ON THE STANDARD ERRORS IN THE FITTING OF POLYNOMIALS TO UNEQUALLY SPACED OBSERVATIONS

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[*Manuscript received November 5, 1952*]

Summary

From the observed values of the independent variable two parameters are derived which specify the departure from uniform spacing. Expressions are obtained for the standard errors of the coefficients and fitted values in terms of these parameters, and numerical tables for the estimation of the errors are given. It is shown that the errors calculated in this way lie within a few per cent. of the exact least squares values for polynomials of the first and second degree, but when the polynomial is of the third degree the deviations may be much greater.

I. Introduction

The problem of fitting polynomials to equally spaced observations has been very thoroughly treated by a number of authors. When the observations are not equally spaced the problem is much more difficult. In fact, each set of observations requires its own separate treatment, and no general information concerning the values of the orthogonal polynomials $T_j(x)$ is available. Information of this type is essential if an adequate discussion is to be given of approximate methods of curve fitting such as those described in an earlier paper (Guest 1952). All that could be done in that paper was to assume that the values $\Sigma T_j^2(x)$ did not differ greatly from those in the equally spaced case. The efficiencies of the methods of grouping described there are calculated on this assumption, and hence may be in error if the spacing is markedly non-uniform.

The aim of the present paper is to prepare the way for a new attack on this problem by discussing the standard errors of the least squares polynomial coefficients and fitted values when the observations are unequally spaced. The procedure adopted is to characterize any particular set of observations by two parameters $\varkappa_2$, $\varkappa_3$. The parameter $\varkappa_2$ is a measure of the departure of the independent variable x from symmetry about the central value, while the parameter $\varkappa_3$ is a measure of the relative concentration of the observations towards the central values of x as opposed to the extreme values. It is shown that $\Sigma T_j^2(x)$ can be expressed approximately as a function of $\varkappa_2$, $\varkappa_3$, and tables have been prepared from which these functions may be obtained. Tables for the calculation of the standard errors of the fitted values are also given.

Although the quantities tabulated were calculated for use in future theoretical discussions, they may also be of use in practical examples, either for the rough calculation of the standard errors or as a check on the values

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obtained by the usual methods. Some examples showing how the values obtained by the present method agree with the exact least squares values will be given.

Unfortunately, the treatment in terms of the parameters $\varkappa_2$, $\varkappa_3$ is not adequate for all possible sets of data. In certain cases additional higher order parameters $\varkappa_4$, $\varkappa_5$ are required. The higher the degree of the polynomial to be fitted the more important will these additional parameters be. However, it is found that the treatment given here is adequate for practically all cases in which the curve is of the first or second degree, and for a large proportion of the cases in which the curve is of the third degree.

II. The Smoothing of the Points of Observation

If the values of the independent variable x at the n points of observation are arranged in order of magnitude, each observation may be identified by a number ε giving its position in the sequence, ε taking the integral or half-integral values from $+\frac{1}{2}(n-1)$ to $-\frac{1}{2}(n-1)$. A point of observation can then be represented by the symbol $x(\varepsilon)$. In the present discussion the system of points $x(\varepsilon)$ will be replaced by a smoothed-out system $X(\varepsilon)$ obtained by fitting a curve of the third degree in ε to the values $x(\varepsilon)$. The smoothed-out system of points is given by the equation

$$X(\varepsilon)=k_0+k_1T_1(\varepsilon)+k_2T_2(\varepsilon)+k_3T_3(\varepsilon), \qquad (1)$$

where

$$k_j=\sum_{\varepsilon} T_j(\varepsilon)x(\varepsilon)/\sum_{\varepsilon} T_j^{\,2}(\varepsilon), \qquad (2)$$

and $T_j(\varepsilon)$ is the orthogonal polynomial of degree j in ε with leading coefficient unity.

The variable $X(\varepsilon)$ can be transformed by a change of origin and scale to give a new variable

$$\xi(\varepsilon)=\varphi\{X(\varepsilon)-k_0\}. \qquad (3)$$

Then equation (1) becomes

$$\xi(\varepsilon)=\varkappa_1\tau_1(\varepsilon)+\varkappa_2\tau_2(\varepsilon)+2\varkappa_3\tau_3(\varepsilon), \qquad (4)$$

where

$$\tau_j(\varepsilon)=n^{-j+1}T_j(\varepsilon), \qquad (5)$$

and

$$\varkappa_1=\varphi k_1,\ \ \varkappa_2=n\varphi k_2,\ \ \varkappa_3=\tfrac{1}{2}n^2\varphi k_3. \qquad (6)$$

The advantage of this notation is that the coefficients $\varkappa_j$ are all of the same order of magnitude.

Since it is desired to compare the smoothed set $\xi(\varepsilon)$ with the equally spaced set ε, the scale factor φ in (3) should be chosen so that the mean interval between successive observations is unity. The mean interval is

$$\frac{1}{n-1}\left\{\xi\left(+\frac{n-1}{2}\right)-\xi\left(-\frac{n-1}{2}\right)\right\}=\varphi\left\{k_1+\frac{(n-2)(n-3)}{10}k_3\right\},$$

and this would give for φ the value $\left\{k_1+\frac{1}{10}(n-2)(n-3)k_3\right\}^{-1}$. However, the term $(n-2)(n-3)$ is found to be inconvenient in subsequent calculations, and the simpler form

$$\varphi=\left\{k_1+\frac{1}{10}n^2k_3\right\}^{-1} \qquad (7)$$

will be adopted. From equations (6) and (7) it follows that

$$\varkappa_1=1-\frac{1}{5}\varkappa_3. \qquad (8)$$

The original set of points $x(\varepsilon)$ is thus replaced by a smoothed-out set characterized by three parameters φ, $\varkappa_2$, $\varkappa_3$. The parameter φ gives the scale factor, while the parameters $\varkappa_2$, $\varkappa_3$ specify the departure from uniform spacing.

III. The Parameters $\varkappa_2$, $\varkappa_3$

Values of k_1, k_2, k_3 may be found by one of the conventional least squares methods for equally spaced data, x being treated as the dependent variable and ε as the independent variable.

A more rapid procedure is to use the method of weighted grouping (Guest 1951). k_2, k_3 are identical with the quantities denoted by a_2, a_3 in the reference quoted, while

$$k_1=a_1+\frac{1}{40}\}n^2-5(\nu+r)^2\}a_3,$$

when $n=3\nu+r$, ν being an integer and r having the value -1, 0, or $+1$.

Very often comparatively rough approximations will suffice. If X_1, X_2, X_3, X_4, X_5 denote the values of X for which ε takes the values $+\frac{1}{2}(n-1)$, $+\frac{1}{4}(n-1)$, 0, $-\frac{1}{4}(n-1)$, $-\frac{1}{2}(n-1)$, then from equation (1)

$$\frac{X_1+X_5-2X_3}{(n-1)^2}=\tfrac{1}{2}k_2, \qquad (9)$$

$$\frac{X_1-X_5-2(X_2-X_4)}{(n-1)^3}=\frac{3}{16}k_3. \qquad (10)$$

Also

$$\frac{X_1-X_5}{n-1}=k_1+\frac{1}{10}(n-2)(n-3)k_3$$
$$=\varphi^{-1}-\frac{1}{10}k_3(5n-6).$$

Hence, if x_i denote the observed values x corresponding to the same five values of ε, the values of the parameters can be estimated from the following formulae :

$$\left.\begin{aligned} k_2&=\frac{2(x_1+x_5-2x_3)}{(n-1)^2},\\ k_3&=\frac{16\{(x_1-x_5)-2(x_2-x_4)\}}{3(n-1)^3},\end{aligned}\right\} \qquad (11)$$

$$\varphi^{-1}=\frac{x_1-x_5}{n-1}+\tfrac{1}{2}nk_3. \qquad (12)$$

$$\left.\begin{aligned} \varkappa_2&=n\varphi k_2,\\ \varkappa_3&=\tfrac{1}{2}n^2\varphi k_3.\end{aligned}\right\} \qquad (13)$$

The significance of the parameters $\varkappa_2$, $\varkappa_3$ can be brought out by combining equations (**11**), (**12**), and (**13**) to give the following approximate equations :

$$\frac{x_1-x_3}{x_1-x_5} \doteqdot \frac{2+\varkappa_2}{4}, \qquad \text{(14)}$$

$$\frac{x_2-x_4}{x_1-x_5} \doteqdot \frac{8-3\varkappa_3}{16}. \qquad \text{(15)}$$

Thus $\varkappa_2$ is a measure of the departure from symmetry about the central value x_3 ($\varepsilon=0$). $\varkappa_3$ is a measure of the relative concentration of the observations towards the centre of the range. For the equally spaced case $\varkappa_2=\varkappa_3=0$. When $\varkappa_2=+1$, the first half of the observations (for which ε is positive) is spread over three-quarters of the range of x. When $\varkappa_3=+4/3$, the central half of the observations (for which $|\varepsilon|$ is less than $\frac{1}{4}(n-1)$) is confined to a quarter of the range of x.

There does not appear to be any very simple criterion which fixes the ranges of values of $\varkappa_2$, $\varkappa_3$ likely to be encountered in practical examples. However, one condition which suggests itself is that the difference $\Delta\xi(\varepsilon)$ should always be positive, that is, the sequence of the smoothed values $X(\varepsilon)$ should be the same as the sequence of the actual values $x(\varepsilon)$.

The finite difference $\Delta\xi(\varepsilon)$ can be obtained from the approximate formula

$$\Delta\xi(\varepsilon) \doteqdot \varkappa_1+2n^{-1}\varkappa_2\varepsilon+2n^{-2}\varkappa_3(3\varepsilon^2-3n^2/20),$$

or

$$\Delta\xi \doteqdot 1+\varkappa_2 e+\tfrac{1}{2}\varkappa_3(3e^2-1), \qquad \text{(16)}$$

where $e=2\varepsilon/n$. It is required to find the condition that $\Delta\xi$ should never be negative in the range $-1<e<+1$.

