NOTE ON BAND TRANSMISSION IN MULTILAYER DIELECTRIC FILTERS*

By S. C. B. Gascoigne†

The existence of pass and stop bands for periodically stratified media, in particular for multilayer dielectric filters, is well known, but there seems to be no way of demonstrating this either without the use of a good deal of algebra or by appeal to relatively unfamiliar ideas, such as those of transmission line theory. Our purpose here is to show that for an infinite medium, this result can be established by a direct adaptation of the Kronig-Penney (1931) model for metal lattices. Accounts of this model are given in a number of textbooks (Rojansky 1942; Brillouin 1946) and also in undergraduate courses in this country, so that it may be considered well known. Mathematical details will accordingly be abbreviated; they may be found in the references cited.

We consider a periodically stratified medium bounded by planes normal to the x-axis. The refractive index is a step function; for $-a<x<0$ it is n_1, for $0<x<b$ it is n_2, and this structure is repeated indefinitely, with period $a+b$.

* Manuscript received May 25, 1959.

† Mount Stromlo Observatory, Canberra.

The differential equation satisfied by the time-independent part of an electromagnetic wave travelling in the x-direction is

$$u''+k^2n^2u=0, \qquad k=2\pi/\lambda.$$

Bounded solutions of this must have the form $u=ve^{i\mu x}$, where μ is real and v periodic with period $a+b$. Substituting for u we find that v must satisfy

$$v_1''+2i\mu v_1'+(\alpha^2-\mu^2)v_1=0, \qquad -a<x<0, \ \alpha=kn_1,$$
$$v_2''+2i\mu v_2'+(\beta^2-\mu^2)v_2=0, \qquad 0<x<b, \ \beta=kn_2,$$

solutions to which are

$$v_1=Ae^{i(\alpha-\mu)x}+Be^{-i(\alpha+\mu)x},$$
$$v_2=Ce^{i(\beta-\mu)x}+De^{-i(\beta+\mu)x},$$

where A, B, C, and D are constants to be fixed. Continuity and periodicity conditions are: $v_1(0)=v_2(0)$, $v_1'(0)=v_2'(0)$, $v_1(-a)=v_2(b)$, $v_1'(-a)=v_2'(b)$, and these

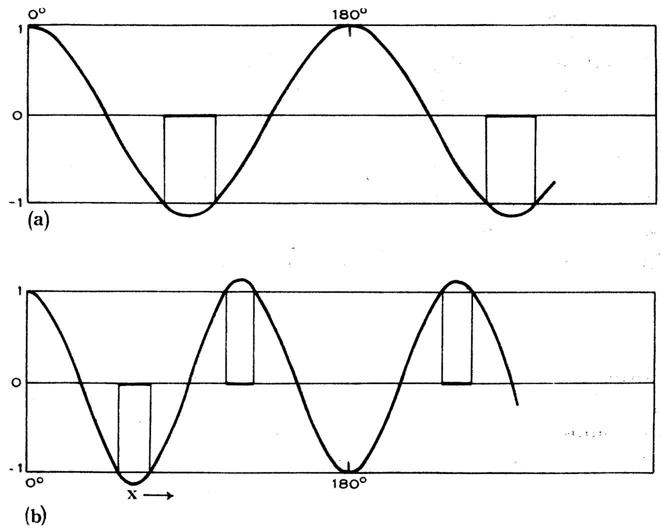

Fig. 1 (a).—Plot of $\cos^2 x-1\cdot133 \sin^2 x$ as a function of x.
Fig. 1 (b).—Plot of $\cos x \cos 2x-1\cdot133 \sin x \sin 2x$.

take the form of four equations linear in A, B, C, and D. These equations have a non-trivial solution only if the determinant of the coefficients vanishes. The determinant reduces to

$$\cos \alpha a \cos \beta b-\frac{\alpha^2+\beta^2}{2\alpha\beta} \sin \alpha a \sin \beta b=\cos \mu(a+b),$$

or to

$$\cos kn_1a \cos kn_2b-\tfrac{1}{2}(n_1/n_2+n_2/n_1) \sin kn_1a \sin kn_2b=\cos \mu(a+b).$$

In this equation kn_1a and kn_2b are angles which measure the phase differences between the layer boundaries. The term $\frac{1}{2}(n_1/n_2+n_2/n_1)$ is clearly greater than unity, and it is then easy to show that there are values of k, and hence of λ, for which the modulus of the left-hand side is also greater than unity. Since μ must be real there can be no solution to the differential equation for these values. They constitute the stop bands or reflection bands.

As an example, suppose that $n_1=1\cdot38$ and $n_2=2\cdot30$ (the values for MgF_2 and ZnS respectively) and suppose also, as is frequently the case, that the optical thicknesses of the layers are equal, each to a quarter of a fixed wavelength λ_0. Thus, $n_1a=n_2b=\frac{1}{4}\lambda_0$, and if we put $x=\frac{1}{2}\pi\lambda_0/\lambda$, the equation becomes

$$\cos^2 x-1\cdot133 \sin^2 x=\cos \mu(a+b).$$

In Figure 1 (*a*) we plot the left-hand side as a function of x. The stop bands, where the left-hand side is less than -1, are indicated by heavy lines along the x-axis. They extend from $75\cdot5$ to $104\cdot5°$, from $255\cdot5$ to $284\cdot5°$, and so on; the corresponding λ-intervals are $1\cdot192$–$0\cdot861\lambda_0$, and $0\cdot352$–$0\cdot316\lambda_0$. λ_0 is near the centre of a stop band of width $0\cdot331\lambda_0$.

If one layer (it is immaterial which) has twice the thickness of the other, the equation becomes

$$\cos x \cos 2x-1\cdot133 \sin x \sin 2x=\cos \mu(a+b),$$

and we have the situation in Figure 1 (*b*). The stop bands extend from $51\cdot3$ to $68\cdot0°$, $112\cdot0$ to $128\cdot7°$, and so on, the corresponding λ-intervals being $1\cdot754$–$1\cdot323\lambda_0$, and $0\cdot804$–$0\cdot699\lambda_0$. λ_0 is now at the centre of a pass band.

Actual filters have, of course, only a finite number of layers, and an exact theory for these becomes complicated. The positions of the bands are very much as indicated above, but the transition between pass and stop bands is more gradual. In particular, the transmission falls appreciably near the edges of the pass bands.

References

Brillouin, L. (1946).—"Wave Propagation in Periodic Structures." p. 180. (McGraw-Hill: New York.)

Kronig, R., and Penney, W. G. (1931).—*Proc. Roy. Soc.* A **130**: 499.

Rojansky, V. (1942).—"Introductory Quantum Mechanics." p. 270. (Prentice-Hall: New York.)