FREE PATH FORMULAE FOR THE ELECTRONIC CONDUCTIVITY OF A WEAKLY IONIZED GAS IN THE PRESENCE OF A UNIFORM AND CONSTANT MAGNETIC FIELD AND A SINUSOIDAL ELECTRIC FIELD

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Summary

A general free path formula is given for the drift velocity of electrons in a weakly ionized gas in a sinusoidal electric field. Most special cases of interest, including the magnetic deflection of an electron stream in a gas, are readily derivable from the general formula.

The results find application in microwave and ionospheric studies of the motion of electrons in gases as well as in experiments on the magnetic deflection of an electron stream.

I. INTRODUCTION

In another paper (Huxley 1957) it was shown that when the method of free paths is correctly applied it is possible to derive accurate formulae for the coefficient of diffusion and the drift velocity of free electrons in a gas in the presence of a constant and uniform electric field. The general case was considered in which the scattering of electrons in encounters with molecules was not restricted to be isotropic and an equivalent mean free path was defined for groups of electrons with approximately the same speeds, depending on the law of scattering appropriate to that speed. The mean free path associated with a group is, in general, a function of the agitational speed of the electrons in the group.

Although these formulae are essential for the interpretation of experiments on the diffusion of electrons in gases by the methods introduced by Townsend (e.g. 1947) there are many instances, as in ionospheric and microwave studies, where formulae for diffusion and drift in the presence of a magnetic field and in alternating electric fields are indispensable. In what follows, an existing free path formula for the drift velocity in a sinusoidal electric field is modified by inference to include the more usual situation in which the mean free path of an electron is not constant, but depends on its speed c.

II. DIFFUSION AND DRIFT IN A CONSTANT AND UNIFORM ELECTRIC FIELD—SUMMARY OF RESULTS (Huxley 1957)

Equivalent Mean Free Path l_{eq} *of an Electron with Speed* c

This is defined to be

$$l_{eq}=l/(1-\overline{\cos\theta_1}), \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

where $l=1/nA$, and n is the number of molecules in unit volume and $A=\pi\sigma^2$ is the collisional cross section of the molecule in collisions with electrons. It

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is implied that an electron at speed c suffers a deflection in an encounter with a molecule when the impact parameter b is less than σ but is undeflected when b exceeds σ.

The term $\overline{\cos\theta_1}$ is the mean of the cosines of the angles of deflection θ_1 in single encounters with molecules of electrons at speed c. Thus

$$\overline{\cos\theta_1}=(2/\sigma^2)\int_0^\sigma \cos\theta(b)b\mathrm{d}b. \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

It follows from equations (1) and (2) that

$$1/l_{eq}=2\pi n\int_0^\sigma\{1-\cos\theta(b)\}b\mathrm{d}b, \quad \ldots\ldots\ldots\ldots\ldots (3)$$

which enables the definition of l_{eq} to be extended to those cases in which molecules and electrons are considered to interact as point centres of force, by allowing the upper limit σ to the integral in (3) to approach infinity provided the integral remains convergent (Huxley 1957). The definition of l_{eq} given in equation (3) and derived by the method of free paths is equivalent to the definition of equivalent mean free path obtained by means of the characteristic equation of Maxwell and Boltzmann (Chapman and Cowling 1952, p. 190).

When the scattering is isotropic and the molecule behaves as a particle with a finite radius σ, $\overline{\cos\theta_1}=0$ and $l_e=l=1/n\pi\sigma^2$. Thus $l_{eq}=l$ and is independent of c.