If $\varkappa_3$ is positive there is a minimum value of $\Delta\xi$ at $e=-\varkappa_2/3\varkappa_3$. When $|\varkappa_2|<3\varkappa_3$ this minimum occurs in the range $-1<e<+1$, and has the value

$$1-\frac{1}{6}\frac{\varkappa_2{}^2}{\varkappa_3}-\tfrac{1}{2}\varkappa_3.$$

Hence, if this is to be positive,

$$\varkappa_2{}^2<3\varkappa_3(2-\varkappa_3). \qquad \text{(17)}$$

When $|\varkappa_2|>3\varkappa_3$, or when $\varkappa_3$ is negative, the least value of $\Delta\xi$ in the range $-1<e<+1$ will occur at one end of the range, the value then being

$$1-|\varkappa_2|+\varkappa_3.$$

If this is to be positive,

$$|\varkappa_2|<1+\varkappa_3. \qquad \text{(18)}$$

Conditions (**17**) and (**18**) may be summarized conveniently as follows :

$$\left.\begin{array}{ll} \text{for } -1 \leqslant \varkappa_3 \leqslant +0\cdot 5, & |\varkappa_2| \leqslant 1+\varkappa_3\,; \\ \text{for } +0\cdot 5 \leqslant \varkappa_3 \leqslant +2, & |\varkappa_2| \leqslant \{3\varkappa_3(2-\varkappa_3)\}^{\frac{1}{2}}. \end{array}\right\} \qquad \text{(19)}$$

If the values of $\varkappa_2$, $\varkappa_3$ lie outside these ranges, then it would certainly be necessary to include terms of higher degree in equation (**1**).

IV. THE ORTHOGONAL POLYNOMIALS $T_j(\xi)$

The orthogonal polynomials $T_j(\xi)$, together with the associated coefficients α_{jk} and β_{jk} defined by the equations

$$T_j(\xi)=\xi^j+\sum_0^{j-1}\alpha_{jk}T_k(\xi), \quad \ldots\ldots\ldots\ldots \quad (20)$$

$$T_j(\xi)=\sum_0^{j}\beta_{jk}\xi^k, \quad \ldots\ldots\ldots\ldots \quad (21)$$

are used in the calculation of the standard errors. Hence it is necessary to obtain formulae for which these quantities may be calculated in terms of the parameters $\varkappa_2$, $\varkappa_3$.

A convenient method of procedure is to expand ξ^j in terms of the orthogonal polynomials $\tau_j(\varepsilon)$, since this permits the rapid calculation of the sums $\Sigma T_j^2(\xi)$. Accordingly ξ^j is written in the form

$$n^{-(j-1)}\xi^j=\sum_{k=1}^{3j}\varkappa_{jk}\tau_k(\varepsilon)+\sum_{m=0}^{j-2}n^{-m}\varkappa_{j-m,0}\xi^m. \quad \ldots\ldots \quad (22)$$

Now the left-hand side of this equation can be rewritten as

$$n^{-1}\xi\{n^{-(j-2)}\xi^{j-1}\}=n^{-1}\sum_1^{3(j-1)}\varkappa_{j-1,k}\tau_k(\varepsilon)\sum_1^{3}\varkappa_{1m}\tau_m(\varepsilon)+\sum_0^{j-3}n^{-(m+1)}\varkappa_{j-m,0}\xi^{m+1},$$

and so

$$\sum_0^{3j}\varkappa_{jk}\tau_k(\varepsilon)=n^{-1}\sum_1^{3(j-1)}\varkappa_{j-1,k}\tau_k(\varepsilon)\sum_1^{3}\varkappa_{1m}\tau_m(\varepsilon). \quad \ldots\ldots\ldots \quad (23)$$

The expression on the right may be evaluated by using the recurrence relation

$$n^{-1}\varepsilon\tau_j(\varepsilon)=\tau_{j+1}(\varepsilon)+\rho_j\tau_{j-1}(\varepsilon), \quad \ldots\ldots\ldots\ldots \quad (24)$$

where

$$\rho_j=\frac{j^2(1-j^2/n^2)}{4(4j^2-1)}.$$

To simplify the calculations the term j^2/n^2 will be neglected, and the form

$$\rho_j=\frac{j^2}{4(4j^2-1)} \quad \ldots\ldots\ldots\ldots \quad (25)$$

will be used.

The expressions required for the calculation of $\varkappa_{ik}$ for values of j from 1 to 3 are listed in Table 1. The explicit expressions for these coefficients in terms of $\varkappa_2$, $\varkappa_3$ are given in Table 2.

From equations (20) and (22), $\tau_j(\xi)$ can be expressed in the form

$$\tau_j(\xi)=\sum_1^{3j}\varkappa_{jk}\tau_k(\varepsilon)+\sum_1^{j-1}\mu_{jm}\tau_m(\xi). \quad \ldots\ldots\ldots\ldots \quad (26)$$

Also the sum $\Sigma\tau_j^2(\xi)$ may be written as

$$\Sigma\tau_j^2(\xi)=\mu_{jj}\Sigma\tau_j^2(\varepsilon). \quad \ldots\ldots\ldots\ldots \quad (27)$$

It remains to calculate μ_{jm}, μ_{jj}.

If equation (**24**) is multiplied by $\tau_{j-1}(\varepsilon)$, it follows that

$$\rho_j \sum_{\varepsilon} \tau_{j-1}{}^2(\varepsilon) = \sum_{\varepsilon} \tau_j(\varepsilon)\{n^{-1}\varepsilon\tau_{j-1}(\varepsilon)\}$$
$$= \sum_{\varepsilon} \tau_j{}^2(\varepsilon),$$

TABLE 1

QUANTITIES REQUIRED IN THE CALCULATION OF THE COEFFICIENTS $\varkappa_{jk}$

ρ_j	$\rho_j = \dfrac{j^2}{4(4j^2-1)}$	$R_j = \prod_1^j \rho_k = \dfrac{(j!)^4}{(2j)!(2j+1)!}$
$\tau_j(\varepsilon)\tau_k(\varepsilon)$	$n^{-1}\tau_1(\varepsilon)\tau_j(\varepsilon) = \tau_{j+1}(\varepsilon) + \rho_j\tau_{j-1}(\varepsilon)$ $n^{-1}\tau_2(\varepsilon)\tau_j(\varepsilon) = \tau_{j+2}(\varepsilon) + \left(\rho_{j+1} + \rho_j - \dfrac{1}{12}\right)\tau_j(\varepsilon) + \rho_j\rho_{j-1}\tau_{j-2}(\varepsilon)$ $n^{-1}\tau_3(\varepsilon)\tau_j(\varepsilon) = \tau_{j+3}(\varepsilon) + \left(\rho_{j+2} + \rho_{j+1} + \rho_j - \dfrac{3}{20}\right)\tau_{j+1}(\varepsilon)$ $+\rho_j\left(\rho_{j+1} + \rho_j + \rho_{j-1} - \dfrac{3}{20}\right)\tau_{j-1}(\varepsilon) + \rho_j\rho_{j-1}\rho_{j-2}\tau_{j-3}(\varepsilon)$	

TABLE 2

THE COEFFICIENTS $\varkappa_{jk}$

$\varkappa_{11} = 1 - \dfrac{1}{5}\varkappa_3$	$\varkappa_{30} = n^{-3}\Sigma\xi^3$
$\varkappa_{12} = \varkappa_2$	$\varkappa_{31} = \dfrac{1}{15} - \dfrac{1}{70}\varkappa_3 + \dfrac{1}{350}\varkappa_3{}^2 - \dfrac{2}{5775}\varkappa_3{}^3 + \varkappa_2{}^2\left(\dfrac{13}{630} - \dfrac{2}{1575}\varkappa_3\right)$
$\varkappa_{13} = 2\varkappa_3$	$\varkappa_{32} = \varkappa_2\left(\dfrac{13}{42} - \dfrac{4}{105}\varkappa_3 + \dfrac{2}{1155}\varkappa_3{}^2 + \dfrac{2}{315}\varkappa_2{}^2\right)$
$\varkappa_{20} = n^{-2}\Sigma\xi^2$	$\varkappa_{33} = 1 - \dfrac{3}{55}\varkappa_3{}^2 + \dfrac{34}{5005}\varkappa_3{}^3 + \varkappa_2{}^2\left(\dfrac{1}{3} - \dfrac{8}{495}\varkappa_3\right)$
$\varkappa_{21} = \dfrac{2}{15}\varkappa_2\left(1 - \dfrac{1}{14}\varkappa_3\right)$	$\varkappa_{34} = \varkappa_2\left(3 + \dfrac{1}{11}\varkappa_3 - \dfrac{5}{143}\varkappa_3{}^2 + \dfrac{1}{11}\varkappa_2{}^2\right)$
$\varkappa_{22} = 1 - \dfrac{1}{7}\varkappa_3 + \dfrac{1}{21}\varkappa_2{}^2$	$\varkappa_{35} = 6\varkappa_3 - \dfrac{15}{13}\varkappa_3{}^2 + \dfrac{3}{65}\varkappa_3{}^3 + \varkappa_2{}^2\left(3 - \dfrac{1}{13}\varkappa_3\right)$
$\varkappa_{23} = 2\varkappa_2\left(1 - \dfrac{1}{9}\varkappa_3\right)$	$\varkappa_{36} = \varkappa_2\left(12\varkappa_3 - \dfrac{7}{5}\varkappa_3{}^2 + \varkappa_2{}^2\right)$
$\varkappa_{24} = 4\varkappa_3 - \dfrac{7}{11}\varkappa_3{}^2 + \varkappa_2{}^2$	$\varkappa_{37} = 12\varkappa_3{}^2 - \dfrac{30}{17}\varkappa_3{}^3 + 6\varkappa_2{}^2\varkappa_3$
$\varkappa_{25} = 4\varkappa_2\varkappa_3$	$\varkappa_{38} = 12\varkappa_2\varkappa_3{}^2$
$\varkappa_{26} = 4\varkappa_3{}^2$	$\varkappa_{39} = 8\varkappa_3{}^3$

and so

$$\Sigma\tau_j{}^2(\varepsilon) = (R_j/R_m)\Sigma\tau_m{}^2(\varepsilon), \quad \ldots\ldots\ldots\ldots \quad \textbf{(28)}$$

where

$$R_j = \prod_1^j \rho_k. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad \textbf{(29)}$$

To calculate μ_{jm}, equation (**26**) is multiplied by $\tau_m(\xi)$ and summed over ξ, giving

$$\mu_{jm}\Sigma\tau_m{}^2(\xi) = -\sum_{\xi}\{\sum_1^{3j}\varkappa_{jk}\tau_k(\varepsilon)\tau_m(\xi)\} \quad \ldots\ldots\ldots\ldots \quad (\mathbf{30})$$

$$= -\sum_{\xi}\{\sum_1^{3j}\varkappa_{jk}\tau_k(\varepsilon)\}\{\sum_1^{3m}\varkappa_{mq}\tau_q(\varepsilon) + \sum_1^{m-1}\mu_{mr}\tau_r(\xi)\}$$

$$= -\{\sum_1^{3m}\varkappa_{jk}\varkappa_{mk}\Sigma\tau_k{}^2(\varepsilon) - \sum_1^{m-1}\mu_{mr}\mu_{jr}\Sigma\tau_r{}^2(\xi)\},$$

and so

$$\mu_{jm} = -\{\sum_1^{3m}\varkappa_{jk}\varkappa_{mk}R_k - \sum_1^{m-1}\mu_{jk}\mu_{mk}\mu_{kk}R_k\}/\mu_{mm}R_m. \quad \ldots\ldots \quad (\mathbf{31})$$

To calculate μ_{jj}, equation (**26**) is squared, and, from (**30**), it follows that

$$\Sigma\tau_j{}^2(\xi) = \sum_1^{3j}\varkappa_{jk}{}^2\Sigma\tau_k{}^2(\varepsilon) - \sum_1^{j-1}\mu_{jm}{}^2\Sigma\tau_m{}^2(\xi).$$

Hence

$$\mu_{jj} = \{\sum_1^{3j}\varkappa_{jk}{}^2R_k - \sum_1^{j-1}\mu_{jm}{}^2\mu_{mm}R_m\}/R_j. \quad \ldots\ldots\ldots\ldots \quad (\mathbf{32})$$

In Table 3 are listed the sums $\sum_m \varkappa_{jm}\varkappa_{km}R_m$ as explicit functions of $\varkappa_2$, $\varkappa_3$, and also the formulae for deriving μ_{jm}, μ_{jj} from these sums, for values of j, k from 1 to 3. In Table 4 the numerical values μ_{jj}, μ_{jm} are tabulated for the range $\varkappa_2{}^2 = 0(0\cdot25)1\cdot0(0\cdot5)2\cdot0$, $\varkappa_3 = -1\cdot0(0\cdot2)+2\cdot0$.

