THE MULTIPLE SCATTERING OF PROTONS IN NUCLEAR EMULSIONS

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Summary

The multiple scattering theories of Williams and Molière have been adapted to give the r.m.s. lateral deflection of protons which lose all their energy in nuclear emulsions. Measurements of 1–5 MeV proton tracks show significant differences from the former theory at low energies and from the latter at higher energies. The introduction of alternative expressions for the minimum angle due to screening does not give a satisfactory explanation of the observed results. It is found, however, that the experimental r.m.s. deflections display the same dependence on maximum single scattering angle as is calculated.

I. Introduction

During the last few years considerable prominence has been given to the multiple scattering theory of Molière (1948). Some authors have obtained results which appear to indicate that Molière's theory is in better agreement with experiment than, for example, the older theory of Williams (1939). At the present time, however, the issue is somewhat confused. This would seem to be largely due to the inherent uncertainties in the experimental measurements and to variations in procedure adopted for the comparison of theory with experiment.

The two main assumptions which underlie the development of multiple scattering theories are, firstly, that the energy of the particles has not changed over the path in the scatterer and, secondly, that the single scattering angles which contribute to the calculated multiple scattering distribution are small enough for sin Θ to be replaced by Θ.

In all experiments performed to date the use of thin scattering foils has been necessary to reduce energy loss but because of this the scattering parameter will be small and there will be considerable statistical uncertainty in the measurements. It seemed desirable, therefore, to find some way of overcoming the first assumption of small energy loss.

The second assumption means that if one wishes to consider particles over an appreciable part of their range there is a limitation to heavy particles. This follows since electrons would be scattered through such large angles that the small angle approximation would not be applicable. Not only this, but for larger angles the penetration depth in the scatterer would not be a satisfactory representation of the path length, as it is taken to be in all the current theories.

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Simple calculation shows that for protons with energies of the order of 5 MeV, which lose the whole of their energy in photographic emulsions, conditions will be such that (with certain reservations) the small angle approximation will be valid up to the end of the range of the particles. This is not true for tracks which exhibit a sudden change in direction attributable to a single scattering event. It is found that the value obtained for the mean of the multiple scattering distribution depends on the upper limit of single scattering angles included in the determination of the mean. This indicates that to obtain a satisfactory basis for comparison the same range of angles must be included in both experimental and theoretical work.

A final consideration is that measurement of length is inherently more accurate than measurement of angle. For this reason it was felt that if the theoretical distribution of the lateral deflection for protons at the end of their range could be obtained, taking account of energy loss and adopting a satisfactory procedure for dealing with the effects of the single scatters, an accurate comparison would be possible with the experimentally observed results. An account of such a comparison is given in the present paper.

II. Calculations

For protons the spin dependent term in the Mott formula for single elastic scattering is negligible if the energies are less than 10 MeV. If the small angle approximation holds then sin Θ may be replaced by Θ and this expression may be written

$$\xi(\Theta)\mathrm{d}\omega=4N\left(\frac{Z^2}{A}\right)r_0^2\frac{m^2}{p^2\beta^2}\frac{\mathrm{d}\omega}{\Theta^4}, \qquad (1)$$

where N is Avogadro's number, r_0 the classical electronic radius, m the mass of the electron, Z and A are respectively the atomic number and the atomic weight of the scattering material, and for low energy non-relativistic protons $p^2\beta^2$ may be expressed in terms of the particle energy E by $p^2\beta^2=4E^2$.

Following Williams, upper and lower limits of Θ are introduced by making use of the concepts of finite size of the nucleus and screening of the nuclear field by the atomic electrons. The expressions obtained by Williams for these limits are

$$\Theta_{\mathrm{min.}}=\frac{\lambda Z^{1/3}}{2\pi(137)^2 r_0}, \qquad \Theta_{\mathrm{max.}}=\frac{\lambda}{2\pi\times 0\cdot 57 r_0 Z^{1/3}}, \qquad (2)$$

in which λ is the de Broglie wavelength of the incident particles.

Making use of the above limits and of the fact that the single scattering cross section (1) is effectively zero for angles outside these limits, the mean square angle for multiple scattering is given by Rossi and Greisen (1941):

$$\langle\Theta^2\rangle_{\mathrm{d}x}=\mathrm{d}x\int_{\Theta_{\mathrm{min.}}}^{\Theta_{\mathrm{max.}}}\Theta^2\xi(\Theta)2\pi\Theta\mathrm{d}\Theta$$

$$=\mathrm{d}x 16\pi N\frac{Z^2}{A}r_0^2\frac{m^2}{p^2\beta^2}\ln(181Z^{-1/3}). \qquad (3)$$

This result may be considerably simplified by introducing a new unit of length, the so-called radiation unit, which eliminates most of the constants in formula (3). The radiation unit is defined as

$$\frac{1}{X_0}=\frac{4}{137}N\left(\frac{Z^2}{A}\right)r_0^2 \ln (181Z^{-1/3}). \qquad (4)$$

Introducing (4) into (3) the mean square angle of multiple scattering over a distance dt radiation units is

$$\langle\Theta^2\rangle_{\mathrm{d}t}=(E_s^2/4E^2)\mathrm{d}t, \qquad (5)$$

in which other constants have been combined into the factor E_s^2 in which E_s has the dimensions of energy. Equation (5) is the well-known Rossi-Greisen relation for the mean square angle. Over a finite range of t for which the energy may not be regarded as constant the mean square angle will be

$$\langle\Theta^2\rangle_t=(E_s^2/4)\int_{E_2}^{E_1}\frac{1}{-(\mathrm{d}E/\mathrm{d}t)}\frac{\mathrm{d}E}{E^2}, \qquad (6)$$

where E_1 and E_2 are the energies at 0 and t respectively.