It is well known that the coefficient of diffusion D and the drift speed W of a group of electrons in this case are given by

$$\left.\begin{aligned} D&=\tfrac{1}{3}l\bar{c},\\ W&=\tfrac{2}{3}(Ee/m)l(\overline{c^{-1}}), \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

where E is the electric field strength and e/m the specific charge of the electron. The terms $\bar{c}$ and $\overline{c^{-1}}$ represent the mean values of the agitational speeds and of their reciprocals taken with respect to the prevailing law of distribution of speeds c. It was shown (Huxley 1957) that in the general case in which the scattering is not isotropic then formulae (4) should be replaced by

$$\left.\begin{aligned} D&=\tfrac{1}{3}(\overline{l_{eq}c}),\\ W&=\frac{Ee}{3m}\left\{\overline{\frac{1}{c^2}\frac{\mathrm{d}}{\mathrm{d}c}(c^2l_{eq})}\right\}, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (5)$$

in which the means are taken with respect to c with l_{eq}, in general, a function of c. The generalized formula for W given in (5) was first derived by Davidson (1954). Equation (5) reduces to equation (1) when $l_{eq}=l$ and is independent of c.

III. Diffusion in a Magnetic Field

It was first shown by Townsend (1915, 1947) that, in a magnetic field B, diffusion normal to the field occurs with a modified coefficient

$$D_\perp=\tfrac{1}{3}l[\overline{c/(1+\omega^2T^2)}], \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

in which $\omega=-Be/m$ is the gyro-angular frequency (positive for electrons) and $T=l/c$. In this formula l is considered to be independent of c.

In the more general formulation l is to be replaced by l_{eq}, which in general is a function of c, whence

$$D_{\perp}=\tfrac{1}{3}[\overline{l_{eq}c/(1+\omega^2T^2)}], \qquad \ldots\ldots\ldots\ldots\ldots \quad (7)$$

where $T=l_{eq}/c$.

This formulation then includes equation (**5**) as a special case.

IV. Electron Drift in the Presence of a Magnetic Field

It is convenient to base the discussion on a formula for the drift velocity of electrons in a gas in a magnetic field **B** and a rotating electric field with constant amplitude perpendicular to the direction of **B** that was given in an earlier paper (Huxley 1951, equation (**13**)). Let the magnetic field **B** be directed along Oz, that is to say, $\mathbf{B}=\mathbf{i}_3B$, and put $\mathbf{E}=\mathbf{i}_1E_x+\mathbf{i}_2E_y$ where $\mathbf{i}_1$, $\mathbf{i}_2$, and $\mathbf{i}_3$ are unit vectors in the directions of the respective coordinate axes. Put $E_x=E\cos pt$ and $E_y=E\sin pt$, so that $E_x+\mathrm{i}E_y=E\exp(\mathrm{i}pt)$ represents an electric vector rotating at angular speed p in a positive sense in the XOY plane.

The drift velocity W is also represented by a rotating vector. Put

$$W_x=W_0\cos(pt+\theta),\quad W_y=W_0\sin(pt+\theta).$$

Then (Huxley 1951)

$$W=W_x+\mathrm{i}W_y$$

$$=W_0\exp\mathrm{i}(pt+\theta)=\tfrac{2}{3}(Ee/m)\left[\overline{\frac{1}{\nu-\mathrm{i}(\omega-p)}\left\{1-\frac{\mathrm{i}(\omega-p)}{2[\nu-\mathrm{i}(\omega-p)]}\right\}}\right]\exp(\mathrm{i}pt), \qquad \ldots\ldots\ldots\ldots\ldots \quad (8)$$

where $\omega=-Be/m$ and e is algebraically positive, that is to say, it is given a negative value for electrons. The speed c is contained in the collisional frequency $\nu=c/l$ of an electron with speed c. The mean is taken with respect to c on the assumption that l is independent of c. When ω and p are zero expression (**8**) reduces to the formula for W in equation (**1**).