TABLE 3

FORMULAE FOR THE DERIVATION OF THE QUANTITIES μ_{jj}, μ_{jm}

$$\sum_1^3\varkappa_{1j}{}^2R_j = [1-0\cdot400000\varkappa_3+0\cdot057143\varkappa_3{}^2+0\cdot066667\varkappa_2{}^2]R_1$$

$$\sum_1^3\varkappa_{1j}\varkappa_{2j}R_j = 0\cdot200000\varkappa_2[1-0\cdot142857\varkappa_3+0\cdot015873\varkappa_2{}^2]R_1$$

$$\left.\begin{array}{l}\displaystyle\sum_1^6\varkappa_{2j}{}^2R_j \\ \\ \displaystyle\sum_1^3\varkappa_{1j}\varkappa_{3j}R_j\end{array}\right\} = [(1-0\cdot285714\varkappa_3+0\cdot085714\varkappa_3{}^2-0\cdot020779\varkappa_3{}^3+0\cdot001912\varkappa_3{}^4) + \varkappa_2{}^2(0\cdot619048-0\cdot076190\varkappa_3+0\cdot003463\varkappa_3{}^2)+\varkappa_2{}^4(0\cdot006349)]R_2$$

$$\sum_1^6\varkappa_{2j}\varkappa_{3j}R_j = 0\cdot571429\varkappa_2[(1-0\cdot150000\varkappa_3+0\cdot013636\varkappa_3{}^2-0\cdot001598\varkappa_3{}^3) + \varkappa_2{}^2(0\cdot205556-0\cdot017677\varkappa_3+0\cdot000178\varkappa_3{}^2)+\varkappa_2{}^4(0\cdot001178)]R_2$$

$$\begin{aligned}\sum_1^9\varkappa_{3j}{}^2R_j = [&(2\cdot037037-0\cdot444444\varkappa_3+0\cdot171717\varkappa_3{}^2-0\cdot071736\varkappa_3{}^3+0\cdot017027\varkappa_3{}^4 \\ &-0\cdot002301\varkappa_3{}^5+0\cdot000136\varkappa_3{}^6) \\ &+\varkappa_2{}^2(3\cdot370370-0\cdot397306\varkappa_3+0\cdot030947\varkappa_3{}^2-0\cdot005565\varkappa_3{}^3+0\cdot000383\varkappa_3{}^4) \\ &+\varkappa_2{}^4(0\cdot342312-0\cdot025273\varkappa_3+0\cdot000464\varkappa_3{}^2)+\varkappa_2{}^6(0\cdot001404)]R_3\end{aligned}$$

$$\mu_{11} = \Sigma\varkappa_{1j}{}^2R_j/R_1$$
$$\mu_{21} = -\{\Sigma\varkappa_{1j}\varkappa_{2j}R_j/R_1\}/\mu_{11}$$
$$\mu_{22} = \{\Sigma\varkappa_{2j}{}^2R_j/R_2\}-15\mu_{21}{}^2\mu_{11}$$
$$\mu_{31} = -\{\Sigma\varkappa_{1j}\varkappa_{3j}R_j/R_2\}/15\mu_{11}$$
$$\mu_{32} = -\{(\Sigma\varkappa_{2j}\varkappa_{3j}R_j/R_2)-15\mu_{31}\mu_{21}\mu_{11}\}/\mu_{22}$$
$$\mu_{33} = \{\Sigma\varkappa_{3j}{}^2R_j/R_3\}-(140/9)\mu_{32}{}^2\mu_{22}-(700/3)\mu_{31}{}^2\mu_{11}$$

TABLE 4

NUMERICAL VALUES OF μ_{jj}, μ_{jm}

μ_{11}

$\varkappa_3$ \ $\varkappa_2^2$	0	0·25	0·50	0·75	1·00	1·50	2·00
—1·0	1·457	1·474	1·490	1·507	1·524	1·557	1·590
—0·8	1·357	1·373	1·390	1·407	1·423	1·457	1·490
—0·6	1·261	1·277	1·294	1·311	1·327	1·361	1·394
—0·4	1·169	1·186	1·202	1·219	1·236	1·269	1·302
—0·2	1·082	1·099	1·116	1·132	1·149	1·182	1·216
0	1·000	1·017	1·033	1·050	1·067	1·100	1·133
+0·2	0·922	0·939	0·956	0·972	0·989	1·022	1·056
+0·4	0·849	0·866	0·882	0·899	0·916	0·949	0·982
+0·6	0·781	0·797	0·814	0·831	0·847	0·881	0·914
+0·8	0·717	0·733	0·750	0·767	0·783	0·817	0·850
+1·0	0·657	0·674	0·690	0·707	0·724	0·757	0·790
+1·2	0·602	0·619	0·636	0·652	0·669	0·702	0·736
+1·4	0·552	0·569	0·585	0·602	0·619	0·652	0·685
+1·6	0·506	0·523	0·540	0·556	0·573	0·606	0·640
+1·8	0·465	0·482	0·498	0·515	0·532	0·565	0·598
+2·0	0·429	0·445	0·462	0·479	0·495	0·529	0·562

μ_{22}

$\varkappa_3$ \ $\varkappa_2^2$	0	0·25	0·50	0·75	1·00	1·50	2·00
—1·0	1·394	1·435	1·478	1·524	1·571	1·670	1·776
—0·8	1·295	1·329	1·366	1·404	1·445	1·532	1·627
—0·6	1·207	1·234	1·264	1·296	1·331	1·406	1·489
—0·4	1·129	1·150	1·173	1·199	1·227	1·290	1·363
—0·2	1·061	1·074	1·091	1·110	1·132	1·184	1·245
0	1·000	1·006	1·016	1·029	1·045	1·085	1·136
+0·2	0·946	0·946	0·949	0·955	0·965	0·994	1·035
+0·4	0·898	0·891	0·887	0·887	0·891	0·910	0·942
+0·6	0·855	0·841	0·831	0·825	0·823	0·832	0·855
+0·8	0·816	0·795	0·779	0·767	0·760	0·759	0·774
+1·0	0·781	0·753	0·731	0·714	0·702	0·692	0·700
+1·2	0·749	0·714	0·686	0·664	0·648	0·631	0·632
+1·4	0·718	0·678	0·645	0·619	0·598	0·575	0·571
+1·6	0·690	0·644	0·607	0·576	0·553	0·524	0·517
+1·8	0·662	0·612	0·571	0·538	0·512	0·480	0·471
+2·0	0·636	0·583	0·539	0·504	0·477	0·443	0·434

TABLE 4 (*Continued*)

μ_{33}

$\varkappa_3$ \ $\varkappa_2^2$	0	0·25	0·50	0·75	1·00	1·50	2·00
—1·0	1·361	1·428	1·505	1·591	1·685	1·900	2·146
—0·8	1·265	1·313	1·370	1·436	1·510	1·684	1·891
—0·6	1·185	1·216	1·256	1·304	1·360	1·498	1·668
—0·4	1·116	1·135	1·160	1·192	1·233	1·337	1·473
—0·2	1·055	1·064	1·078	1·098	1·124	1·199	1·303
0	1·000	1·003	1·009	1·019	1·033	1·081	1·157
+0·2	0·948	0·949	0·949	0·951	0·957	0·981	1·031
+0·4	0·897	0·898	0·897	0·894	0·892	0·898	0·924
+0·6	0·847	0·851	0·850	0·845	0·838	0·828	0·833
+0·8	0·796	0·806	0·807	0·801	0·792	0·769	0·756
+1·0	0·745	0·761	0·766	0·762	0·752	0·720	0·691
+1·2	0·692	0·716	0·727	0·726	0·716	0·679	0·637
+1·4	0·639	0·671	0·688	0·692	0·684	0·643	0·592
+1·6	0·587	0·627	0·650	0·658	0·653	0·612	0·554
+1·8	0·537	0·583	0·612	0·625	0·623	0·584	0·524
+2·0	0·491	0·541	0·575	0·592	0·593	0·558	0·499

μ_{31}

$\varkappa_3$ \ $\varkappa_2^2$	0	0·25	0·50	0·75	1·00	1·50	2·00
—1·0	—0·0638	—0·0710	—0·0781	—0·0850	—0·0918	—0·1052	—0·1181
—0·8	—0·0636	—0·0712	—0·0785	—0·0858	—0·0929	—0·1068	—0·1201
—0·6	—0·0638	—0·0717	—0·0794	—0·0870	—0·0944	—0·1088	—0·1227
—0·4	—0·0644	—0·0727	—0·0807	—0·0886	—0·0963	—0·1113	—0·1257
—0·2	—0·0653	—0·0740	—0·0824	—0·0907	—0·0987	—0·1143	—0·1292
0	—0·0667	—0·0757	—0·0846	—0·0932	—0·1016	—0·1177	—0·1331
+0·2	—0·0684	—0·0779	—0·0872	—0·0962	—0·1049	—0·1217	—0·1376
+0·4	—0·0705	—0·0805	—0·0902	—0·0996	—0·1087	—0·1262	—0·1426
+0·6	—0·0730	—0·0836	—0·0937	—0·1035	—0·1130	—0·1311	—0·1481
+0·8	—0·0760	—0·0870	—0·0976	—0·1079	—0·1177	—0·1364	—0·1539
+1·0	—0·0792	—0·0908	—0·1019	—0·1126	—0·1229	—0·1422	—0·1602
+1·2	—0·0829	—0·0950	—0·1066	—0·1177	—0·1283	—0·1483	—0·1667
+1·4	—0·0868	—0·0995	—0·1116	—0·1231	—0·1340	—0·1545	—0·1734
+1·6	—0·0908	—0·1041	—0·1167	—0·1286	—0·1399	—0·1609	—0·1800
+1·8	—0·0949	—0·1088	—0·1218	—0·1340	—0·1456	—0·1671	—0·1865
+2·0	—0·0989	—0·1132	—0·1267	—0·1393	—0·1511	—0·1729	—0·1925