In the calculation of the radiation unit for photographic emulsion it must be remembered that the definition (4) represents the constants in the scattering formula and, in view of the fact that this describes individual processes, a fallacious result will be obtained if mean values of Z and A for the particular compound are substituted into (4). The value obtained in this way for the radiation unit in the C2 emulsion is of the order of 3×10^4 μ. Using the correct method and working out a mean value for the quantity $(\ln 181/Z^{\frac{1}{3}})Z^2/A$ for substitution into (4) a value of $7\cdot09\times10^4$ μ is obtained.

Equation (6) makes it possible to determine the value of the mean square angle for multiple scattering when the energy of the particles does not remain constant over the path in the scattering material. In order to obtain an explicit value some form of range-energy relation must be inserted into the formula. The usual empirical relation used in emulsion work is

$$E=\gamma R^{\delta}, \qquad (7)$$

from which by differentiation there results

$$\frac{\mathrm{d}E}{\mathrm{d}R}=\alpha E^{\beta}, \qquad (8)$$

where the constants have the values $\gamma=0\cdot262$, $\delta=0\cdot575$, $\alpha=0\cdot0558$, and $\beta=-0\cdot74$.

If the effect of the multiple scattering on the range is neglected, then, denoting the residual range in radiation units by s and the total range by S, it is clear that

$$t'+s=S,$$

where t' is the penetration depth in the scatterer. If n is the number of microns per radiation unit, (7) becomes

$$E=\gamma\{n(S-t')\}^{\delta}. \qquad (7a)$$

Using these results (**6**) may be integrated to yield

$$\Theta_{\text{r.m.s.}}=\sqrt{\langle\Theta^2\rangle}=\frac{E_s}{2}\,\frac{(\gamma n^{\delta})^{-(1+\beta)/2}S^{-\delta(1+\beta)/2}}{\{(1+\beta)n\alpha\}^{\frac{1}{2}}}\left[\left(1-\frac{t'}{S}\right)^{-\delta(1+\beta)}-1\right]^{\frac{1}{2}}. \quad \text{.. (9)}$$

A formula having the same form as this equation has been derived by Wilson (1947) but is not as satisfactory for two reasons. Firstly, the empirical range-energy relation is more accurate for energies above 2 MeV than the approximation to the Bethe-Bloch formula used by Wilson and, secondly, this author, in his derivation, has made an approximation which, according to his paper, introduces an error of about 5 per cent.

The relation (**7**) is satisfactory above 3 MeV, but for lower energies the experimental range-energy results cannot be fitted using a constant γ. For this reason (**9**) is valid only for particles with energy high enough so that the part of the range during which the energy falls from 2 MeV to zero is small compared with the total range. If this condition is not satisfied then the integration of (**6**) must be done numerically using empirical range-energy results.

As has been stated in the introduction, comparison with the experimental values is possible only if, instead of taking a theoretical upper limit for single scattering angles, values are chosen to conform with the experimental procedure. For this reason the lateral deflection has been calculated using upper limits of 5, 10, 15, 20, and 25°. Since it is more convenient to measure projected quantities rather than space quantities, use is made of the fact that the mean square projected angle is half the mean square space angle.

Under these conditions equation (**5**) becomes

$$\langle\theta^2\rangle_{dt}=\frac{dtE_s^2}{16E^2}\cdot\frac{\ln(\varphi\times10^4\sqrt{E}/2\cdot03)}{\ln 181Z^{-1/3}}, \quad \text{........ (10)}$$

where θ is now a projected angle and φ takes on the arbitrary values given above. From this result the root mean square value of the lateral deflection from the original direction is readily obtained

$$y_{\text{r.m.s.}}=\int_0^t \tan\,\theta_{\text{r.m.s.}}dt'. \quad \text{................ (11)}$$

The required value of $y_{\text{r.m.s.}}$ is found by inserting the root mean square value of θ from (**10**) into (**11**). For protons of initial energy 10 MeV or less the values of $y_{\text{r.m.s.}}$ must be obtained by numerical integration.

In the evaluation of (**11**) successively greater values of φ are introduced and from these results a curve of root mean square lateral deflection versus upper limit of single scattering angle may be plotted. Curves of this kind for four values of initial energy are given in Figure 4.

