NOTE ON THE FACTORIAL MOMENTS OF STANDARD DISTRIBUTIONS*

By R. B. Potts†

The factorial moments of the standard discontinuous distributions can be readily derived by means of a generating function, and it is well known that the formulae for these moments are much simpler than those for the more usual power moments. It is the purpose of the present note to draw attention to an interesting alternative method of deriving these factorial moments.

If $\varphi(x)$ is the distribution function of a discrete variate x which is capable of only integral values, then the rth factorial moment about the origin is defined to be

$$T(r)=\sum_{j=0}^{\infty}\frac{(r+j)!}{j!}\varphi(r+j) \quad\quad (1)$$

(the summation may be finite). The formula inverse to (1) is

$$\varphi(x)=\sum_{j=0}^{\infty}\frac{(-1)^j}{x!j!}T(x+j), \quad\quad (2)$$

which expresses the distribution function $\varphi(x)$ in terms of the xth and higher factorial moments. It is this relation which will be used in deriving the factorial moments of the standard discontinuous distributions; there is, incidentally, no relation as simple as (2) which expresses the distribution function in terms of the power moments.

Binomial Distribution

For the binomial distribution,

$$\varphi(x)=\binom{n}{x}p^x(1-p)^{n-x}, \quad\quad (3)$$

which gives, by expanding $(1-p)^{n-x}$,

$$\varphi(x)=\sum_{j=0}^{n-x}\frac{(-1)^j}{x!j!}\frac{n!}{(n-x-j)!}p^{x+j}. \quad\quad (4)$$

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† Department of Mathematics, University of Adelaide.

By comparison with (2), the factorial moments of the binomial distribution are given by

$$T(r)=\frac{n!}{(n-r)!}p^r. \qquad (5)$$

Poisson Distribution

For the Poisson distribution,

$$\varphi(x)=\frac{m^x e^{-m}}{x!}=\sum_{j=0}^{\infty}\frac{(-1)^j}{x!j!}m^{x+j}, \qquad (6)$$

by expanding the exponential. Hence, from (2), the factorial moments are

$$T(r)=m^r. \qquad (7)$$

Polya Distribution

For this distribution,

$$\varphi(x)=(1+b)(1+2b)\ldots.\{1+(x-1)b\}a^x(1+ab)^{-x-b^{-1}}/x!, \qquad (8)$$

from which one obtains

$$\varphi(x)=\sum_{j=0}^{\infty}\frac{(-1)}{x!j!}(1+b)(1+2b)\ldots.\{1+(x+j-1)b\}a^{x+j}. \qquad (9)$$

Hence, from (2), the factorial moments of the Polya distribution are

$$T(r)=(1+b)(1+2b)\ldots.\{1+(r-1)b\}a^r. \qquad (10)$$

Hypergeometric Distribution

The distribution function is, in the usual notation,

$$\varphi(x)=\binom{Np}{x}\binom{N-Np}{n-x}\bigg/\binom{N}{n}. \qquad (11)$$

From the identity

$$\binom{N-Np}{n-x}=\sum_{j=0}^{n-x}(-1)^j\binom{Np-x}{j}\binom{N-x-j}{n-x-j}, \qquad (12)$$

it follows that

$$\varphi(x)=\sum_{j=0}^{n-x}\frac{(-1)^j}{x!j!}\frac{(Np)!}{(Np-x-j)!}\frac{n!}{(n-x-j)!}\frac{(N-x-j)!}{N!}, \qquad (13)$$

and therefore, from (2),

$$T(r)=\frac{(Np)!}{(Np-r)!}\frac{n!}{(n-r)!}\frac{(N-r)!}{N!}. \qquad (14)$$

Recent work on number distributions occurring in cosmic ray cascade theory has revealed the especial importance of the factorial moments, and such relations as (2), for discontinuous distributions which are more complex than the simple standard ones considered above.