TABLE 4 (*Continued*)

μ_{21}
(Sign of μ_{21} opposite to that of $\varkappa_2$)

$\varkappa_3$ \ $\varkappa_2^2$ / $\lvert\varkappa_2\rvert$	0 0	0·25 0·500	0·50 0·707	0·75 0·866	1·00 1·000	1·50 1·225	2·00 1·414
—1·0	0	0·0778	0·1092	0·1327	0·1521	0·1835	0·2089
—0·8	0	0·0814	0·1142	0·1387	0·1588	0·1914	0·2176
—0·6	0	0·0853	0·1195	0·1451	0·1660	0·1998	0·2267
—0·4	0	0·0895	0·1253	0·1519	0·1737	0·2086	0·2365
—0·2	0	0·0940	0·1314	0·1592	0·1818	0·2180	0·2467
0	0	0·0988	0·1379	0·1669	0·1905	0·2280	0·2575
+0·2	0	0·1039	0·1449	0·1752	0·1997	0·2385	0·2688
+0·4	0	0·1094	0·1524	0·1839	0·2094	0·2495	0·2806
+0·6	0	0·1152	0·1602	0·1931	0·2196	0·2610	0·2928
+0·8	0	0·1213	0·1685	0·2028	0·2302	0·2728	0·3053
+1·0	0	0·1278	0·1772	0·2129	0·2412	0·2850	0·3181
+1·2	0	0·1345	0·1861	0·2232	0·2525	0·2973	0·3308
+1·4	0	0·1414	0·1952	0·2336	0·2638	0·3095	0·3433
+1·6	0	0·1483	0·2043	0·2439	0·2748	0·3213	0·3552
+1·8	0	0·1550	0·2130	0·2538	0·2853	0·3323	0·3661
+2·0	0	0·1613	0·2211	0·2628	0·2949	0·3420	0·3755

μ_{32}
(Sign of μ_{32} opposite to that of $\varkappa_2$)

$\varkappa_3$ \ $\varkappa_2^2$ / $\lvert\varkappa_2\rvert$	0 0	0·25 0·500	0·50 0·707	0·75 0·866	1·00 1·000	1·50 1·225	2·00 1·414
—1·0	0	0·1580	0·2202	0·2657	0·3024	0·3600	0·4044
—0·8	0	0·1648	0·2299	0·2776	0·3161	0·3764	0·4228
—0·6	0	0·1711	0·2389	0·2889	0·3292	0·3925	0·4410
—0·4	0	0·1766	0·2472	0·2994	0·3416	0·4081	0·4590
—0·2	0	0·1813	0·2544	0·3089	0·3531	0·4231	0·4767
0	0	0·1852	0·2606	0·3173	0·3636	0·4373	0·4940
+0·2	0	0·1880	0·2656	0·3245	0·3730	0·4507	0·5107
+0·4	0	0·1898	0·2694	0·3303	0·3811	0·4631	0·5270
+0·6	0	0·1905	0·2718	0·3348	0·3879	0·4746	0·5427
+0·8	0	0·1903	0·2729	0·3379	0·3933	0·4851	0·5578
+1·0	0	0·1891	0·2728	0·3397	0·3974	0·4944	0·5722
+1·2	0	0·1871	0·2716	0·3401	0·4002	0·5026	0·5857
+1·4	0	0·1845	0·2694	0·3395	0·4017	0·5096	0·5981
+1·6	0	0·1814	0·2666	0·3379	0·4022	0·5153	0·6089
+1·8	0	0·1783	0·2634	0·3358	0·4018	0·5196	0·6176
+2·0	0	0·1754	0·2604	0·3336	0·4010	0·5223	0·6233

It will be observed that the expressions given in Table 3 for $\Sigma\varkappa_{2k}{}^2R_k$ and for $\Sigma\varkappa_{1k}\varkappa_{3k}R_k$ are the same. This is a special case of the general equality

$$\sum_1^{3m}\varkappa_{jk}\varkappa_{mk}R_k = \sum_1^{3(m+q)} \varkappa_{j-q,k}\varkappa_{m+q,k}R_k, \quad \ldots\ldots\ldots\ldots \quad (33)$$

which may be proved by induction.

TABLE 5

THE COEFFICIENTS α_{jk}, β_{jk}

$\alpha_{10}=0$	$\beta_{10}=0$
$\alpha_{21}=n\mu_{21}$	$\beta_{21}=n\mu_{21}$
$\alpha_{20}=-n^2\mu_{11}/12$	$\beta_{20}=-n^2\mu_{11}/12$
$\alpha_{32}=n\mu_{32}$	$\beta_{32}=n\mu_{32}$
$\alpha_{31}=n^2(\mu_{31}-\mu_{11}/12)$	$\beta_{31}=n^2(\mu_{31}+\mu_{32}\mu_{21}-\mu_{11}/12)$
$\alpha_{30}=n^3\mu_{21}\mu_{11}/12$	$\beta_{30}=n^3\mu_{11}(\mu_{21}-\mu_{32})/12$

The coefficients α_{jk}, β_{jk} may be determined by combining equations (**20**), (**22**), and (**26**). The values so obtained are listed in Table 5 for values of j from 1 to 3.

V. THE STANDARD ERRORS

(*a*) *The Orthogonal Coefficients* a_j

If the least squares polynomial is written in the form

$$u_p(\xi)=\sum_{j=0}^{p} a_jT_j(\xi), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (34)$$

the standard error of the coefficient a_j is given by

$$\sigma^2(y)/\sigma^2(a_j)=\sum_{\xi}T_j{}^2(\xi),$$

where $\sigma(y)$ is the standard error of an observation. Thus

$$\sigma^2(a_j)=\mu_{jj}{}^{-1}\{\sigma^2(y)/\Sigma T_j{}^2(\varepsilon)\}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (35)$$

The values μ_{jj} are given in Table 4. It will be seen that for all three values of j negative values of $\varkappa_3$ yield values for μ_{jj} greater than unity and positive values of $\varkappa_3$ yield values less than unity. That is, the standard errors are reduced if the observations are crowded towards the extremes of the range and increased if they are crowded towards the centre of the range.

The effect of $\varkappa_2$ can be summarized as follows. The standard error $\sigma(a_1)$ decreases only slowly as $\varkappa_2{}^2$ increases. The standard error $\sigma(a_2)$ for negative values of $\varkappa_3$ decreases as $\varkappa_2{}^2$ increases, while for positive values of $\varkappa_3$ it increases as $\varkappa_2{}^2$ increases. The standard error $\sigma(a_3)$ for negative values of $\varkappa_3$ decreases as $\varkappa_2{}^2$ increases, but varies only slowly for positive values of $\varkappa_3$. Both $\sigma(a_2)$ and $\sigma(a_3)$ are almost independent of $\varkappa_2$ near $\varkappa_3=0$.

If the values $\varkappa_2$, $\varkappa_3$ likely to be encountered in practical examples are assumed to lie between -1 and $+1$, then the corresponding range of μ_{jj} is from $1\cdot685$ to $0\cdot657$. Thus the range of variation of the standard errors, which are proportional to $\mu_{jj}{}^{-\frac{1}{2}}$, is from $0\cdot77$ to $1\cdot23$, and the standard errors $\sigma(a_j)$ will lie within 25 per cent. of the values for the equally spaced case.

The coefficient a_j is identical with the coefficient b_{jj} in the power series expansion of the polynomial of degree j,

$$u_j(\xi)=\sum_{k=0}^{j} b_{jk}\xi^k. \qquad \text{(36)}$$

The values of the coefficients a_j are independent of the choice of origin for the variable.

(b) The Fitted Values

The standard error of the fitted value $u_p(\xi)$ is given by the expression

$$\begin{aligned}\sigma^2[u_p(\xi)]/\sigma^2(y) &= \sum_{j=0}^{p}\{T_j{}^2(\xi)/\sum_{\xi} T_j{}^2(\xi)\}\\ &= n^{-1}\sum_{j=0}^{p}\{(\sum_{k=0}^{j}\beta_{jk}\xi^k)^2/\mu_{jj}R_j n^{2j}\}\\ &= n^{-1}\sum_{j=0}^{p}\chi_{j0}, \text{ say.}\end{aligned}$$

If the substitution $e=2\xi/n$ is made, it is found that the quantities χ_{j0} are the following functions of e:

$$\chi_{00}=1,$$

$$\chi_{10}=3e^2/\mu_{11}, \qquad \text{(37)}$$

$$\chi_{20}=1\cdot25(3e^2+6en^{-1}\beta_{21}+12n^{-2}\beta_{20})^2/\mu_{22}, \qquad \text{(38)}$$

$$\chi_{30}=1\cdot75(5e^3+10e^2n^{-1}\beta_{32}+20en^{-2}\beta_{31}+40n^{-3}\beta_{30})^2/\mu_{33}. \qquad \text{(39)}$$

The quantities $n^{-(j-k)}\beta_{jk}$ are functions of the coefficients μ, as given in Table 5. The standard error may be written in the form

$$\sigma[u_p(\xi)]=n^{-\frac{1}{2}}\rho_{p0}[e,\varkappa_2,\varkappa_3]\sigma(y). \qquad \text{(40)}$$

For the case $p=1$, the function ρ_{p0} is just $(1+3e^2/\mu_{11})^{\frac{1}{2}}$. Thus the standard error is a minimum at the value $\xi=0$ (i.e. at $x=\Sigma x/n$), and increases symmetrically on each side of this value. If the variable $k=e\mu_{11}^{-\frac{1}{2}}$ is introduced, then $\rho_{10}{}^2=1+3k^2$ and the function $\rho_{10}(k)$ is identical with that tabulated in a previous paper (Guest 1950). When the value $\Sigma T_1{}^2(x)$ is known from the least squares calculations, k is given by

$$k=\left\{\frac{n}{3\Sigma T_1{}^2(x)}\right\}^{\frac{1}{2}}\left\{x-\frac{\Sigma x}{n}\right\},$$

and

$$\sigma[u_1(x)]=n^{-1}\rho_{10}(k)\sigma(y),$$

where $\rho_{10}(k)$ may be read off directly from the tables.