The values of the lateral deflection are found from the Molière theory using the formula for the mean multiple scattering angle after particles have traversed a distance σ g/cm² of scattering material:

$$\theta_m=\frac{22\cdot9Z}{2\sqrt{\pi}E}\sqrt{\frac{\sigma}{A}}\{B^{1/2}+0\cdot982B^{-1/2}-0\cdot117B^{-3/2}\}, \quad \text{...... (12)}$$

where the parameter B is related to a second parameter Ω_b which is given by

$$\log_{10}\Omega_b = 8\cdot 215 + \log_{10}\left[Z^{-2/3}\left(\frac{\sigma}{A}\right)\alpha^2/(1\cdot 13+3\cdot 76\alpha^2)\right]. \quad \ldots\ (13)$$

The relation between B and Ω_b is

$$\Omega_b = 1\cdot 167 e^B/B, \quad \ldots\ (14)$$

and

$$\alpha = 2\cdot 095/\sqrt{E}$$

for protons in photographic emulsion.

Since the energy is a function of σ these formulae will only be true over an infinitesimal range of σ.

For finite σ we will have

$$\log_{10}\Omega_b = 8\cdot 215 + \log_{10}\int_0^{\sigma}\frac{0\cdot 0269 d\sigma'}{1\cdot 13E+16\cdot 5}, \quad \ldots\ (15)$$

in which the constants for the particular scattering substance have been inserted. By integration of (**15**) and the use of (**14**) the values of B corresponding to a series of values of σ from the beginning to near the end of the range of the protons are obtained. It is then possible to use the relation

$$\theta_m^2 = \int_0^{\sigma}\frac{(15\cdot 87)^2}{E^2}\{B^{1/2}+0\cdot 982B^{-1/2}-0\cdot 117B^{-3/2}\}^2 d\sigma' \quad \ldots\ (16)$$

to find values of θ_m along the path.

TABLE 1

THEORETICAL AND EXPERIMENTAL VALUES OF LATERAL DEFLECTION

Energy (MeV)	4·87	3·91	2·66	1·35
R.m.s. deflection (15° limit) (μ)				
Experiment	11·0±0·2	8·0±0·3	4·38±0·06	1·70±0·03
Rossi-Greisen B $\theta_{min.}$ eqn. (**17**)	11·16	7·96	4·80	2·07
Rossi-Greisen C $\theta_{min.}$ eqn. (**2**)	10·78	7·71	4·60	1·96
Rossi-Greisen E $\theta_{min.}$ eqn. (**18**)	10·39	7·41	4·37	1·73
Mean deflection (all tracks) (μ)				
Experiment	8·9±0·3	6·7±0·3	3·5±0·1	1·34±0·05
Molière	8·19	5·92	3·38	1·40

Now a defect of the Molière theory is that it is impossible to find a mean square scattering angle. In fact the general expression for the kth moment of the distribution becomes infinite for even values of k. It is thus only possible to find mean values of the various quantities using Molière's results, and a comparison with experiment is only possible after expressing the experimental results as means.

Owing to the fact that the Molière expression for the kth moment is obtained by integrating over all angles from 0 to ∞, and that it is only because of this choice of limits that the complicated expression for the general moment reduces to the simple form (**12**), the lateral deflection has not been worked out in this case for a series of upper limits on the single scattering angle considered.

Values of the mean lateral deflection are found using equation (**11**) with $\theta_{\text{r.m.s.}}$ replaced by θ_m of (**16**). Results for the four different proton energies are plotted to give the curve in Figure 5.

All the calculated results have been collected in Table 1.

III. Experimental Procedure

Tracks of monoenergetic protons were obtained by exposing Ilford C2 emulsions to protons from the reactions $^{9}Be(d,p)^{10}Be$, $^{6}Li(d,p)^{7}Li$, and $^{12}C(d,p)^{13}C$. The exposures were made in a camera which has been described by Martin *et al.*

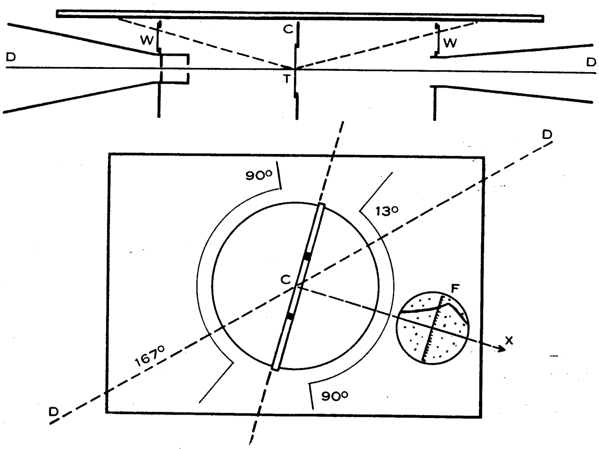

Fig. 1.—Photographic plate showing location in camera and method of predicting initial direction of tracks. *DD*, deuteron beam ; *T*, target spot ; *W*, " Cellophane " windows ; *F*, microscope field of view.

