technique which is mathematically tractable while yielding some modelling potential, which, in the proposed forms, resides in a free function of radial distance from the symmetry axis. I shall call $g(U)$ the 'magnetic structure function'—its departure from one determines the distortion of the poloidal magnetic field from the dipole form caused by the toroidal motions of the magnetospheric charged particles. In the dipole case $g \equiv 1 \equiv f$, and the various equations in this sub-section take on their familiar dipolar forms, as in Section 2*b* above.

The quasi-dipole formalism of Section 3 above contains functions $\alpha(U,Q)$ and $\beta(U,Q)$, which are linked by the partial differential equation obtained on putting the right-hand side of (25) equal to zero to ensure the vanishing of $\operatorname{div}\mathbf{B}$. I intend to use that formalism later as the basis for an improved distorted dipole approximation, with additional free functions entering through expansion of $\alpha(U,Q)$ in powers of QU, with $1 - QUg(U)$ appearing as the first approximation.

(4b) First Derivatives of Distorted Dipole Fields

The expressions in (30) for $\alpha(U,Q)$ and $\beta(U,Q)$ specifying a distorted dipole field have the following first partial derivatives with respect to U (at constant Q) and Q (at constant U):

$$\alpha_{,U} = -Q(g + Ug'), \qquad \alpha_{,Q} = -Ug,$$
$$\beta_{,U} = -(3Q/2)(f + Uf'), \qquad \beta_{,Q} = -3Uf/2\,, \tag{39}$$

with a dash denoting d/dU.

Use of a magnetic Stokes stream function means that $\nabla.\mathbf{B} = 0$ is automatically satisfied. Substituting $\alpha = 1 - QUg$ and $\beta = 1 - 3QUf/2$, together with the relevant derivatives from (39), into (25) for the quasi-dipole $\nabla.\mathbf{B}$ yields $(3f - 3g - Ug')$ for the right-hand side; the definition (31) of f in terms of g shows that this vanishes, as expected.

Substituting the same expressions for α and β, together with the relevant derivatives from (39), into (28) for the quasi-dipole $(\nabla\times\mathbf{B})_\phi$ yields

$$(\nabla\times\mathbf{B})_\phi = -(2\bar{P}Q/XU^2)[6(1-f) - U df/dU - 15QU(g - f^2)/2]\,. \tag{40}$$

The expressions in (29) for $\bar{P}$ and QU in the distorted dipole formalism, together with $U \equiv X^{\frac{2}{3}}$, show that the coefficient before the braces in (40) is equal to $-2X/D^5$.

The distorted dipole stream function leads to a toroidal component of $\mathrm{curl}\,\mathbf{B}$ containing the magnetic structure function $g(U)$ and its first and second derivatives. Steadily rotating axisymmetric systems have no displacement current, so the magnetic field and the electric current density are linked by Ampère's law: $\nabla\times\mathbf{B} = \mathbf{j}$ in terms of the dimensionless quantities. The expression in (40) for the toroidal component of $\mathrm{curl}\,\mathbf{B}$ must match the toroidal component of the electric current density in the region, as yielded by a magnetosphere model.

5. Field-line Curvature

(5a) Quasi-dipole Fields

The quasi-dipole field lines, described by (16) for B_X and B_Z, have slope

$$dZ/dX = B_Z/B_X = 2\beta/[3(QU\alpha)^{\frac{1}{2}}]\,. \tag{41}$$

Forming the second derivative from $dZ/dX = B_Z/B_X$ gives

$$d^2Z/dX^2 = B_{Z,X}/B_X - (B_Z/B_X)(B_{X,X}/B_X)\,; \tag{42}$$

inserting B_X from (16) and B_Z/B_X from (41) into this leads to

$$(3X/2)(QU\alpha)^{\frac{1}{2}} d^2Z/dX^2 = (X^3/2\bar{P})B_{Z,X} - \beta X B_{X,X}/B_X\,. \tag{43}$$