In order to modify equation (**8**) so that $l=l_{eq}$ is a function of c a formula is sought that reduces to equation (**8**) when l_{eq} is independent of c and to the formula for W in equation (**5**) when l_{eq} is a function of c but ω and p are zero. The appropriate form can be recognized immediately when equation (**8**) is expressed in the equivalent form

$$W=W_0\exp\mathrm{i}(pt+\theta)=\frac{Ee}{m}\left\{\overline{\frac{1}{3c^2}\frac{\mathrm{d}}{\mathrm{d}c}\frac{c^3}{\nu-\mathrm{i}(\omega-p)}}\right\}\exp(\mathrm{i}pt). \quad \ldots\ldots \quad (9)$$

The required extension is achieved merely by defining ν in equation (**9**) to be c/l_{eq} where l_{eq} is in general a function of c, instead of c/l as in equation (**8**). A number of special cases of interest are implicit in equation (**9**) and these are now considered.

V. Magnetic Deflection of an Electron Stream

In equation (9) put $p=0$, then

$$W_0 \exp(\mathrm{i}\theta)=\frac{Ee}{3m}\cdot\overline{\frac{1}{c^2}\frac{\mathrm{d}}{\mathrm{d}c}\left[\frac{c^3}{\nu-\mathrm{i}\omega}\right]}=\frac{Ee}{3m}\cdot\overline{\frac{1}{c^2}\frac{\mathrm{d}}{\mathrm{d}c}\left[\frac{c^3(\nu+\mathrm{i}\omega)}{\nu^2+\omega^2}\right]}. \quad \ldots\ldots (10)$$

Choose a coordinate system in which **B** is parallel to Oz and **E** to Ox, then, from equation (10),

$$\left.\begin{aligned} W_x=W_0\cos\theta&=\frac{Ee}{3m}\cdot\overline{\frac{1}{c^2}\frac{\mathrm{d}}{\mathrm{d}c}\left(\frac{\nu c^3}{\nu^2+\omega^2}\right)},\\ W_y=W_0\sin\theta&=\frac{Ee}{3m}\cdot\overline{\frac{1}{c^2}\frac{\mathrm{d}}{\mathrm{d}c}\left(\frac{\omega c^3}{\nu^2+\omega^2}\right)}.\end{aligned}\right\} \quad \ldots\ldots\ldots (11)$$

Whence

$$W_y/W_x=\tan\theta=\overline{\frac{1}{c^2}\frac{\mathrm{d}}{\mathrm{d}c}\left(\frac{\omega c^3}{\nu^2+\omega^2}\right)}\Bigg/\overline{\frac{1}{c^2}\frac{\mathrm{d}}{\mathrm{d}c}\left(\frac{\nu c^3}{\nu^2+\omega^2}\right)}, \quad \ldots (12)$$

where θ is the angle through which **W** is deflected by **B**.

In laboratory measurements of θ (Huxley and Zaazou 1949; Hall 1955) the conditions are such that $\nu^2 \gg \omega^2$, in which event,

$$\left.\begin{aligned} &W_x\to\frac{Ee}{3m}\cdot\overline{\frac{1}{c^2}\frac{\mathrm{d}}{\mathrm{d}c}(c^2 l_{eq})}=W,\\ &W\tan\theta\to W_y\to\frac{Ee}{3m}\omega\,\overline{\frac{1}{c^2}\frac{\mathrm{d}}{\mathrm{d}c}(cl_{eq}^2)}=-\frac{EB}{3}\left(\frac{e}{m}\right)^2\cdot\overline{\frac{1}{c^2}\frac{\mathrm{d}}{\mathrm{d}c}(cl_{eq}^2)},\\ &\text{to which may be added}\\ &D=\tfrac{1}{3}(\overline{l_{eq}c}).\end{aligned}\right\} \quad \ldots (13)$$