(1949). It contains a photographic plate parallel to the incident deuteron beam and 1 cm above it (Fig. 1). A copper ring supports the plate and the target holder and has " Cellophane " windows which stop the scattered deuterons but allow the more energetic reaction products to reach the outer parts of the plate. Those particles which reach the plate at a distance of 4 cm from the point C vertically above the target spot strike the emulsion at an angle of 13° to the plane of the surface. At various positions on one plate, therefore, a range of angles from 13 to 167° relative to the incident deuteron direction is obtainable. This, together with the use of one or more thicknesses of " Cellophane ", provides a range of proton energies from 1 to 5 MeV.

The reactions used were chosen since they have accurately known Q-values and characteristic energy spectra which allow a group of protons with a particular energy to be distinguished without measuring the length of each track. The use of thin targets and suitable control of the bombarding energy ensures that the reaction products will have a small spread in energy. The length of exposure was chosen in each case to give a high density of tracks in the emulsion. The limiting condition on the track density was provided by the requirement that tracks should not interfere with one another. With a high density the required number of tracks can be obtained in a small area of plate over which the change in energy due to change in angle relative to the direction of the deuteron beam will be small. For the same reason the diameter of the deuteron beam is restricted by diaphragms to 0·020 in.

In order to test the dependence of the width of the scattering distributions on experimental conditions, emulsions from various batches and of various thicknesses were used. They were vacuum dried to constant weight prior to exposure and maintained under vacuum during exposure. A constant value of 3·94 g/c.c. was then assumed for the emulsion density in the calculation of the

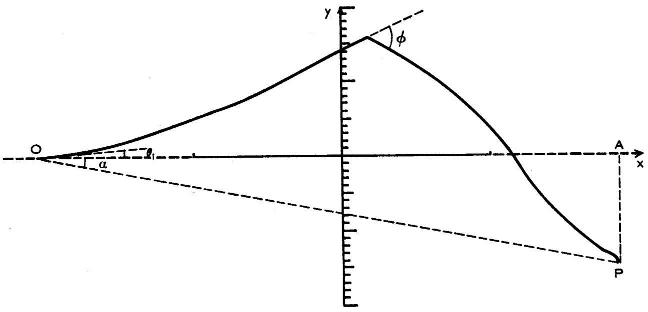

Fig. 2.—Schematic representation of a track and eyepiece graticule. θ_i, initial angle; φ, single scattering angle; α, average angle, x, predicted initial direction; y, lateral deflection.

combined effect of the component elements on the scattering distribution. The processing of the emulsions was varied to some extent, but as far as possible care was taken to prevent distortion and reticulation.

From a knowledge of the relative positions of the deuteron beam, target spot, and photographic plate one may predict the initial direction of tracks which appear on any part of the plate. During exposure the plate rests on the upper edge of the target holder and this is placed accurately at 45° to the direction of the incident beam. The holder prevents scattered deuterons from blackening the plate except where two V-slots occur in the top of the holder at known distances from the point C (Fig. 1). In this way the position of the slots is registered on the plate. After processing, therefore, the position of the point C and the direction of the target holder can be ascertained.

As a check on the geometry, a mask was constructed to fit in the top of the camera, and marked with the various positions and directions. During loading of the camera a brief exposure to light produces an image of the markings on the surface of the emulsion nearest the glass backing.

The measurements were made with a microscope fitted with a polar stage on which the plates were mounted with the target direction along the 45° line and with the point C over the centre of rotation. The stage can be moved along a line joining the centre of the objective and the centre of the stage. Since the tracks appear to originate at a point above the centre of the stage, their initial direction will coincide with the direction of linear motion, no matter what part of the plate is studied. Location of the plate was checked in each case by observing the behaviour of the image of the mask as the stage moved. In all cases the angular settings of the stage agreed with those of the mask to within $\frac{1}{2}°$. This is consistent with the accuracy possible in the construction of the mask.

In order to measure the lateral deflection of each track from its initial direction use was made of an eyepiece graticule which has a scale and its perpendicular bisector engraved on it (Fig. 2). The orientation of the eyepiece was such that the perpendicular to the scale lay in the direction of linear motion of the stage and was then assumed to indicate the correct initial direction for all tracks. The displacement of the end of each track from this direction may be determined by moving the track so that first the beginning and then the end passes across the scale, and taking the difference between the readings for each position. Each reading was made to the nearest half division, so that the lateral deflection was accurate to the nearest division. The magnification is about 1500 for all tracks, since with the procedure adopted it does not matter if the tracks are more than one field of view in length. With this magnification, one scale division corresponds to less than 1 μ and is of the order of magnitude of the average grain size, so that no gain in accuracy could be achieved by trying to make measurements more accurately than this.

The actual initial direction of each track, θ_i, could be checked by observing the scale reading at a certain distance from the beginning of the track, chosen so that a deflection of half a division would correspond to an angle of 1°. In this way a rough estimate of the angle was made, allowing for any curvature in this first section of the track. A check of this kind is necessary since an appreciable number of protons are scattered in the " Cellophane " and therefore all particles do not arrive at the emulsion with the same initial direction.

