CONDUCTION OF HEAT IN THE SEMI-INFINITE SOLID, WITH A SHORT TABLE OF AN IMPORTANT INTEGRAL

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Summary

The solution of a problem in heat conduction is expressed in terms of the integral $\int_0^U e^{-\alpha(1+u^2)}\frac{du}{1+u^2}$. This integral is tabulated for various values of α and U.

I. Introduction

The temperature distribution in a semi-infinite solid is determined for the following conditions: initial temperature zero, the plane $x=0$ held at the constant temperature θ for a time T then made impervious to heat.

II. Calculations and Results

To maintain the temperature θ at $x=0$ heat must be supplied at the rate $\theta K(\varkappa\pi t)^{-\frac{1}{2}}$ per unit time per unit area. The temperature distribution for $x>0$ will therefore be the same as in an infinite solid with a continuous plane source at $x=0$ liberating heat at the rate $2\theta K(\varkappa\pi t)^{-\frac{1}{2}}$ per unit time per unit area for $t<T$.

For $t<T$, the temperature is given by

$$v(x,t)=\frac{\theta}{\pi}\int_0^t e^{-x^2/4\varkappa(t-\tau)}\frac{d\tau}{\tau^{\frac{1}{2}}(t-\tau)^{\frac{1}{2}}},$$

that is,

$$v(x,t)=\theta\left\{1-\operatorname{erf}\frac{x}{2\sqrt{\varkappa t}}\right\}, \quad \ldots\ldots\ldots\ldots \quad (1)$$

while, for $t>T$,

$$v(x,t)=\frac{\theta}{\pi}\int_0^T e^{-x^2/4\varkappa(t-\tau)}\frac{d\tau}{\tau^{\frac{1}{2}}(t-\tau)^{\frac{1}{2}}}. \quad \ldots\ldots \quad (2)$$

The temperature at $x=0$ is given by

$$v(0,t)=\begin{cases}\theta, & 0<t<T\\ \dfrac{2}{\pi}\theta\sin^{-1}\left(\dfrac{T}{t}\right)^{\frac{1}{2}}, & t>T\end{cases}. \quad \ldots\ldots \quad (3)$$

(For these results see Carslaw and Jaeger (1947, pp. 57, 222).)

The substitution

$$\tau=\frac{u^2}{1+u^2}t$$

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TABLE 1

$$\int_0^U e^{-\alpha(1+u^2)}\frac{du}{1+u^2}$$

U \ α	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
0·1	0·09015	0·08155	0·07376	0·06672	0·06035	0·05459	0·04938	0·04467	0·04040	0·03655
0·2	0·17838	0·16119	0·14566	0·13162	0·11894	0·10748	0·09713	0·08777	0·07931	0·07167
0·3	0·26295	0·23723	0·21403	0·19310	0·17422	0·15719	0·14182	0·12795	0·11545	0·10416
0·4	0·34254	0·30837	0·27761	0·24993	0·22501	0·20259	0·18240	0·16422	0·14786	0·13314
0·5	0·41626	0·37374	0·33557	0·30132	0·27058	0·24299	0·21822	0·19599	0·17603	0·15812
0·6	0·48366	0·43290	0·38751	0·34692	0·31062	0·27814	0·24908	0·22308	0·19982	0·17900
0·7	0·54464	0·48580	0·43340	0·38673	0·34515	0·30809	0·27507	0·24562	0·21937	0·19596
0·8	0·59940	0·53264	0·47347	0·42100	0·37447	0·33317	0·29652	0·26398	0·23508	0·20940
0·9	0·64829	0·57380	0·50812	0·45017	0·39903	0·35385	0·31393	0·27864	0·24742	0·21979
1·0	0·69176	0·60975	0·53784	0·47475	0·41935	0·37066	0·32783	0·29013	0·25693	0·22765
1·1	0·73033	0·64100	0·56318	0·49529	0·43600	0·38415	0·33877	0·29900	0·26411	0·23348
1·2	0·76448	0·66808	0·58465	0·51232	0·44950	0·39486	0·34726	0·30574	0·26946	0·23772
1·3	0·79470	0·69148	0·60276	0·52634	0·46035	0·40327	0·35377	0·31078	0·27336	0·24074
1·4	0·82144	0·71164	0·61797	0·53781	0·46901	0·40979	0·35870	0·31449	0·27616	0·24286
1·5	0·84509	0·72899	0·63069	0·54714	0·47586	0·41482	0·36238	0·31720	0·27815	0·24431
1·6	0·86601	0·74388	0·64130	0·55469	0·48123	0·41865	0·36511	0·31914	0·27953	0·24530
1·7	0·88454	0·75666	0·65010	0·56076	0·48542	0·42153	0·36710	0·32051	0·28048	0·24595
1·8	0·90095	0·76759	0·65739	0·56562	0·48865	0·42369	0·36854	0·32147	0·28112	0·24638
1·9	0·91549	0·77693	0·66340	0·56948	0·49114	0·42529	0·36956	0·32213	0·28154	0·24665
2·0	0·92838	0·78491	0·66833	0·57254	0·49303	0·42646	0·37029	0·32258	0·28182	0·24682
2·5	0·97404	0·81009	0·68228	0·58029	0·49735	0·42887	0·37165	0·32335	0·28225	0·24707
3·0	0·99920	0·82094	0·68698	0·58234	0·49825	0·42928	0·37183	0·32343	0·28229	0·24708
∞	1·02843	0·82795	0·68892	0·58291	0·49843	0·42933	0·37184	0·32343	0·28229	0·24709