Suppose that the distribution function for the speeds c is given in the form $f(c)c^2\mathrm{d}c$. Laboratory studies of electronic motion in gases lead to measurements of W, D, and $\tan\theta$, whence the behaviour of

$$\left.\begin{aligned} \overline{\frac{1}{c^2}\frac{\mathrm{d}}{\mathrm{d}c}(c^2 l_{eq})}&=-\int_0^\infty \frac{\mathrm{d}f}{\mathrm{d}c}l_{eq}c^2\mathrm{d}c,\\ \overline{\frac{1}{c^2}\frac{\mathrm{d}}{\mathrm{d}c}(cl_{eq}^2)}&=-\int_0^\infty \frac{\mathrm{d}f}{\mathrm{d}c}(cl_{eq}^2)\mathrm{d}c,\\ (\overline{cl_{eq}})&=\int_0^\infty fl_{eq}c^3\mathrm{d}c \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (14)$$

may be studied in order to deduce the dependence of the distribution function f on the mean energy $\frac{1}{2}m\overline{c^2}$ of the electrons and of the mean free path l_{eq} upon the speed c of an electron.

Since l_{eq}, when E is fixed, is inversely proportional to the molecular concentration n, which itself at constant temperature is proportional to the pressure of the gas, it follows from equation (13) that nW_y/W is constant. This relationship has received experimental confirmation (Hall 1955).

VI. Conductivity of an Electron Gas in a Sinusoidal Electric Field

It is assumed throughout that fluctuations in the mean energy of agitation of the electrons are unimportant. Consider an alternating electric field whose component along Ox is $E_x = X_0 \cos(pt+\alpha)$. It can be represented as the sum of two electric fields with equal amplitudes rotating at angular speeds $+p$ and $-p$, thus $E_x = \frac{1}{2}X \exp(\mathrm{i}pt) + \frac{1}{2}X^* \exp(-\mathrm{i}pt)$, where $X = X_0 \exp(\mathrm{i}\alpha)$.

The components of the drift velocity in the XOY plane, according to equations (**8**) and (**10**), are now to be written

$$W_x + \mathrm{i}W_y = (W_x^+ + \mathrm{i}W_y^+) + (W_x^- + \mathrm{i}W_y^-)$$
$$= \frac{1}{6}\frac{e}{m} \cdot \overline{\frac{1}{c^2}\frac{\mathrm{d}}{\mathrm{d}c} \cdot c^3\left[\frac{X\exp(\mathrm{i}pt)}{\nu - \mathrm{i}(\omega - p)} + \frac{X^*\exp(-\mathrm{i}pt)}{\nu - \mathrm{i}(\omega + p)}\right]}.$$

The corresponding current densities are $J_x = neW_x$ and $J_y = neW_y$. If the components $E = Y\cos(pt+\beta)$ and $E = Z\cos(pt+\gamma)$ are also present it can be seen that the complex current density $J(J_x, J_y, J_z)$, whose real parts represent the true current density, is given by

$$\left.\begin{aligned}
&\{J\} = \begin{Bmatrix} J_x \\ J_y \\ J_z \end{Bmatrix} = \|\ \sigma^+\ \| \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} \exp(\mathrm{i}pt) + \|\ \sigma^-\ \| \begin{Bmatrix} X^* \\ Y^* \\ Z \end{Bmatrix} \exp(-\mathrm{i}pt), \\
&\text{in which} \\
&\|\ \sigma^+\ \| = \begin{Vmatrix} \sigma_{xx}^+ & \sigma_{xy}^+ & 0 \\ \sigma_{yx}^+ & \sigma_{yy}^+ & 0 \\ 0 & 0 & \sigma_{zz} \end{Vmatrix}, \quad \|\ \sigma^-\ \| = \begin{Vmatrix} \sigma_{xx}^- & \sigma_{xy}^- & 0 \\ \sigma_{yx}^- & \sigma_{yy}^- & 0 \\ 0 & 0 & \sigma_{zz} \end{Vmatrix} \\
&\text{and} \\
&X = X_0\exp(\mathrm{i}\alpha), \quad Y = Y_0\exp(\mathrm{i}\beta), \quad Z = Z_0\exp(\mathrm{i}\gamma), \\
&\sigma_{xx}^+ = \sigma_{yy}^+ = \mathrm{i}\sigma_{xy}^+ = -\mathrm{i}\sigma_{yx}^+ = (ne^2/6m).\overline{\frac{1}{c^2}\frac{\mathrm{d}}{\mathrm{d}c} \cdot \frac{c^3}{\nu - \mathrm{i}(\omega - p)}}, \\
&\sigma_{xx}^- = \sigma_{yy}^- = \mathrm{i}\sigma_{xy}^- = -\mathrm{i}\sigma_{yx}^- = (ne^2/6m) \cdot \overline{\frac{1}{c^2}\frac{\mathrm{d}}{\mathrm{d}c} \cdot \frac{c^3}{\nu - \mathrm{i}(\omega + p)}}, \\
&\sigma_{zz} = (ne^2/3m)\left\{\overline{\frac{1}{c^2}\frac{\mathrm{d}}{\mathrm{d}c}\left[\frac{c^3}{\nu + \mathrm{i}p}\right]}\right\}.
\end{aligned}\right\} \quad (\mathbf{15})$$

