CORRECTING FOR GAUSSIAN AERIAL SMOOTHING

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Summary

Let a two-dimensional survey with a Gaussian aerial beam establish values at intervals of $\sqrt{2}$ standard deviations. Then the correction for aerial smoothing is simply calculated as the difference between the value to be corrected and the mean of the neighbouring four values.

I. Introduction

An important part of the reduction of radio-astronomical observations consists in correcting for the blurring which arises from the finite extent of the aerial beam. Methods for doing this in the one-dimensional case have been discussed by Bracewell and Roberts (1954), and in principle, and indeed in practice, the methods may be extended to two dimensions. It was shown, however, that some of the detail is irretrievably lost; and, as there is considerable labour involved in calculating the correction, which is in any case only partial, there has been a tendency in recent publications of two-dimensional observations to omit any correction for aerial smoothing (e.g. McGee and Bolton 1954).

There appears therefore to be scope for new methods of correction involving procedures less elaborate than the calculation of two-dimensional convolutions or two-dimensional Fourier transforms. A sacrifice of accuracy would seem reasonable in order to obtain this simplicity, especially where the correction to be applied is small. A simple, approximate method of correction would be valuable for quickly seeing the effect of the correction and for deciding whether the labour of calculating a small correction would be worth while.

In some current researches, the labour involved in applying the established methods is already prohibitive as a result of the vast increase in the mass of observational data yielded by the high-resolution aerials employed. The Mills radiometer (Mills and Little 1953) is a case in point. An attempt has therefore been made to find a simple, approximate two-dimensional method of correction for a Gaussian aerial beam as used in that case.

The approach to the problem was to list simple numerical operations on a two-dimensional array of data (graphical or tabular) and to explore the possibilities of each in turn. The operation of taking finite differences is the one which has proved fruitful. It is applicable to aerial beams other than Gaussian.

As a by-product, the present theory provides, in the one-dimensional case, a sound alternative to a formula given by Eddington (1913) for correcting the blurring effect of a Gaussian distribution of errors. Eddington's series, which

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is widely known and used in astronomy (though not apparently in radio astronomy), has proved valuable in practice, but suffers from being theoretically doubtful (Jeffreys 1938) and from the fact that in actual use the series of derivatives is evaluated as a series of finite differences.

II. Theory

Some preliminary notes are needed on the theory of finite differences in relation to Fourier theory. If from a given function another is derived by taking finite differences, then its spectrum is related to that of the given function in a way which will now be shown ; first for functions of one variable, and then for functions of two variables.

Let

$$^{\alpha}\Delta f(x) \equiv f(x+\tfrac{1}{2}\alpha) - f(x-\tfrac{1}{2}\alpha),$$

i.e. $^{\alpha}\Delta f(x)$ represents the result of taking the difference between two values of $f(x)$, at values of x separated by an interval α.

Now the Fourier transform of $f(x+x_0)$ is, by the shift theorem, $\exp(2\pi i x_0 s)\bar{f}(s)$, where $\bar{f}(s)$ is the Fourier transform of $f(x)$. Hence

$$\begin{aligned}\text{F.T. of } {}^{\alpha}\Delta f(x) &= e^{\pi i \alpha s}\bar{f}(s) - e^{-\pi i \alpha s}\bar{f}(s)\\ &= 2i \sin(\pi\alpha s)\bar{f}(s).\end{aligned}$$

Thus, *the effect of differencing a function of one variable at interval* α *is to multiply the spectrum by* $2i \sin \pi\alpha s$.

In the two-dimensional case let $f(x, y)$ have a two-dimensional Fourier transform $\bar{f}(s_x, s_y)$. For differences at interval α in the x-direction we write

$$^{\alpha}\Delta_x f(x, y) = f(x+\tfrac{1}{2}\alpha, y) - f(x-\tfrac{1}{2}\alpha, y),$$

and for differences at interval β in the y-direction

$$^{\beta}\Delta_y f(x, y) = f(x, y+\tfrac{1}{2}\beta) - f(x, y-\tfrac{1}{2}\beta).$$

As an example of the notation for differences of higher order,

$$\begin{aligned}^{\alpha\alpha}\Delta_{xx} f(x, y) &= f(x+\tfrac{1}{2}\alpha, y) - 2f(x, y) + f(x-\tfrac{1}{2}\alpha, y),\\ ^{\alpha\beta}\Delta_{xy} f(x, y) &= f(x+\tfrac{1}{2}\alpha, y+\tfrac{1}{2}\beta) - f(x-\tfrac{1}{2}\alpha, y+\tfrac{1}{2}\beta)\\ &\quad + f(x-\tfrac{1}{2}\alpha, y-\tfrac{1}{2}\beta) - f(x+\tfrac{1}{2}\alpha, y-\tfrac{1}{2}\beta).\end{aligned}$$