U \ α	1·1	1·2	1·3	1·4	1·5	1·6	1·7	1·8	1·9	2·0
0·1	0·03306	0·02990	0·02705	0·02446	0·02213	0·02002	0·01811	0·01638	0·01481	0·01340
0·2	0·06477	0·05853	0·05289	0·04779	0·04319	0·03903	0·03527	0·03187	0·02880	0·02603
0·3	0·09398	0·08479	0·07651	0·06903	0·06228	0·05620	0·05071	0·04575	0·04128	0·03725
0·4	0·11988	0·10794	0·09720	0·08752	0·07881	0·07097	0·06391	0·05756	0·05183	0·04668
0·5	0·14203	0·12759	0·11462	0·10297	0·09251	0·08312	0·07468	0·06711	0·06030	0·05419
0·6	0·16036	0·14368	0·12875	0·11538	0·10340	0·09268	0·08308	0·07448	0·06678	0·05988
0·7	0·17508	0·15645	0·13982	0·12499	0·11174	0·09992	0·08936	0·07993	0·07150	0·06398
0·8	0·18657	0·16628	0·14824	0·13219	0·11790	0·10519	0·09387	0·08379	0·07481	0·06680
0·9	0·19532	0·17365	0·15444	0·13741	0·12230	0·10889	0·09699	0·08642	0·07702	0·06867
1·0	0·20183	0·17903	0·15889	0·14109	0·12535	0·11141	0·09907	0·08814	0·07844	0·06985
1·1	0·20655	0·18286	0·16200	0·14361	0·12739	0·11307	0·10041	0·08923	0·07933	0·07056
1·2	0·20991	0·18553	0·16411	0·14529	0·12872	0·11412	0·10125	0·08989	0·07985	0·07098
1·3	0·21225	0·18734	0·16552	0·14638	0·12956	0·11478	0·10176	0·09028	0·08016	0·07122
1·4	0·21385	0·18855	0·16643	0·14706	0·13008	0·11517	0·10205	0·09051	0·08033	0·07134
1·5	0·21492	0·18933	0·16700	0·14749	0·13039	0·11540	0·10222	0·09063	0·08042	0·07141
1·6	0·21562	0·18983	0·16736	0·14774	0·13057	0·11552	0·10231	0·09070	0·08046	0·07144
1·7	0·21607	0·19014	0·16757	0·14789	0·13067	0·11559	0·10236	0·09073	0·08049	0·07146
1·8	0·21636	0·19033	0·16770	0·14797	0·13073	0·11563	0·10239	0·09075	0·08050	0·07147
1·9	0·21653	0·19045	0·16777	0·14802	0·13076	0·11565	0·10240	0·09075	0·08050	0·07147
2·0	0·21664	0·19051	0·16781	0·14804	0·13078	0·11566	0·10240	0·09076	0·08051	0·07147
2·5	0·21678	0·19059	0·16786	0·14807	0·13079	0·11567	0·10241	0·09076	0·08051	0·07147
3·0	0·21679	0·19059	0·16786	0·14807	0·13079	0·11567	0·10241	0·09076	0·08051	0·07147
∞	0·21679	0·19059	0·16786	0·14807	0·13079	0·11567	0·10241	0·09076	0·08051	0·07147

TABLE 1 (*Continued*)