The corresponding matrix components of the conductivity implicit in the Appleton-Hartree equations for radio wave propagation in the ionosphere are:

$$\left.\begin{aligned}
&\sigma_{xx}^+ = \sigma_{yy}^+ = \mathrm{i}\sigma_{xy}^+ = -\mathrm{i}\sigma_{yx}^+ = (ne^2/2m)\frac{1}{\nu - \mathrm{i}(\omega - p)}, \\
&\sigma_{xx}^- = \sigma_{yy}^- = \mathrm{i}\sigma_{xy}^- = -\mathrm{i}\sigma_{yx}^- = (ne^2/2m)\frac{1}{\nu - \mathrm{i}(\omega + p)}, \\
&\sigma_{zz} = \frac{ne^2}{m}\,\frac{1}{(\nu + \mathrm{i}p)}.
\end{aligned}\right\} \quad \ldots\ldots \ (\mathbf{16})$$

These equations agree with equations (**15**) in the special case only of $\nu=0$. In other cases equations (**16**) may be in error by factors which can be of the order of a few units, depending on the nature of the dependence of l_{eq}, contained in $\nu=c/l_{eq}$, upon c and upon the law of distribution of the speeds c. These errors are important in studies of the motions of electrons in gases by microwave methods since it appears to be the practice to interpret the experimental measurements in such studies by means of equations equivalent to equations (**16**) with $\omega=0$.

The special case of $\omega=0$ is implicit in the expression for σ_{zz} in equations (**15**). The complex current density **J** in a sinusoidal field **E** cos pt, with $\mathbf{B}=0$, is

$$\mathbf{J}=\mathbf{E}(ne^2/3m)\overline{\left\{\frac{1}{c^2}\,\frac{\mathrm{d}}{\mathrm{d}c}\left[\frac{c^3\exp(\mathrm{i}pt)}{\nu+\mathrm{i}p}\right]\right\}}.$$

When l is independent of c this becomes

$$\mathbf{J}=\mathbf{E}(ne^2/m)\overline{\left\{\frac{1}{(\nu+\mathrm{i}p)}\left[1-\tfrac{1}{3}\left(\frac{\nu}{\nu+\mathrm{i}p}\right)\right]\exp(\mathrm{i}pt)\right\}}. \quad \ldots\ldots \quad (\mathbf{17})$$

It has been shown by Pfister (1955) that formula (**17**) is equivalent to a formula derived by Margenau for this case, in which l_{eq} is independent of c and the speeds c are distributed according to Maxwell's law.

VII. Power Communicated by the Field to Unit Volume of the Medium

In terms of the components of the field and the complex current density this is

$$P=\tfrac{1}{2}R_e[E_xJ_x^*+E_yJ_y^*+E_zJ_z^*],$$

whose value in the general case may be derived from equation (**15**).

VIII. Other Recent Investigations

A series of investigations of the conductivity of weakly ionized gases, based on the methods of Maxwell and Boltzmann, have been made by Bayet (1956 and references therein) the results of which would seem to be consistent in general with Huxley (1951, 1957) and the present paper.

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