By the two-dimensional shift theorem, the two-dimensional Fourier transform of $f(x+x_0, y+y_0)$ is

$$\exp[i2\pi(x_0 s_x + y_0 s_y)]\bar{f}(s_x, s_y).$$

Hence

$$\begin{aligned}\text{2-dim. F.T. of } {}^{\alpha}\Delta_x f(x, y) &= e^{\pi i \alpha s_x}\bar{f}(s_x, s_y) - e^{-\pi i \alpha s_x}\bar{f}(s_x, s_y)\\ &= 2i \sin(\pi\alpha s_x)\bar{f}(s_x, s_y),\\ \text{2-dim. F.T. of } {}^{\beta}\Delta_y f(x, y) &= 2i \sin(\pi\beta s_y)\bar{f}(s_x, s_y),\\ \text{,, ,, ,, } {}^{\alpha\alpha}\Delta_{xx} f(x, y) &= (2i \sin \pi\alpha s_x)^2\bar{f}(s_x, s_y),\\ \text{,, ,, ,, } {}^{\alpha\beta}\Delta_{xy} f(x, y) &= (2i \sin \pi\alpha s_x)(2i \sin \pi\beta s_y)\bar{f}(s_x, s_y).\end{aligned}$$

With these necessary properties established, we turn to the study of the integral equation

$$T_a(x, y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(x-u, y-v)\ T(u, v)dudv,$$

where $T(x, y)$ is the true distribution of temperature and $T_a(x, y)$ is the observed distribution. To represent a Gaussian aerial pattern we take

$$A(x, y)=\frac{1}{2\pi\sigma\tau}\exp\left\{-\left(\frac{x^2}{2\sigma^2}+\frac{y^2}{2\tau^2}\right)\right\},$$

where σ and τ are the standard deviations in the x and y directions respectively.

Given T_a and A it is now required to deduce T. By the two-dimensional convolution theorem

$$\overline{T}_a(s_x, s_y)=\overline{A}(s_x, s_y)\overline{T}(s_x, s_y),$$

where $\overline{T}_a$, $\overline{A}$, and $\overline{T}$ are the two-dimensional Fourier transforms of T_a, A, and T respectively. Since

$$\overline{A}(s_x, s_y)=\exp\{-2\pi^2(\sigma^2 s_x{}^2+\tau^2 s_y{}^2)\},$$

it follows that

$$\overline{T}(s_x, s_y)=\exp\{2\pi^2(\sigma^2 s_x{}^2+\tau^2 s_y{}^2)\}\overline{T}_a(s_x, s_y).$$

Now using the formula

$$\exp\theta^2=1+\sin^2\theta+\frac{5}{6}\sin^4\theta+\frac{61}{90}\sin^6\theta+\ldots,\qquad \theta^2<\left(\frac{\pi}{2}\right)^2,$$

we have

$$\begin{aligned}\overline{T}(s_x, s_y)&=\left(1+\sin^2\pi\alpha s_x+\frac{5}{6}\sin^4\pi\alpha s_x+\ldots\right)\\&\quad\times\left(1+\sin^2\pi\beta s_y+\frac{5}{6}\sin^4\pi\beta s_y+\ldots\right)\overline{T}_a(s_x, s_y)\\&=\left(1+\sin^2\pi\alpha s_x+\sin^2\pi\beta s_y+\frac{5}{6}\sin^4\pi\alpha s_x\right.\\&\quad\left.+\frac{5}{6}\sin^4\pi\beta s_y+\sin^2\pi\alpha s_x\sin^2\pi\beta s_y+\ldots\right)\overline{T}_a(s_x, s_y),\end{aligned}$$

where $\alpha=\sqrt{2}\sigma$, $\beta=\sqrt{2}\tau$, $\alpha^2 s_x{}^2<\frac{1}{4}$, and $\beta^2 s_y{}^2<\frac{1}{4}$.