For the case $p=2$, χ_{20} is roughly symmetrical about the value $e=-\mu_{21}$, and the major effect of the parameter $\varkappa_2$ is to make ρ_{20} roughly symmetrical about $e=\varkappa_2/5$. For this reason it is an advantage to introduce a new variable k, such that

$$k=e-\varkappa_2/5. \qquad \text{(41)}$$

The function $\rho_{20}[k,\varkappa_2,\varkappa_3]$ is given in Table 6 for the range $k=-1\cdot4(0\cdot2)+1\cdot4$, $\varkappa_2=-1\cdot0(0\cdot5)+1\cdot0$, $\varkappa_3=-1\cdot0(0\cdot25)+1\cdot0$.

Similarly, for the case $p=3$, it is an advantage to tabulate in terms of the variable k given by (**41**) rather than the variable e. The function $\rho_{30}[k, \varkappa_2, \varkappa_3]$ is given in Table 6 for the same range as the function ρ_{20}.

It is found that when $|\varkappa_2|$ is large the values of ρ_{20} and ρ_{30} near $|k|=0\cdot5$ are increased for points with $k\varkappa_2$ positive and decreased for points with $k\varkappa_2$ negative. The parameter $\varkappa_3$ has a much less marked effect. In general the values of ρ_{20} and ρ_{30} are increased when $\varkappa_3$ is positive and decreased when $\varkappa_3$ is negative.

The procedure for the rough estimation of the standard errors of the fitted values when the polynomial is of the second or third degree may be summarized as follows:

(i) Suppose the observations to be numbered, in order of x, by the values ε from $+\frac{1}{2}(n-1)$ to $-\frac{1}{2}(n-1)$. Write down the values for x_1, x_5 for $\varepsilon=\pm\frac{1}{2}(n-1)$; x_2, x_4 for $\varepsilon=\pm\frac{1}{4}(n-1)$; x_3 for $\varepsilon=0$, interpolating where necessary.

(ii) Calculate

$$\begin{aligned}
\psi_2&=2(x_1+x_5-2x_3),\\
\psi_3&=2\cdot67\{(x_1-x_5)-2(x_2-x_4)\},\\
\psi_1&=(x_1-x_5)+\psi_3/(n-1),\\
\varphi&=(n-1)/\psi_1,\\
\varkappa_2&=\{1+1/n\}\{\psi_2/\psi_1\},\\
\varkappa_3&=\{1+2/n\}\{\psi_3/\psi_1\}.
\end{aligned}$$

(iii) Calculate

$$k=\frac{2\varphi}{n}x-\left(\frac{2\varphi}{n}\,\frac{\Sigma x}{n}+\frac{\varkappa_2}{5}\right),$$

for each value of x at which the standard error is required.

(iv) Find $\rho_{p0}(k, \varkappa'_2, \varkappa'_3)$ by interpolation in Table 6 for the values $\varkappa'_2$, $\varkappa'_3$ nearest to $\varkappa_2$, $\varkappa_3$.

(v) Then

$$\sigma[u_p(x)]\doteqdot n^{-\frac{1}{2}}\rho_{p0}(k, \varkappa'_2, \varkappa'_3)\sigma(y).$$

(*c*) *The Polynomial Coefficients*

Suppose that at the origin of the variable x the variable e has the value e_0. If the least squares polynomial is written as

$$u_p(x)=\sum_{j=0}^{p} c_{pj}x^j, \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{42})$$

then the standard error of the coefficient c_{pj} is given by an expression of the form

$$\sigma[c_{pj}(e_0,\varkappa_2,\varkappa_3)]=\varphi^j n^{-(j+\frac{1}{2})}\rho_{pj}(e_0,\varkappa_2,\varkappa_3)\sigma(y). \quad \ldots \quad (\mathbf{43})$$

The functions ρ_{pj} for $j=1$ are given by

$$\begin{aligned}
\rho_{11}{}^2&=12/\mu_{11}\\
\rho_{21}{}^2&=\rho_{11}{}^2+180(e_0+n^{-1}\beta_{21})^2/\mu_{22},\\
\rho_{31}{}^2&=\rho_{21}{}^2+175(3e_0{}^2+4e_0n^{-1}\beta_{32}+4n^{-2}\beta_{31})^2/\mu_{33}.
\end{aligned}$$

For $j=2$ the functions are given by

$$\begin{aligned}
\rho_{22}{}^2&=180/\mu_{22},\\
\rho_{32}{}^2&=\rho_{22}{}^2+700(3e_0+2n^{-1}\beta_{32})^2/\mu_{33}.
\end{aligned}$$

Table 6

VALUES OF ρ_{20} AND ρ_{30}

		ρ_{20}														
		$k\varkappa_2$ negative								$k\varkappa_2$ positive						
	$\lvert k \rvert$	1·4	1·2	1·0	0·8	0·6	0·4	0·2	0	0·2	0·4	0·6	0·8	1·0	1·2	1·4
$\varkappa_3 = +1\cdot00$	$\lvert\varkappa_2\rvert = 1\cdot0$	7·59	5·46	3·68	2·27	1·36	1·16	1·39	1·61	1·69	1·72	1·91	2·52	3·62	5·15	7·07
	0·5	7·54	5·51	3·81	2·48	1·57	1·20	1·26	1·38	1·44	1·51	1·81	2·55	3·71	5·26	7·16
	0	7·32	5·39	3·79	2·54	1·71	1·33	1·28	1·30	1·28	1·33	1·71	2·54	3·79	5·39	7·32
$\varkappa_3 = +0\cdot75$	$\lvert\varkappa_2\rvert = 1\cdot0$	7·02	5·03	3·36	2·07	1·30	1·19	1·42	1·62	1·68	1·68	1·85	2·43	3·48	4·96	6·79
	0·5	7·11	5·17	3·56	2·30	1·48	1·21	1·30	1·42	1·45	1·48	1·74	2·43	3·55	5·06	6·90
	0	7·01	5·14	3·59	2·40	1·63	1·32	1·32	1·35	1·32	1·32	1·63	2·40	3·59	5·14	7·01
$\varkappa_3 = +0\cdot50$	$\lvert\varkappa_2\rvert = 1\cdot0$	6·49	4·62	3·07	1·90	1·25	1·23	1·46	1·63	1·66	1·64	1·78	2·33	3·34	4·75	6·52
	0·5	6·70	4·85	3·31	2·13	1·41	1·23	1·35	1·46	1·47	1·47	1·68	2·31	3·39	4·84	6·63
	0	6·70	4·88	3·39	2·26	1·56	1·32	1·35	1·39	1·35	1·32	1·56	2·26	3·39	4·88	6·70
$\varkappa_3 = +0\cdot25$	$\lvert\varkappa_2\rvert = 1\cdot0$	6·00	4·24	2·80	1·74	1·22	1·27	1·50	1·64	1·66	1·62	1·73	2·23	3·19	4·55	6·24
	0·5	6·30	4·53	3·07	1·98	1·36	1·26	1·40	1·50	1·50	1·46	1·62	2·20	3·22	4·62	6·35
	0	6·38	4·63	3·19	2·12	1·50	1·33	1·40	1·45	1·40	1·33	1·50	2·12	3·19	4·63	6·38
$\varkappa_3 = 0$	$\lvert\varkappa_2\rvert = 1\cdot0$	5·54	3·89	2·56	1·61	1·21	1·32	1·54	1·66	1·66	1·60	1·67	2·14	3·05	4·34	5·97
	0·5	5·91	4·22	2·85	1·84	1·32	1·29	1·45	1·55	1·53	1·46	1·56	2·09	3·06	4·40	6·07
	0	6·05	4·37	3·00	1·99	1·44	1·35	1·44	1·50	1·44	1·35	1·44	1·99	3·00	4·37	6·05
$\varkappa_3 = -0\cdot25$	$\lvert\varkappa_2\rvert = 1\cdot0$	5·11	3·57	2·34	1·50	1·22	1·36	1·58	1·68	1·66	1·58	1·63	2·04	2·90	4·14	5·69
	0·5	5·52	3·92	2·63	1·71	1·30	1·33	1·50	1·59	1·56	1·47	1·52	1·98	2·89	4·18	5·77
	0	5·73	4·11	2·81	1·87	1·41	1·37	1·49	1·55	1·49	1·37	1·41	1·87	2·81	4·11	5·73
$\varkappa_3 = -0\cdot50$	$\lvert\varkappa_2\rvert = 1\cdot0$	4·71	3·27	2·14	1·42	1·24	1·41	1·61	1·71	1·67	1·58	1·59	1·95	2·76	3·93	5·41
	0·5	5·15	3·64	2·43	1·61	1·29	1·38	1·55	1·63	1·59	1·48	1·49	1·88	2·73	3·95	5·48
	0	5·40	3·86	2·62	1·76	1·38	1·40	1·54	1·61	1·54	1·40	1·38	1·76	2·62	3·86	5·40
$\varkappa_3 = -0\cdot75$	$\lvert\varkappa_2\rvert = 1\cdot0$	4·33	2·99	1·96	1·35	1·26	1·46	1·65	1·73	1·69	1·58	1·56	1·87	2·62	3·72	5·13
	0·5	4·80	3·37	2·24	1·52	1·29	1·42	1·60	1·68	1·63	1·50	1·46	1·79	2·57	3·72	5·18
	0	5·07	3·61	2·45	1·67	1·37	1·44	1·59	1·66	1·59	1·44	1·37	1·67	2·45	3·61	5·07
$\varkappa_3 = -1\cdot00$	$\lvert\varkappa_2\rvert = 1\cdot0$	3·99	2·74	1·81	1·31	1·30	1·51	1·69	1·76	1·70	1·58	1·53	1·79	2·48	3·52	4·86
	0·5	4·45	3·11	2·07	1·45	1·31	1·47	1·65	1·72	1·66	1·52	1·45	1·70	2·41	3·50	4·88
	0	4·75	3·36	2·28	1·58	1·37	1·48	1·64	1·70	1·64	1·48	1·37	1·58	2·28	3·36	4·75