U \ α	2·5	3·0	4·0	5·0	U \ α	2·5	3·0	4·0	5·0
0·1	0·00811	0·00491	0·00180	0·00066	1·1	0·03956	0·02240	0·00734	0·00246
0·2	0·01568	0·00945	0·00343	0·00125	1·2	0·03969	0·02244	0·00735	,,
0·3	0·02228	0·01333	0·00477	0·00171	1·3	0·03976	0·02246	,,	,,
0·4	0·02766	0·01640	0·00577	0·00204	1·4	0·03979	0·02247	,,	,,
0·5	0·03178	0·01866	0·00645	0·00224	1·5	0·03980	,,	,,	,,
0·6	0·03475	0·02021	0·00688	0·00236	1·6	0·03981	,,	,,	,,
0·7	0·03677	0·02120	0·00712	0·00242	1·7	0·03981	,,	,,	,,
0·8	0·03806	0·02180	0·00724	0·00244	1·8	0·03982	,,	,,	,,
0·9	0·03885	0·02213	0·00730	0·00245	1·9	,,	,,	,,	,,
1·0	0·03931	0·02231	0·00733	0·00246	2·0	,,	,,	,,	,,
					∞	,,	,,	,,	,,

reduces the integral (**2**) to the form

$$v(x,\,t)=\frac{2\theta}{\pi}\int_0^{\sqrt{T/(t-T)}} e^{-x^2(1+u^2)/4\varkappa t}\,\frac{du}{1+u^2}, \qquad \ldots\ldots \quad (\mathbf{4})$$

that is, to the form

$$v(x,\,t)=\frac{2\theta}{\pi}\int_0^{U} e^{-\alpha(1+u^2)}\,\frac{du}{1+u^2}, \qquad \ldots\ldots\ldots\ldots\ldots \quad (\mathbf{5})$$

with

$$U=\sqrt{\frac{T}{t-T}}, \qquad \alpha=\frac{x^2}{4\varkappa t}. \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{6})$$

The integral (**5**) has been calculated from rearranged forms of the series expansions quoted by Lightfoot (1930)

$$\int_0^U e^{-\alpha(1+u^2)}\frac{du}{1+u^2}=e^{-\alpha}\sum_{n=0}^{\infty}\frac{(-1)^n U^{2n+1}}{2n+1}\left\{1+\frac{\alpha}{1!}+\ldots\ldots+\frac{\alpha^n}{n!}\right\}$$

for $U\leqslant 1$,

$$\int_0^U e^{-\alpha(1+u^2)}\frac{du}{1+u^2}=\frac{\pi}{2}\,\{1-\operatorname{erf}\sqrt{\alpha}\,\operatorname{erf}U\sqrt{\alpha}\}$$

$$-e^{-\alpha U^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)\,U^{2n+1}}\left\{1+\frac{\alpha U^2}{1!}+\ldots\ldots+\frac{(\alpha U^2)^n}{n!}\right\}$$

for $U>1$,

and the results are given in Table 1. The calculations were made using six figures and differencing did not show errors greater than two in the last figure. The results were then rounded off to five figures so that the final table should not have rounding-off errors greater than 0·7 in the last figure.

Since little arithmetic is involved in passing from the integral to the temperature, no table of temperature distribution is given. The results are presented graphically in Figure 1, $x/(2\sqrt{\kappa t})$ being plotted against $\sqrt{t/T}$ for fixed values of $v(x, t)/\theta$.

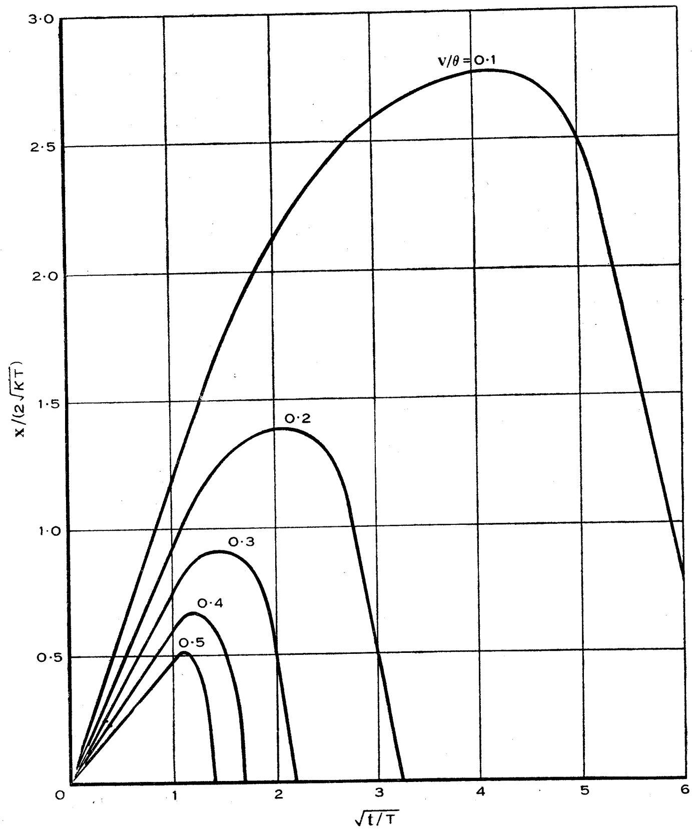

Fig. 1

III. References

Carslaw, H. S., and Jaeger, J. C. (1947).—" Conduction of Heat in Solids." (Oxford Univ. Press.)

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