Using (27) for $B_{Z,X}$ and (22) for $B_{X,X}/B_X$ in (43) yields the following result for the rate of change of the slope:

$$(9X\alpha/2)(QU\alpha)^{\frac{1}{2}}d^2Z/dX^2 = -\delta\,, \tag{44}$$

where

$$\delta(U,Q) \equiv \alpha\beta(2\beta+1) + \beta(U\alpha_{,U} + 2Q\beta\alpha_{,Q}) - 2\alpha(U\beta_{,U} + 2Q\beta\beta_{,Q})\,. \tag{45}$$

Taking dZ/dX from (41) gives

$$1 + (dZ/dX)^2 = (B_p/B_X)^2 = 4\Delta/9QU\alpha\,, \tag{46}$$

where the quasi-dipole forms of B_p and B_X in (16) have been used; $\Delta(U,Q) \equiv \beta^2 + 9QU\alpha/4$, as before. Equation (46) shows that

$$(9X\alpha/2)(QU\alpha)^{\frac{1}{2}}[1 + (dZ/dX)^2]^{\frac{3}{2}} = (4U^{\frac{1}{2}}/3Q)\Delta^{\frac{3}{2}}\,. \tag{47}$$

Inserting (47) and (44) for the numerator and denominator in the standard formula for the radius of curvature of a plane curve, equation (14) above, yields

$$\rho = (4U^{\frac{1}{2}}/3Q)\Delta^{\frac{3}{2}}/|\delta| \tag{48}$$

for the dimensionless radius of curvature of the quasi-dipole magnetic field lines.

(5b) Distorted Dipole Fields

For a distorted dipole field, α and β are given in (30), and their first partial derivatives with respect to U and Q are given by (39). Inserting those derivatives into the quasi-dipole form (45) for δ produces

$$\delta = [\alpha\beta + QU(3\alpha f - \beta g)](2\beta+1) + QU(3\alpha f' - \beta g')U \tag{49}$$

for the distorted dipole case; inserting α and β themselves, from (30), into this gives

$$\begin{aligned}\delta = {} & [1 - QU(2g - 3f/2)](1 - QUf) \\ & + QU[3(1 - QUg)f' - (1 - 3QUf/2)g']U\,.\end{aligned} \tag{50}$$

The second relation in (31) between f and g, which can be written $Ug'/3 \equiv f - g$, can be used to substitute for g' in (50), resulting in the alternative form

$$\delta = 1 - QU(5f/2 - g) + 3QU(1 - QUg)Uf' + (QU)^2(3f - 5g/2)f\,. \tag{51}$$

The dimensionless radius of curvature of distorted dipole field lines is given by (48), with Δ from (30) and δ from (50) or (51).

(5c) Dipole Fields

In the case of a dipole field, $g \equiv 1 \equiv f$ and $QU = \sin^2\theta$, so (30) and (50) for Δ and δ reduce to

$$\Delta = 1 - 3QU/4 = (1 + 3\cos^2\theta)/4\,, \tag{52}$$

$$\delta = (1 - QU/2)(1 - QU) = (1 + \cos^2\theta)(\cos^2\theta)/2\,. \tag{53}$$

Hence, on using $U^{\frac{1}{2}}/Q = X/QU = R/\sin\theta$, equation (48) for the dimensionless radius of curvature yields the dipolar formulas (15), in Section 2*b* above, for ρ in terms of either R-θ or U-Q coordinates. Also, since $\alpha = 1 - QU = \cos^2\theta$ and $\beta = 1 - 3QU/2 = (3\cos^2\theta - 1)/2$ for a dipole field, substituting (53) for the dipolar form of δ into (44) for the second derivative leads to the expressions (13), quoted in Section 2*b* above, for the rate of change of the slope of dipole field lines in R-θ or U-Q coordinates.

6. Concluding Remarks

The formalism presented in this paper is a description of an axisymmetric magnetic field structure applicable to the magnetosphere of a steadily rotating magnetised neutron star whose rotation and magnetic axes coincide. The poloidal part of the magnetic field is thought of as the dipole field of the star modified by the toroidal currents in the magnetosphere. The formalism includes a free function, which is to be chosen so that the toroidal electric current density calculated from a magnetosphere model will match the curl of the poloidal field calculated here. A subsequent paper will explore this process for a magnetosphere model of the type introduced by MRΩ^2.

Acknowledgments

I thank the referee for a careful review with a number of helpful suggestions.

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Manuscript received 8 January, accepted 17 September 1998