		ρ_{30}														
		$k\varkappa_2$ negative								$k\varkappa_2$ positive						
	$\lvert k \rvert$	1·4	1·2	1·0	0·8	0·6	0·4	0·2	0	0·2	0·4	0·6	0·8	1·0	1·2	1·4
$\varkappa_3=+1\cdot00$	$\lvert\varkappa_2\rvert=1\cdot0$	19·7	11·8	6·27	2·81	1·38	1·41	1·50	1·61	1·95	2·32	2·45	2·58	4·06	7·83	14·0
	0·5	18·1	10·8	5·74	2·75	1·62	1·48	1·37	1·39	1·73	2·07	2·18	2·55	4·61	8·90	15·5
	0	16·9	9·91	5·20	2·62	1·89	1·76	1·49	1·30	1·49	1·76	1·89	2·62	5·20	9·91	16·9
$\varkappa_3=+0\cdot75$	$\lvert\varkappa_2\rvert=1\cdot0$	18·5	11·0	5·71	2·49	1·35	1·50	1·56	1·62	1·91	2·26	2·38	2·48	3·89	7·54	13·5
	0·5	17·3	10·2	5·37	2·52	1·56	1·52	1·43	1·42	1·71	2·01	2·10	2·43	4·39	8·51	14·9
	0	16·2	9·42	4·91	2·46	1·82	1·74	1·51	1·35	1·51	1·74	1·82	2·46	4·91	9·42	16·2
$\varkappa_3=+0\cdot50$	$\lvert\varkappa_2\rvert=1\cdot0$	17·3	10·1	5·14	2·17	1·36	1·59	1·62	1·63	1·88	2·20	2·31	2·39	3·72	7·23	13·0
	0·5	16·4	9·63	4·98	2·30	1·53	1·57	1·49	1·46	1·70	1·97	2·04	2·31	4·17	8·12	14·2
	0	15·4	8·94	4·61	2·31	1·77	1·74	1·54	1·39	1·54	1·74	1·77	2·31	4·61	8·94	15·4
$\varkappa_3=+0\cdot25$	$\lvert\varkappa_2\rvert=1\cdot0$	15·9	9·19	4·55	1·89	1·42	1·69	1·68	1·64	1·85	2·15	2·25	2·30	3·55	6·92	12·4
	0·5	15·5	9·02	4·58	2·09	1·53	1·63	1·56	1·50	1·70	1·95	1·99	2·20	3·94	7·74	13·6
	0	14·7	8·46	4·31	2·15	1·74	1·75	1·58	1·45	1·58	1·75	1·74	2·15	4·31	8·46	14·7
$\varkappa_3=0$	$\lvert\varkappa_2\rvert=1\cdot0$	14·6	8·26	3·97	1·66	1·50	1·77	1·73	1·66	1·84	2·12	2·20	2·21	3·37	6·60	11·9
	0·5	14·6	8·37	4·15	1·89	1·56	1·71	1·62	1·55	1·71	1·94	1·96	2·09	3·70	7·34	13·0
	0	14·0	7·97	4·00	2·01	1·73	1·78	1·62	1·50	1·62	1·78	1·73	2·01	4·00	7·97	14·0
$\varkappa_3=-0\cdot25$	$\lvert\varkappa_2\rvert=1\cdot0$	13·2	7·32	3·41	1·51	1·60	1·85	1·78	1·68	1·84	2·09	2·15	2·13	3·19	6·26	11·3
	0·5	13·6	7·70	3·73	1·73	1·61	1·78	1·69	1·59	1·74	1·94	1·94	1·99	3·46	6·94	12·4
	0	13·2	7·47	3·68	1·88	1·74	1·82	1·67	1·55	1·67	1·82	1·74	1·88	3·68	7·47	13·2
$\varkappa_3=-0\cdot50$	$\lvert\varkappa_2\rvert=1\cdot0$	11·8	6·41	2·89	1·43	1·71	1·92	1·81	1·71	1·85	2·08	2·12	2·06	3·01	5·90	10·7
	0·5	12·6	7·00	3·30	1·61	1·69	1·86	1·74	1·64	1·76	1·96	1·94	1·91	3·22	6·52	11·8
	0	12·5	6·94	3·36	1·77	1·78	1·87	1·72	1·61	1·72	1·87	1·78	1·77	3·36	6·94	12·5
$\varkappa_3=-0\cdot75$	$\lvert\varkappa_2\rvert=1\cdot0$	10·4	5·55	2·44	1·43	1·80	1·97	1·84	1·73	1·86	2·07	2·09	1·99	2·82	5·54	10·1
	0·5	11·5	6·29	2·89	1·54	1·77	1·93	1·80	1·68	1·80	1·98	1·95	1·84	2·97	6·09	11·1
	0	11·6	6·39	3·03	1·69	1·82	1·93	1·77	1·66	1·77	1·93	1·82	1·69	3·03	6·39	11·6
$\varkappa_3=-1\cdot00$	$\lvert\varkappa_2\rvert=1\cdot0$	9·15	4·75	2·06	1·48	1·88	2·00	1·86	1·76	1·88	2·07	2·08	1·94	2·64	5·17	9·46
	0·5	10·5	5·57	2·50	1·53	1·86	1·99	1·84	1·72	1·83	2·01	1·97	1·79	2·72	5·63	10·4
	0	10·8	5·83	2·71	1·65	1·88	1·98	1·82	1·70	1·82	1·98	1·88	1·65	2·71	5·83	10·8

VI. Illustrative Examples

(a) *Exact Calculations*

(i) *Example* 1

This example was used in a previous paper (Guest 1952). There are 16 observations, at the following values of x:

$$+7\cdot9 \quad +7\cdot2 \quad +6\cdot25 \quad +6\cdot1 \quad +4\cdot8 \quad +3\cdot5 \quad +2\cdot8 \quad +0\cdot8$$
$$-1\cdot25 \quad -2\cdot2 \quad -3\cdot65 \quad -4\cdot4 \quad -5\cdot5 \quad -6\cdot4 \quad -7\cdot3 \quad -9\cdot0.$$

The parameters are

$$\varphi=0\cdot95576, \quad \varkappa_2=-0\cdot1227, \quad \varkappa_3=-0\cdot6415.$$

For the coefficients μ_{jj} interpolation in Table 4 gives

$$\mu_{11}=1\cdot282\ (1\cdot287), \quad \mu_{22}=1\cdot227\ (1\cdot229), \quad \mu_{33}=1\cdot204\ (1\cdot585).$$

The true values, obtained from the calculation of $\varphi^{2j}\Sigma T_j^{\,2}(x)/\Sigma T_j^{\,2}(\varepsilon)$, are shown in brackets.

Interpolation in Table 4 gives for the coefficients μ_{jk}

$$\mu_{21}=+0\cdot0207, \quad \mu_{32}=+0\cdot0417, \quad \mu_{33}=-0\cdot0643.$$

Application of formulae (**37**) to (**40**), with $e=0\cdot11947x+0\cdot0026$, leads to the functions ρ_{20}, ρ_{30} tabulated below, the true values being shown in brackets.

x	ρ_{20}	ρ_{30}
−10	3·52 (3·55)	5·8 (5·2)
− 7·5	1·96 (1·99)	2·0 (2·0)
− 5	1·38 (1·38)	1·87 (1·73)
− 2·5	1·53 (1·54)	1·83 (1·69)
0	1·64 (1·65)	1·64 (1·68)
+ 2·5	1·48 (1·48)	1·86 (1·89)
+ 5	1·36 (1·36)	1·74 (1·68)
+ 7·5	2·12 (2·11)	2·3 (2·4)
+10	3·84 (3·84)	7·2 (7·1)

(ii) *Example* 2

The 67 observations are those of Jaeger and von Steinwehr (1921), obtained in their determination of the mechanical equivalent of heat. Proceeding as in Example 1, the following values are obtained.

x:	+29·60	+28·36	+26·96	+25·79	+25·56	+24·34	+23·09	+19·41
	+17·19	+16·64	+15·79	+15·75	+14·39	+14·32	+13·98	+12·54
	+11·49	+10·60	+ 9·19	+ 9·15	+ 7·76	+ 6·33	+ 6·24	+ 5·55
	+ 5·36	+ 4·82	+ 4·11	+ 3·96	+ 3·24	+ 2·53	+ 1·80	+ 1·41
	+ 1·15	+ 1·13	+ 0·01	− 0·05	− 0·25	− 1·17	− 1·18	− 1·45
	− 1·58	− 1·58	− 2·59	− 2·67	− 3·61	− 3·69	− 5·01	− 5·12
	− 6·00	− 6·03	− 6·97	− 7·13	− 7·40	− 7·41	− 7·94	− 8·39
	− 8·55	− 8·67	− 9·98	−11·05	−11·19	−11·49	−12·59	−13·62
	−13·85	−14·35	−15·25.					

$$\varphi = 1\cdot 43378, \quad \varkappa_2 = 0\cdot 5871, \quad \varkappa_3 = 0\cdot 7403.$$

$$\mu_{11} = 0\cdot 759\ (0\cdot 760), \quad \mu_{22} = 0\cdot 803\ (0\cdot 810), \quad \mu_{33} = 0\cdot 820\ (0\cdot 770).$$

$$\mu_{21} = -0\cdot 1391, \quad \mu_{32} = -0\cdot 2250, \quad \mu_{31} = -0\cdot 0899.$$

$$e = 0\cdot 042799x - 0\cdot 129.$$

x	ρ_{20}	ρ_{30}
−20	4·27 (4·31)	7·6 (7·8)
−15	2·77 (2·77)	3·55 (3·61)
−10	1·70 (1·69)	1·70 (1·69)
− 5	1·22 (1·22)	1·49 (1·51)
0	1·28 (1·27)	1·46 (1·47)
+ 5	1·43 (1·43)	1·43 (1·43)
+10	1·49 (1·49)	1·71 (1·73)
+15	1·51 (1·51)	2·05 (2·10)
+20	1·79 (1·78)	2·15 (2·18)
+25	2·53 (2·52)	2·53 (2·53)
+30	3·76 (3·77)	4·8 (4·9)

(iii) *Example* 3

This is an example used by Kendall (1948).

x: +101 +87 +76 +62 +53 +41 +29 +13
+ 3 − 6 −18 −29 −35 −40 −45 −50.

$$\varphi = 0\cdot 103080, \quad \varkappa_2 = +0\cdot 4706, \quad \varkappa_3 = -0\cdot 3920.$$

$$\mu_{11} = 1\cdot 181\ (1\cdot 181), \quad \mu_{22} = 1\cdot 145\ (1\cdot 155), \quad \mu_{33} = 1\cdot 130\ (1\cdot 232).$$

$$\mu_{21} = -0\cdot 0844, \quad \mu_{32} = -0\cdot 1664, \quad \mu_{31} = -0\cdot 0718.$$

$$e = 0\cdot 012885x - 0\cdot 195.$$

x	ρ_{20}	ρ_{30}
− 60	2·89 (2·89)	4·51 (4·52)
− 40	1·67 (1·68)	1·68 (1·68)
− 20	1·28 (1·28)	1·73 (1·73)
0	1·45 (1·46)	1·79 (1·80)
+ 20	1·61 (1·61)	1·62 (1·62)
+ 40	1·56 (1·56)	1·78 (1·74)
+ 60	1·45 (1·44)	1·97 (1·93)
+ 80	1·75 (1·75)	1·87 (1·87)
+100	2·77 (2·81)	3·28 (3·30)
+120	4·45 (4·49)	8·0 (7·9)

(*b*) *Approximate Calculations*

In practical examples it would require too much time to undertake a least squares determination of the parameters φ, $\varkappa_2$, $\varkappa_3$. The approximate scheme described in Section V (*b*) based on formulae (**11**), (**12**), and (**13**), will now be applied to the three examples treated in Section VI (*a*).