A second eyepiece scale was used in conjunction with an eyepiece protractor to measure the approximate magnitude of any single scattering events which occurred. Reasonable accuracy could be obtained for large angles but, for angles less than 10°, not only was the accuracy of measurement poor but also the chance of detecting such events became increasingly less. No attempt was made to measure angles less than 5° and these were only included when obvious.

A total of about 8000 tracks was measured and the characteristics of each tabulated. If any sign of distortion appeared no tracks were measured in that neighbourhood. Measurements were made at the four energies selected for comparison with the calculations. These values were obtained by measuring the range of about 300 tracks at each energy, and then using the range-energy relation for C2 emulsions. The values found checked satisfactorily with those calculated from the energetics of the particular reaction used and the energy lost in traversing the " Cellophane ".

IV. Results

At each proton energy histograms are plotted for the lateral deflections of those tracks which have a correct initial angle, a separate histogram being plotted for each group of tracks having a particular range of magnitude of single scattering events. A typical histogram is shown in Figure 3 and compared with a Gaussian distribution which has the same standard deviation. The characteristic features of a scattering distribution are indicated by the increased number of tracks at the centre and extremes, while at the intermediate deflections

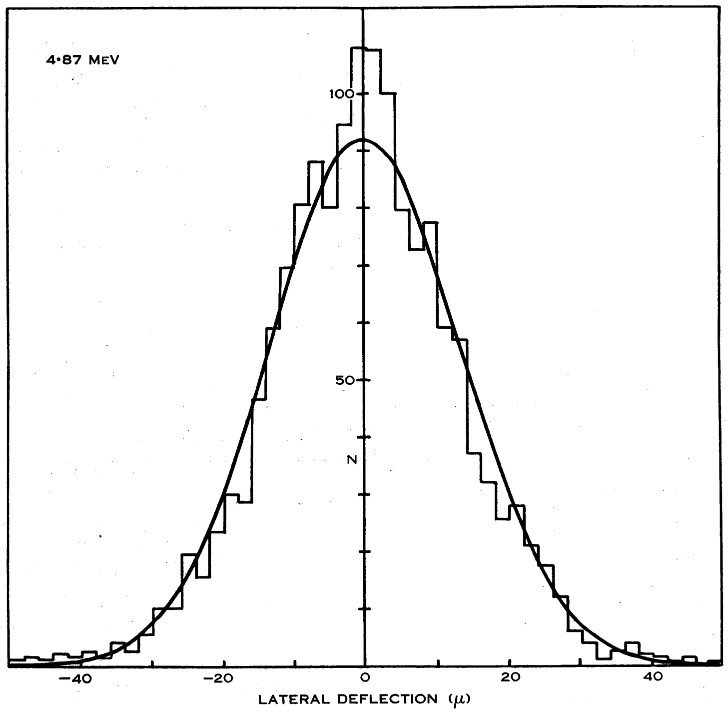

Fig. 3.—Typical histogram of number of tracks *v.* lateral deflection. Smooth curve is the Gaussian with the same r.m.s. deflection.

there are less than would be found for a Gaussian distribution. The ratio of standard deviation to mean for the experimental curves is found to be larger than the value $\sqrt{\pi/2}$ for a Gaussian distribution, the increase being about 3 per cent. at the higher energies investigated, and nearly 20 per cent. at the lowest energy.

The increase in standard deviation at each energy, as increasing single scattering angles are included, is shown in Figure 4, together with the probable errors calculated from the number of tracks associated with each measurement. A direct comparison can be made between these results and the calculations by the Rossi-Greisen method, but not with the mean values which are obtained from the theory cf Molière. The Molière values indicated in Figure 5 are obtained using the observed values of the ratio of the standard deviation to the mean.

The experimental results at the two higher energies are both a combination of two separate measurements, the difference between them being less than the probable errors for each. The results at 3·91 MeV have been analysed in two different ways in order to check the measurements of initial angle. The tracks