Hence, taking transforms, and provided that $\overline{T}(s_x, s_y)$ does not extend outside the central rectangle of breadth α^{-1} and height β^{-1},

$$\begin{aligned}T(x, y)=T_a(x,y)&-\frac{1}{4}\alpha\alpha\Delta_{xx}T_a(x, y)-\frac{1}{4}\beta\beta\Delta_{yy}T_a(x, y)\\&+\frac{5}{96}\alpha\alpha\alpha\alpha\Delta_{xxxx}T_a(x, y)+\frac{5}{96}\beta\beta\beta\beta\Delta_{yyyy}T_a(x, y)\\&+\frac{1}{16}\alpha\alpha\beta\beta\Delta_{xxyy}T_a(x, y)+\ldots\ .\end{aligned}$$

This equation states that it is possible to obtain the true distribution by the application to the observed distribution of a series of corrections formed by differencing the observations. If $T(x, y)$ contains Fourier components of spatial frequencies for which $|s_x| > \frac{1}{2}\alpha^{-1}$ or $|s_y| > \frac{1}{2}\beta^{-1}$ they will be severely reduced by Gaussian smoothing and will not be recoverable by the present process; however, their presence will not interfere with the application of the method. An approximately Gaussian aerial beam produced with a finite array will not receive spatial frequencies beyond a certain limit, and in the case of Mills's aerial this limit agrees, within a few per cent., with the natural limit set by the method.

Bearing in mind the purpose for which the equation was established, we now limit attention to the formulae resulting from the retention of two terms only. Then

$$\begin{aligned} T(x, y) & \\ \simeq T_a(x, y) & - \frac{1}{4}[T_a(x+\tfrac{1}{2}\alpha, y) + T_a(x-\tfrac{1}{2}\alpha, y) - 2T_a(x, y)] \\ & - \frac{1}{4}[T_a(x, y+\tfrac{1}{2}\beta) + T_a(x, y-\tfrac{1}{2}\beta) - 2T_a(x, y)] \\ = T_a(x, y) & + \left[T_a(x, y) - \frac{T_a(x+\frac{1}{2}\alpha, y) + T_a(x-\frac{1}{2}\alpha, y) + T_a(x, y+\frac{1}{2}\beta) + T_a(x, y-\frac{1}{2}\beta)}{4}\right]. \qquad \ldots\ldots (1) \end{aligned}$$

$$\begin{aligned} = T_a(x, y) & + \frac{[T_a(x, y) - T_a(x+\frac{1}{2}\alpha, y)] + [T_a(x, y) - T_a(x-\frac{1}{2}\alpha, y)]}{4} \\ & + \frac{[T_a(x, y) - T_a(x, y+\frac{1}{2}\beta)] + [T_a(x, y) - T_a(x, y-\frac{1}{2}\beta)]}{4}. \qquad \ldots\ldots (2) \end{aligned}$$

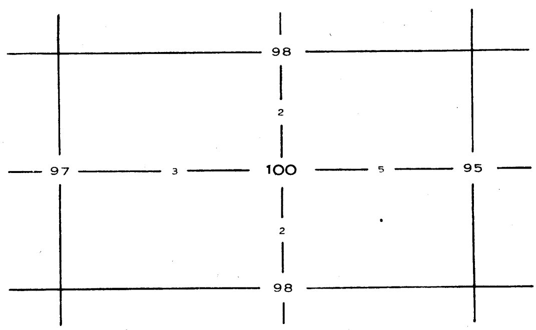

Fig. 1.—How the correction to the value 100 is calculated.

The correcting procedure based on formula (**1**) is as follows. *Tabulate the values of* $T_a(x, y)$ *in a two-dimensional array at intervals of* $\sqrt{2}\sigma$ *and* $\sqrt{2}\tau$ *in the* x *and* y *directions respectively. Subtract the mean of the four values surrounding any particular point from the value at that point, and the result is the correction to be applied to that value.* For example, in Figure 1 the mean of the four values surrounding the value 100 is 97. The corrected value is therefore 103. The alternative arrangement of formula (**2**) gives the correction as the mean excess of the value 100 over the four surrounding values, namely, $\frac{1}{4}(5+2+3+2)=3$.