(i) *Example* 1

$$x_1 = +7\cdot 9,\ x_3 = -0\cdot 225,\ x_2 = +5\cdot 1,$$
$$x_5 = -9\cdot 0,\qquad x_4 = -4\cdot 7.$$

$$\psi_2 = 2\{x_1 + x_5 - 2x_3\} = -1\cdot 30,$$
$$\psi_3 = \frac{8}{3}\{(x_1 - x_5) - 2(x_2 - x_4)\} = -7\cdot 2,$$
$$\psi_1 = (x_1 - x_5) + \psi_3/(n-1) = 16\cdot 42.$$
$$\varphi = (n-1)/\psi_1 = 0\cdot 914\ (0\cdot 956),$$
$$\varkappa_2 = \{1 + 1/n\}\{\psi_2/\psi_1\} = -0\cdot 084\ (-0\cdot 123),$$
$$\varkappa_3 = \{1 + 2/n\}\{\psi_3/\psi_1\} = -0\cdot 499\ (-0\cdot 641).$$

Interpolation in Table 4 gives

$$\mu_{11} = 1\cdot 22,\ \mu_{22} = 1\cdot 17,\ \mu_{33} = 1\cdot 15,$$

and hence the values $\Sigma T_j^{\,2}(x) = \varphi^{-2j}\mu_{jj}\Sigma T_j^{\,2}(\varepsilon)$ are $\Sigma T_1^{\,2}(x) = 497$ (479), $\Sigma T_2^{\,2}(x) = 958 \times 10$ (841), $\Sigma T_3^{\,2}(x) = 179 \times 10^3$ (189).

On interpolating in Table 6, with $\varkappa_2 = 0$ and $\varkappa_3 = -0\cdot 5$, and using (**41**), the following values are obtained.

$$k = 0\cdot 1142x + 0\cdot 019.$$

x	k	ρ_{20}	ρ_{30}
−10	−1·12	3·4 (3·6)	5·5 (5·2)
− 7·5	−0·84	1·93 (1·99)	2·1 (2·0)
− 5	−0·55	1·38 (1·38)	1·80 (1·73)
− 2·5	−0·27	1·49 (1·54)	1·77 (1·69)
0	+0·02	1·60 (1·65)	1·62 (1·68)
+ 2·5	+0·30	1·47 (1·48)	1·80 (1·89)
+ 5	+0·59	1·38 (1·36)	1·78 (1·68)
+ 7·5	+0·88	2·10 (2·11)	2·4 (2·4)
+10	+1·16	3·6 (3·8)	6·2 (7·1)

(ii) *Example* 2

$$x_1 = +29\cdot 60,\ x_5 = -15\cdot 25,\ x_3 = +1\cdot 13,\ x_2 = +11\cdot 05,\ x_4 = -6\cdot 50.$$
$$\psi_2 = +24\cdot 18,\ \psi_3 = +26\cdot 00,\ \psi_1 = +45\cdot 24.$$
$$\varphi = 1\cdot 459\ (1\cdot 434),\ \varkappa_2 = +0\cdot 543\ (0\cdot 587),\ \varkappa_3 = +0\cdot 592\ (0\cdot 740).$$
$$\mu_{11} = 0\cdot 80,\ \mu_{22} = 0\cdot 84,\ \mu_{33} = 0\cdot 85.$$
$$\Sigma T_1^{\,2}(x) = 94 \times 10^2\ (93),\ \Sigma T_2^{\,2}(x) = 139 \times 10^4\ (144),\ \Sigma T_3^{\,2}(x) = 191 \times 10^6\ (191).$$
$$k = 0\cdot 0436x - 0\cdot 240.$$

x	k	ρ_{20}	ρ_{30}
−20	−1·11	4·2 (4·3)	7·5 (7·8)
−15	−0·89	2·66 (2·77)	3·5 (3·6)
−10	−0·68	1·70 (1·69)	1·84 (1·69)
− 5	−0·46	1·28 (1·22)	1·56 (1·51)
0	−0·24	1·32 (1·27)	1·51 (1·47)
+ 5	−0·02	1·45 (1·43)	1·46 (1·43)
+10	+0·20	1·47 (1·49)	1·70 (1·73)
+15	+0·41	1·48 (1·51)	1·97 (2·10)
+20	+0·63	1·77 (1·78)	2·08 (2·18)
+25	+0·85	2·58 (2·52)	2·8 (2·5)
+30	+1·07	3·9 (3·8)	5·6 (4·9)

(iii) *Example* 3

$$x_1 = +101,\ x_5 = -50,\ x_3 = +8,\ x_2 = +55,\ x_4 = -30\cdot 5.$$

$$\psi_2 = +70,\ \psi_3 = -53\cdot 3,\ \psi_1 = 147\cdot 45.$$

$$\varphi = 0\cdot 1017\ (0\cdot 1031),\ \varkappa_2 = +0\cdot 506\ (0\cdot 471),\ \varkappa_3 = -0\cdot 411\ (0\cdot 392).$$

$$\mu_{11} = 1\cdot 19,\ \mu_{22} = 1\cdot 15,\ \mu_{33} = 1\cdot 14.$$

$$\Sigma T_1{}^2(x) = 391 \times 10^2\ (378),\ \Sigma T_2{}^2(x) = 614 \times 10^5\ (585),\ \Sigma T_3{}^2(x) = 93 \times 10^9\ (93).$$

$$k = 0\cdot 01271x - 0\cdot 293.$$

x	k	ρ_{20}	ρ_{30}
− 60	−1·06	2·8 (2·9)	4·4 (4·5)
− 40	−0·80	1·61 (1·68)	1·61 (1·68)
− 20	−0·55	1·31 (1·28)	1·73 (1·73)
0	−0·29	1·47 (1·46)	1·79 (1·80)
+ 20	−0·04	1·61 (1·61)	1·66 (1·62)
+ 40	+0·22	1·58 (1·56)	1·78 (1·74)
+ 60	+0·47	1·48 (1·44)	1·95 (1·93)
+ 80	+0·72	1·72 (1·75)	1·92 (1·87)
+100	+0·98	2·64 (2·81)	3·1 (3·3)
+120	+1·23	4·2 (4·5)	7·3 (7·9)

VII. Deviations of the Errors from the Exact Least Squares Values

It is necessary to examine whether the smoothed set of points $X(\varepsilon)$ will be an adequate substitute for the actual set $x(\varepsilon)$ in the calculation of the standard errors. The departures $x(\varepsilon) - X(\varepsilon)$ may be represented by additional terms $k_4T_4(\varepsilon) + k_5T_5(\varepsilon)$. Thus to obtain a closer approximation to the observed points the expression $\xi(\varepsilon)$ can be written in the form

$$\xi(\varepsilon) = \xi_3(\varepsilon) + 4\varkappa_4\tau_4(\varepsilon) + 8\varkappa_5\tau_5(\varepsilon), \quad \ldots\ldots\ldots\ldots \quad (\mathbf{44})$$

where $\xi_3(\varepsilon)$ is the expression given in equation (**4**).

If the function $\xi(\varepsilon)$ given by (**44**) is to represent the points of observation, it must satisfy the condition $\Delta\xi(\varepsilon) \geqslant 0$. Thus it is required to find, for a given $\xi_3(\varepsilon)$, the ranges of the parameters $\varkappa_4$, $\varkappa_5$ for which this condition is satisfied. The ranges of these parameters and the effect they have on μ_{jj} will depend to some extent on the value of n. However, since no great accuracy is required, it is simplest to take some definite value of n, say 50, and make use of the tabulated values of the orthogonal polynomials for this number of observations. A table of finite difference values for the polynomials can then be drawn up and used to estimate the allowable ranges of $\varkappa_4$, $\varkappa_5$.

Table 7 gives the finite difference values for $n = 50$. As an illustration of the method of procedure, the calculation of the maximum positive value of $\varkappa_5$ when $\varkappa_2 = \varkappa_3 = 0$ will be outlined. The finite difference is

$$\Delta\xi(\varepsilon) = 1 + \varkappa_4\{4\Delta\tau_4(\varepsilon)\} + \varkappa_5\{8\Delta\tau_5(\varepsilon)\} \geqslant 0,$$

and so

$$\varkappa_5\{8\Delta\tau_5(\varepsilon)\} \geqslant -1 - \varkappa_4\{4\Delta\tau_4(\varepsilon)\},$$

and, since $\Delta\tau_5(\varepsilon)$ is an even function of ε, $\Delta\tau_4(\varepsilon)$ is an odd function of ε,

$$\varkappa_5\{8\Delta\tau_5(\varepsilon)\} \geqslant -1 + |\ \varkappa_4\{4\Delta\tau_4(\varepsilon)\}\ |.$$

Now, from Table 7, the minimum value of $8\Delta\tau_5(\varepsilon)$ s $-0\cdot156$. Hence the maximum positive value of $\varkappa_5$ occurs when $\varkappa_4=0$, and has the value $1/0\cdot156=6\cdot4$.

Examination of the table of finite differences leads in a similar manner to the following conclusions :

(*a*) When $\xi_3(\varepsilon)=\varepsilon$ (i.e. when $\varkappa_2=\varkappa_3=0$), values of $\varkappa_5$ from $-1\cdot4$ to $+6\cdot4$ can occur. The maximum value of $|\varkappa_4|$ depends on the value of $\varkappa_5$, being zero at the extreme values of $\varkappa_5$ and reaching a maximum of 3 when $\varkappa_5=+3$.

(*b*) The only major effect of the parameter $\varkappa_3$ is to permit larger negative values of $\varkappa_5$ (maximum negative $\varkappa_5=-3\cdot0$ when $\varkappa_3=+1\cdot3$).