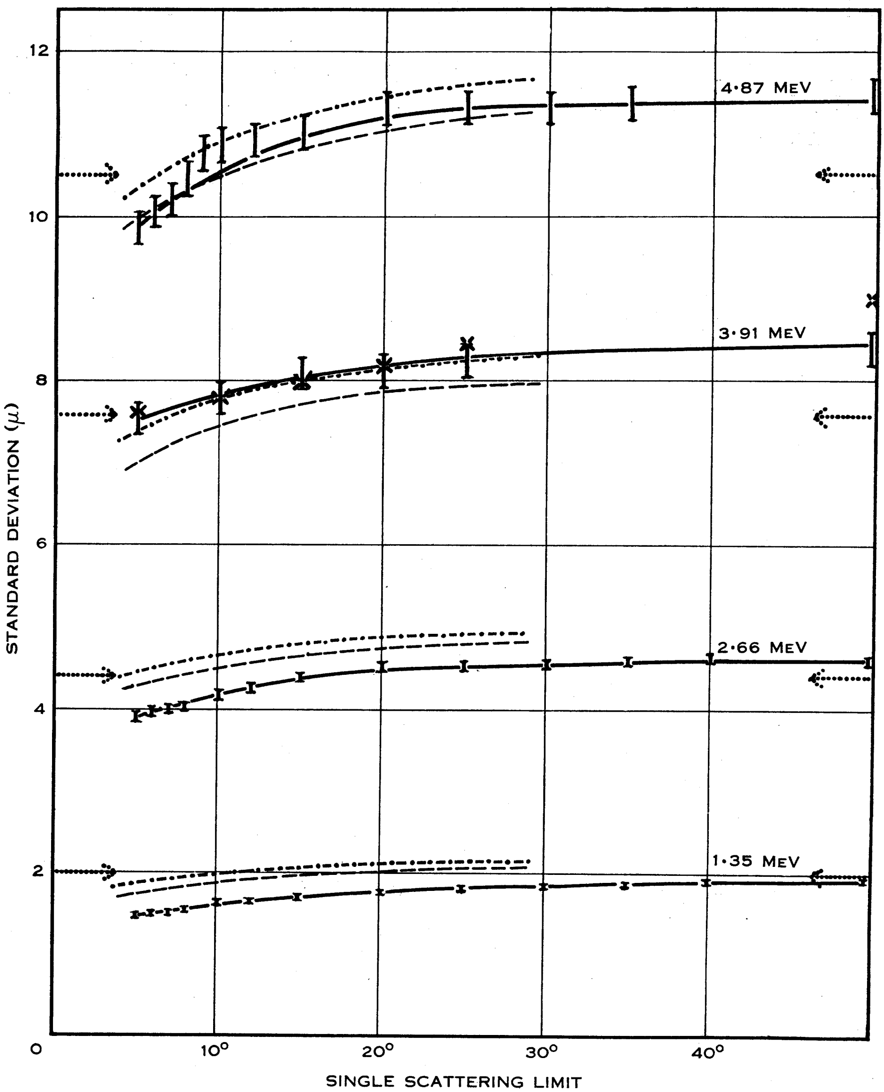

Fig. 4.—Variation of r.m.s. deflection with single scattering limit. Full curves, experimental values ; dashed curves, Rossi-Greisen with $\theta_{\mathrm{min.}}$ from equation (**17**) ; dot and dash curves, Rossi-Greisen with $\theta_{\mathrm{min.}}$ from equation (**2**) ; arrows, Molière values.

are divided into groups according to their estimated initial angle and each group analysed separately. The observed centre of each group agrees with that expected from the initial angle, and the standard deviations calculated about these centres are also consistent. The results obtained by combining standard deviations for all initial angles are plotted in Figure 4 for comparison with the zero-angle results.

The centres of the distributions for zero-angle tracks should be zero, and in all cases the average deflection when divided by the mean range indicates an initial angle of less than $\frac{1}{2}°$. This verifies that the plates have been located during the experiment to this degree of accuracy. Each non-zero average deflection has been used to give a small correction to the standard deviations which should therefore be free from the effect of errors in the geometry of the experiment.

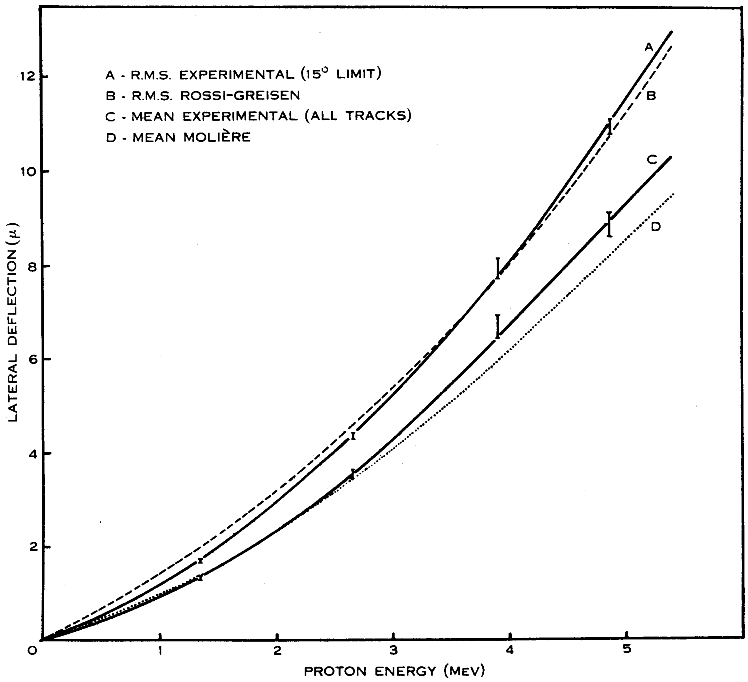

Fig. 5.—Dependence of mean and r.m.s. deflection on proton energy.

The effect of loss of tracks by scattering out of the emulsion has been investigated empirically, since an accurate calculation would be difficult. The number of tracks lost depends on the distance of the point A (Fig. 2) from both surfaces of the emulsion and can be obtained by direct observation as well as from the number of tracks having a sufficiently large lateral deflection. The distribution in radial deflection was assumed to be approximately Gaussian with a variance twice that observed for the projected lateral deflection. The tracks having a small range of radial deflection were assumed to be distributed uniformly over an annular strip which may be divided into a number of segments. The projected deflection was plotted for the tracks in each segment, the centre of which lay outside the emulsion, and then summed for all annular strips contributing to the total distribution. The variance obtained in this way for the lost tracks can be used as a correction to the variance of the observed distribution in lateral deflection. The correction was found to be ½ per cent. for the one plate for which the loss of tracks was considerable, and negligible for the others.