III. Trial of the Formula

It is sufficient to test the formula in the one-dimensional case, where it reduces to

$$T(x)=T_a(x)-\frac{1}{4}\alpha\alpha\Delta_{xx}T_a(x).$$

A convenient distribution T in Figure 2 was taken and smoothed with a Gaussian aerial pattern to obtain the T_a shown. For comparison purposes, T_a was then smoothed so that the dotted curve, representing a single stage of restoration by

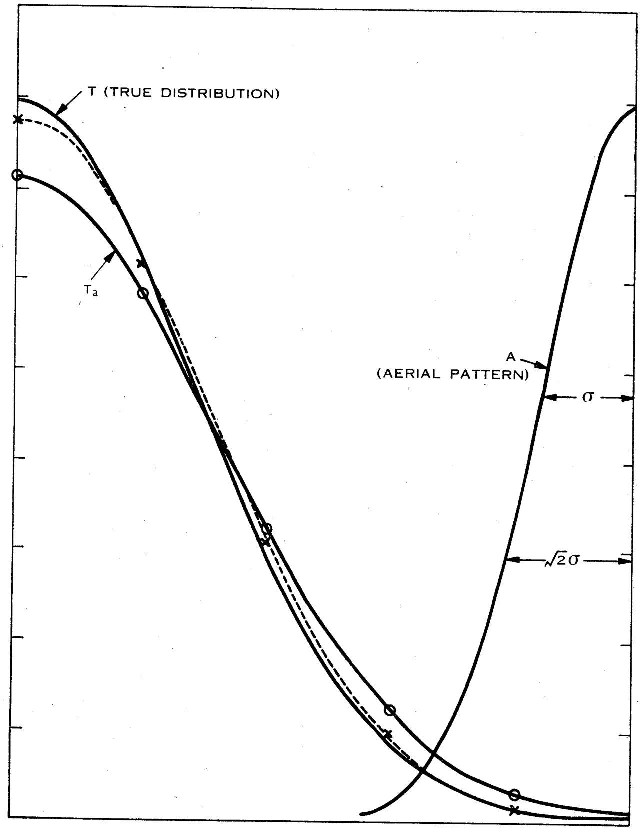

Fig. 2.—Numerical trial of correction by one stage of differencing.
○ Values of T_a used for differencing.
× Values corrected by differencing.
--- One stage of correction by successive substitutions.

successive substitutions, could be plotted. T_a was then read off at the six points marked by circles, and the above formula applied to obtain the corrected values shown as crosses.

This test shows that, in this case, the simple formula gives the order of magnitude of the correction correctly and that it is as good as a single stage of correction by successive substitutions, which is of course much harder to compute, even in the one-dimensional case.

IV. Relation to Eddington's Formula

The one-dimensional solution is

$$T(x)=T_a(x)-\frac{1}{4}\Delta^2 T_a(x)+\frac{5}{96}\Delta^4 T_a(x)+\ldots .$$

This equation may be compared with Eddington's formula for dealing with the same problem

$$T(x)=T_a(x)-\frac{1}{4}T_a''(x)+\frac{1}{32}T_a''''(x)-\ldots .$$

In practical applications the differential coefficients are replaced by finite differences and, if only one correction term is taken, the two procedures may be shown to be the same since the Gaussian function assumed for the purposes of Eddington's formula has a standard deviation of $1/\sqrt{2}$.

The derivation of Eddington's formula is known to be doubtful (see Jeffreys 1938). The method of deriving a finite-difference formula by Fourier methods seems to be a better approach.

V. Discussion

In approximate terms we may say that the negative second difference measures the upward convexity (or downward curvature) of the data. Where this is a maximum, as in the vicinity of a peak, the effect of the correction is greatest and is in such a sense as to increase the height of the peak. Near points of inflection the correction is small and changes sign. These features may be observed on Figure 3 which shows two curves, one obtained from the other by smoothing. The correction clearly agrees closely in sense and magnitude with the curvature.

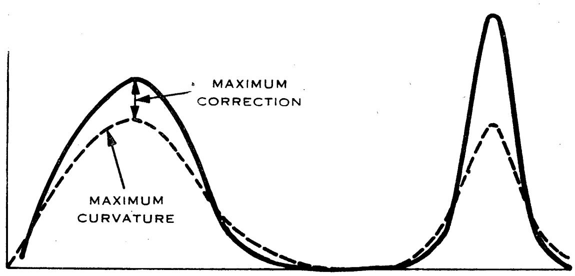

Fig. 3.—A curve (- - -) obtained from another (——) by smoothing.

The precise interval over which the differences are to be taken is prescribed by the theory. But if the data do not show much change in distances of several beam widths, then a coarser interval may be desirable to save numerical work. Also, small adjustments to the interval may be desired for numerical or graphical reasons. Differences taken over an interval m times wider than prescribed (ma instead of a) will be approximately m^2 times greater, since $\sin^2 \pi mas \simeq m^2 \sin^2 \pi as$. Within the limitations of this approximation, adjustments may be made to the interval, the factor m^2 being allowed for.

VI. References

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