TABLE 7

FINITE DIFFERENCES AND VALUES OF μ'_{jj} FOR $n=50$

Finite Differences

ε	$\Delta\tau_2$	$2\Delta\tau_3-\frac{1}{5}\Delta\tau_1$	$4\Delta\tau_4$	$8\Delta\tau_5$
$23\frac{1}{2}$	0·96	0·883	0·948	0·711
$22\frac{1}{2}$	0·92	0·770	0·771	0·504
$21\frac{1}{2}$	0·88	0·662	0·611	0·332
$20\frac{1}{2}$	0·84	0·559	0·467	0·192
$19\frac{1}{2}$	0·80	0·461	0·340	0·079
$18\frac{1}{2}$	0·76	0·367	0·228	—0·006
$17\frac{1}{2}$	0·72	0·278	0·131	—0·071
$16\frac{1}{2}$	0·68	0·194	0·048	—0·115
$15\frac{1}{2}$	0·64	0·115	—0·023	—0·142
$14\frac{1}{2}$	0·60	0·040	—0·081	—0·156
$13\frac{1}{2}$	0·56	—0·029	—0·127	—0·156
$12\frac{1}{2}$	0·52	—0·094	—0·163	—0·148
$11\frac{1}{2}$	0·48	—0·154	—0·190	—0·132
$10\frac{1}{2}$	0·44	—0·209	—0·205	—0·110
$9\frac{1}{2}$	0·40	—0·260	—0·214	—0·083
$8\frac{1}{2}$	0·36	—0·305	—0·215	—0·055
$7\frac{1}{2}$	0·32	—0·346	—0·208	—0·026
$6\frac{1}{2}$	0·28	—0·382	—0·195	0·004
$5\frac{1}{2}$	0·24	—0·413	—0·178	0·030
$4\frac{1}{2}$	0·20	—0·439	—0·155	0·056
$3\frac{1}{2}$	0·16	—0·461	—0·128	0·077
$2\frac{1}{2}$	0·12	—0·478	—0·099	0·095
$1\frac{1}{2}$	0·08	—0·490	—0·068	0·108
$\frac{1}{2}$	0·04	—0·497	—0·034	0·115
$-\frac{1}{2}$	0	—0·500	0	0·118
$-1\frac{1}{2}$	—0·04	—0·497	0·034	0·115

TABLE 7 (*Continued*)

Effect of Parameters $\varkappa_4$, $\varkappa_5$

(a) $\varkappa_2=\varkappa_3=0$

$\varkappa_5$ \ $\varkappa_4$	μ'_{22}: 0	μ'_{22}: 1·0	μ'_{22}: 2·4	μ'_{33}: 0	μ'_{33}: 1·0	μ'_{33}: 2·4
—1·4	1·02	—	—	0·81	—	—
0	1·00	1·03	—	1·00	1·05	—
1	1·01	1·05	—	1·24	1·25	—
4	1·16	1·20	1·40	2·50	2·39	2·15
6·4	1·42	—	—	4·27	—	—

(b) $\varkappa_3\neq0$

$\varkappa_3$	$\varkappa_5$	μ'_{22}	μ'_{33}
—1	+6	1·07	2·91
+1	+6	1·69	4·38
+1·3	—3	0·88	0·45

(c) $\varkappa_2\neq0$

$\varkappa_2$	$\varkappa_4$	μ'_{22}	μ'_{33}
—1·2	+2·2	0·76	0·84

(d) Higher terms included

$\xi_3=\varepsilon$	μ'_{22}	μ'_{33}
$\varkappa_5=+7\cdot7$ (+term in $\tau_9(\varepsilon)$)	1·71	7·05
$\varkappa_4=+2\cdot3$ (+term in $\tau_6(\varepsilon)$)	1·14	0·99

(*c*) The effect of the parameter $\varkappa_2$ is to allow slightly higher values of $\varkappa_4$ when $\varkappa_5$ is small (maximum $\varkappa_4=+2\cdot2$ when $\varkappa_2=-1\cdot2$).

(*d*) Somewhat higher values of $\varkappa_4$, $\varkappa_5$ are possible if higher order terms are added to equation (**44**). The maximum value of $\varkappa_5$ is unaffected by the addition of a term $\tau_7(\varepsilon)$, and is increased to $+7\cdot7$ by the addition of a term $\tau_9(\varepsilon)$. The maximum value of $|\,\varkappa_4\,|$ is not greatly altered by the addition of terms $\tau_6(\varepsilon)$ and $\tau_8(\varepsilon)$.

The quantities μ'_{jj}, obtained by calculating the values $\mu_{jj}=\Sigma T_j^{\,2}(x)/\Sigma T_j^{\,2}(\varepsilon)$ and dividing by the tabulated values μ_{jj} for the same $\varkappa_2$, $\varkappa_3$, provide a measure of the deviations of the standard errors from the exact least squares values. Values of μ'_{jj} for various values of the parameters are listed in Table 7.

The general conclusions to be drawn from this table concerning the probable deviations of the standard errors from the exact least squares values might be set out as follows :

	2nd Degree Polynomial	3rd Degree Polynomial
(i) Likely deviations ..	3%	12%
(ii) Possible deviations ..	15%	50%
(iii) Improbable deviations	30%	Factor of 2

It should be realized that high values of $\varkappa_5$ correspond to a very wide spacing at the ends of the range and a very close spacing near $\varepsilon = \pm\frac{1}{4}(n-1)$, and so very high values of $\varkappa_5$ are not likely to occur in practical examples.

TABLE 8

VALUES OF μ_{jj} FOR SMALL n

	n	9	16	25	∞
(a) $\varkappa_3 = +1$	μ_{11}	0·652	0·656	0·657	0·657
	μ_{22}	0·714	0·761	0·772	0·781
	μ_{33}	0·537	0·675	0·713	0·745
(b) $\varkappa_3 = -1$	n	9	16	25	∞
	μ_{11}	1·458	1·457	1·457	1·457
	μ_{22}	1·444	1·408	1·401	1·394
	μ_{33}	1·564	1·414	1·386	1·361

The parameters $\varkappa_4$, $\varkappa_5$ will also produce errors in the rough approximations to φ, $\varkappa_2$, $\varkappa_3$, obtained by the use of equations (**11**) to (**13**). The approximation to $\varkappa_2$ is found to be increased by $0 \cdot 10\varkappa_4$, and the approximation to φ to be reduced by $0 \cdot 05\varkappa_5$. The approximation to $\varkappa_3$ is increased by an amount varying from $0 \cdot 05\varkappa_5$ to $0 \cdot 15\varkappa_5$, the larger amount corresponding to the cases in which $\varkappa_3$ is negative.

When $\varkappa_4$, $\varkappa_5$ are small, the effect they have on μ_{22}, μ_{33} can be estimated from the leading terms in these parameters. These terms are found to be :

(i) for μ_{22}, $0 \cdot 0496\varkappa_4^2$;

(ii) for μ_{33}, $0 \cdot 156\varkappa_4^2 + 0 \cdot 192\varkappa_5$.

For the examples discussed in Section VI, the parameters and the corrections to μ_{33} estimated from the leading terms are set out below.

(i) Example 1

$$\varkappa_4 = -0 \cdot 522, \quad \varkappa_5 = +1 \cdot 802.$$

$$\mu_{33} = 1 \cdot 204 + 0 \cdot 390 = 1 \cdot 59 \text{ (exact } 1 \cdot 58).$$

(ii) Example 2

$$\varkappa_4 = +0 \cdot 061, \ \varkappa_5 = -0 \cdot 094.$$

$$\mu_{33} = 0 \cdot 820 - 0 \cdot 017 = 0 \cdot 80 \ \text{(exact } 0 \cdot 77).$$

(iii) Example 3

$$\varkappa_4 = +0 \cdot 107, \ \varkappa_5 = +0 \cdot 606.$$

$$\mu_{33} = 1 \cdot 130 + 0 \cdot 119 = 1 \cdot 25 \ \text{(exact } 1 \cdot 23).$$

Only in Example 1, where the parameter $\varkappa_5$ is rather large, is the correction to μ_{33} important.

The effect produced by the neglect of terms j^2/n^2 in the calculation of the quantities μ_{jj} also requires consideration. Evaluation of the true values μ_{jj} for small values of n leads to the figures shown in Table 8. From this table it is seen that the dependence of μ_{jj} on n is small, except for the quantity μ_{33} when n is less than 16.

VIII. Conclusion

There are two separate questions which require some discussion. Firstly, whether the standard errors obtained from the exactly calculated values φ, $\varkappa_2$, $\varkappa_3$ are in good agreement with the true least squares values. Secondly, whether the standard errors obtained by rough estimation of the parameters are of use in practical examples.

As regards the first point, it seems clear that the standard errors of a_1, a_2 will differ from the calculated least squares values by at most a few per cent. In the examples given in Section VI the difference is less than 1 per cent. in each case. For the coefficient a_3 differences of up to 10 per cent. will commonly occur, and larger differences may occur in certain cases. In the examples given here the differences are 13, 3, and 4 per cent. For the fitted values when the polynomial is of the second degree the agreement should always be to within a few per cent.—in the examples given the differences are always less than 2 per cent. When the polynomial is of the third degree, the percentage differences will be less than the difference for the coefficient a_3 in the region of interpolation ($|k| < 1$), and of the same order as for this coefficient in the region of extrapolation ($|k| > 1$). For the examples of Section VI the maximum deviations are 11, 3, and 2 per cent.

As was shown in Section VII, the principal cause of these divergences is the neglect of the higher order parameters $\varkappa_4$ and $\varkappa_5$. It does not seem feasible, however, to include the effects of these parameters in a treatment of the kind given here. The necessity for calculating four parameters and for tabulating the various quantities in terms of them would make the treatment so time-consuming and involved that it would be of little practical use. Consequently it is felt that the present treatment will provide a picture of the behaviour of the standard errors of quantities occurring in curve fitting by the method of least squares which is sufficiently detailed to serve as a basis for the calculations of the efficiencies of other methods of fitting.

Concerning the second point mentioned in the opening paragraph of this section, the principal use of the methods described here would be in the estimation of the standard errors of the fitted values. The exact calculation of these errors for a number of points is somewhat tiresome, since the orthogonal polynomials have to be determined for each point. For this reason Table 6 should prove of use in practical examples where estimates of the standard errors of the fitted values are required. It should be remembered, however, that the estimates cannot be relied on when a third degree polynomial is fitted to a series of observations which are very widely spaced at the extremes of the range and very closely spaced near $\varepsilon = \pm\frac{1}{4}(n-1)$.

For the examples given in Section VI the differences from the exact values are less than 10 per cent. for $|k|<0\cdot8$, and less than 15 per cent. for $|k|>0\cdot8$. The larger deviations for $|k|>0\cdot8$ are due principally to the error in the estimated value of φ.

IX. References

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