At the lower energies grouping of the tracks into one-division intervals has an appreciable effect on the standard deviation and so Sheppard's correction has been applied (Kendall 1947). The final values for a 15° single scattering limit are included in Table 1, together with the values of the mean deflection obtained by including all events. The dependence of these values on proton energy is illustrated in Figure 5, in which the probable errors used are a combination of the statistical uncertainties and the uncertainty in proton energy.

V. Discussion

The use of arbitrary values of $\theta_{\text{max.}}$ in the calculation of multiple scattering by the Rossi-Greisen method gives results which vary in the same way as the measurements, using the same $\theta_{\text{max.}}$ values (dot and dashed curves in Fig. 4). At small angles the experimental values fall a little faster but this may be due to systematic errors in the detection and measurement of these angles. The two low energy curves are seen to be appreciably higher than the experimental ones.

Since the form of the single scattering expression in equation (**1**) should be reliable for the protons used, we need to investigate the form of $\theta_{\text{min.}}$ used in the equation of type (**3**) for projected mean square angle. The dot and dash curves in Figure 4 have been obtained using the value for the projected $\theta_{\text{min.}}$ given in (**2**). This angle was calculated by Williams on the basis of a simple consideration of the screening of the nuclear field by the atomic electrons. The value found is a direct consequence of the use of the Born approximation in the derivation of the single scattering formula.

On the basis of a more refined consideration of the screening effect Williams has obtained the result

$$\theta_{\text{min.}}=\frac{1\cdot 75\lambda Z^{1/3}}{2\pi(137)^2 r_0}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\mathbf{17})$$

In this estimate account is taken of the fact that the cut-off due to screening is not abrupt but is spread over a range of angles. This effective screening angle has been obtained using the calculations of Bullard and Massey (1930) for the Thomas-Fermi field of the atomic electrons.

The dashed curves of Figure 4 have been calculated using this value for $\theta_{\text{min.}}$. They are seen to agree with the experimental results more satisfactorily than the other curves, although the variation with $\theta_{\text{max.}}$ remains unaltered. A comparison (Fig. 5) of the results for a particular value of $\theta_{\text{max.}}$ and using $\theta_{\text{min.}}$ from (**17**) shows reasonable agreement with experiment above $3\cdot 5$ MeV, but at lower energies the calculated results are still considerably higher.

A comparison of the experimental results with the theory of Molière, however, indicates the opposite effect (curve *D*). There is reasonable agreement at low energies, but the calculations are low at the higher energies. Both these discrepancies are outside the experimental errors and indicate that neither approach to the multiple scattering of protons can adequately explain the observed results.

This is indicated more clearly in Figure 6, in which the average angle α (Fig. 2) is plotted against energy. In this figure curves B and C represent the results for $\theta_{\text{min.}}$ given by (**2**) and (**17**) respectively.

Curve E has been obtained by working out the lateral deflection again, using the Rossi-Greisen method and the Molière value for $\theta_{\text{min.}}$:

$$\theta_{\text{min.}}=\frac{1\cdot 142mZ^{1/3}}{137M\beta}\left\{1\cdot 13+\frac{3\cdot 76Z^2}{(137)^2\beta^2}\right\}^{1/2}, \quad \ldots\ldots \quad (\mathbf{18})$$

where M is the mass of the proton.

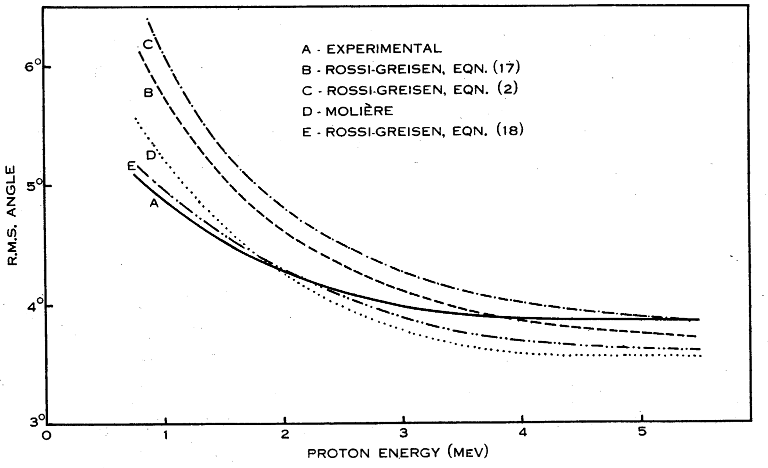

Fig. 6.—Dependence of average angle on energy.

It is interesting to note that this curve lies fairly close to curve D found from the Molière theory itself. The slope of these two curves is almost the same although curve E is closer to the experimental curve.

The characteristic screening angle does not occur in the Molière theory in the form of a lower limit but is included in the approximate single scattering expression. The dependence of the Molière values on screening angle is thus quite complicated and it is of interest that the values obtained in this way are less satisfactory than those represented by curve E. This may be due to the approximations which Molière has been obliged to make.

The shape of the curves in Figure 6 is very similar to those presented by Groetzinger, Berger, and Ribe (1950) and Groetzinger, Humphrey, and Ribe (1952) for the multiple scattering of electrons with energies up to 2 MeV. In both cases the theoretical results are too high at the lowest energies, and decrease too rapidly with increasing energy.

Using 115 MeV electrons Corson (1950) claims better agreement with the theory of Snyder and Scott (1949) than with the Rossi-Greisen theory. Corson uses a value of $2\cdot 92$ cm for the radiation unit in Ilford G5 emulsions and obtains a value of $0\cdot 28°/100\,\mu$ for the mean multiple scattering deflection using the

Rossi-Greisen theory. The value of the radiation unit for the emulsion obtained using the correct averaging procedure is 7·41 cm, and gives a mean deflection of $0\cdot18°/100\ \mu$. Using the Snyder-Scott theory which does not involve the use of radiation units he finds $0\cdot20°/100\ \mu$. In contrast to Corson's conclusion, therefore, his experimental value of $(0\cdot17° \pm 0\cdot02)/100\ \mu$ is in better agreement with the Rossi-Greisen result than with that obtained from the more complicated theories of Molière or Snyder and Scott.

Molière's theory, however, is presented in such a way that it is necessary to choose with care the method of evaluation for any particular case. This is illustrated by the work of Spencer and Blanchard (1954). These authors have shown that the Molière distribution may be improved by the introduction of a more accurate single scattering expression for the large angles which fits the multiple scattering distribution for small angles better than the Molière large angle formula. With this modification they obtain a curve for the multiple scattering of 15·7 MeV electrons which differs from the observed shape obtained by Hanson *et al.* (1951). This is in contrast with the claim made by Hanson *et al.* that their results were in good agreement with Molière's.

Other experiments on the scattering of electrons and protons do not show any systematic agreement with either theory, although where the theory of Williams gives wider distributions than the Molière theory the experimental values are usually lower than the former. The differences are seldom found to be greater than 10 to 15 per cent. and often are of the same magnitude as the experimental errors.

VI. Conclusions

Using conditions under which a simple single scattering law is valid, the calculation of multiple scattering by the Rossi-Greisen method gives a satisfactory dependence on the maximum scattering angle. It is found, however, that none of the usual expressions for the minimum angle gives an energy dependence which will explain the observed results.

Use of the more complicated Molière expression for the screening angle in the Rossi-Greisen calculations gives a better fit than the $\sqrt{E}$ dependence of the Williams expressions but it still has too strong a dependence on energy. Further measurements are required to investigate these effects for particles of higher energy.

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Appendix I

Range Straggling due to Multiple Scattering

A procedure similar to that in Section II may be adopted to obtain an estimate of the effect of multiple scattering on the straggling of protons in the energy range considered. Because of multiple scattering some particles follow more devious tracks than others and the perpendicular penetration depth in the scattering material will not be the same for all. In actual fact the measured range will not be the penetration depth (OA, Fig. 2), but the distance between the point of entry of the particle into the emulsion and the point where the track ends (OP).

The path length corresponding to the root mean square angle will be

$$L=\int_0^t \frac{\mathrm{d}t'}{\cos \Theta_{\mathrm{r.m.s.}}}. \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (19)$$

If the distance t is small enough so that energy loss may be neglected one may use equation (5) to obtain the result

$$L=t-\left(\frac{E_s}{2E}\right)^2\frac{t^2}{2\times 2!}+5\left(\frac{E_s}{2E}\right)^4\frac{t^3}{3\times 4!}+\ldots \qquad \ldots\ldots \quad (20)$$

For most applications, however, this is not a satisfactory approximation and one must use instead the relation (6) to obtain

$$L=\int_0^t \cos^{-1}\left[\frac{E_s}{2}\sqrt{\left\{\int_{E_2}^{E_1}\frac{1}{-(\mathrm{d}E/\mathrm{d}t)}\frac{\mathrm{d}E}{E^2}\right\}}\right]\mathrm{d}t'. \qquad \ldots\ldots \quad (21)$$

This expression has been evaluated for a series of values of initial proton energy. The values of L so found may be regarded as a measure of the mean value of the path length for all particles. The difference between this and the range of the particles will be a convenient measure of the magnitude of the multiple scattering effect on the straggling. Thus it is required to work out $L-S$, where S is the OP of Figure 2. Values of L have been expressed as a percentage of the range and it is found that for protons with energy of the order of 10 MeV the effect is about $\frac{1}{2}$ per cent. For protons with energy $0 \cdot 5$ MeV the value has increased only to $1 \cdot 18$ per cent. In view of the fact that the observed straggling for protons with these energies at the end of their range is about 6 per cent., it is seen that the major contribution to the total straggling must come from the variations in ionization energy loss.