DRIFT OF A CHARGED PARTICLE IN A MAGNETIC FIELD OF CONSTANT GRADIENT

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[*Manuscript received June* 8, 1959]

Summary

A simple expression for the drift velocity of a charged particle moving in an inhomogeneous magnetic field has been obtained by Alfvén, who, in his first-order theory, considered the inhomogeneity as a small perturbation of a uniform field.

In this paper, by use of a different approach, an exact solution is obtained for the drift velocity of a charged particle moving in a magnetic field of constant gradient, $B_z=\lambda x$. The method easily yields as approximations Alfvén's result and the case of circular orbit, and includes the case of zero mean field, for which perturbation methods are inappropriate.

I. MOTION IN A MAGNETIC FIELD OF CONSTANT GRADIENT

Consider first the motion of an electron, and suppose the magnetic field is in the z-direction and is represented by $B_z=\lambda x$. Then, from the non-relativistic equation of motion we have, in the absence of electric field,

$$m\mathrm{d}\mathbf{w}/\mathrm{d}t=-(e/c)\mathbf{w}\times\mathbf{B}, \quad \ldots\ldots\ldots\ldots \quad (1)$$

where the electron charge is $-e$.

The z-component of $\mathbf{w}$ is constant, and need not be considered explicitly. The other components vary as

$$\left.\begin{aligned} m\mathrm{d}w_x/\mathrm{d}t&=-(e/c)\lambda x w_y,\\ m\mathrm{d}w_y/\mathrm{d}t&=+(e/c)\lambda x w_x. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (2)$$

$w^2=w_x^2+w_y^2$ is constant, and, in the notation of Figure 1, $w_x=w\cos\psi$, $w_y=w\sin\psi$. Substitution of these relations in (2) gives

$$\mathrm{d}\psi/\mathrm{d}t=e\lambda x/mc. \quad \ldots\ldots\ldots\ldots \quad (3)$$

Since $w=\mathrm{d}s/\mathrm{d}t$, where s denotes distance along the electron trajectory

$$\mathrm{d}\psi/\mathrm{d}s=e\lambda x/mcw. \quad \ldots\ldots\ldots\ldots \quad (4)$$

Thus

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}s^2}=\frac{e\lambda}{mcw}\frac{\mathrm{d}x}{\mathrm{d}s}=\frac{e\lambda}{mcw}\cos\psi, \quad \ldots\ldots\ldots\ldots \quad (5)$$

and so

$$\left(\frac{\mathrm{d}\psi}{\mathrm{d}s}\right)^2=\frac{2e\lambda}{mcw}\sin\psi+\text{constant}. \quad \ldots\ldots\ldots\ldots \quad (6)$$

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From (4) and (6)

$$x=\pm\sqrt{\{x_0^2+(2mcw/e\lambda)\sin\psi\}}, \quad \ldots\ldots\ldots\ldots\ldots \quad (7)$$

where x_0 is the value of x when $\psi=0$.

For x positive, (3) gives

$$t=\frac{mc}{e\lambda}\int\frac{d\psi}{\sqrt{\{x_0^2+(2mcw/e\lambda)\sin\psi\}}}. \quad \ldots\ldots\ldots\ldots \quad (8)$$

Also, since

$$\frac{dy}{ds}=\sin\psi, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (9)$$

(4) gives

$$y=\frac{mcw}{e\lambda}\int\frac{\sin\psi\,d\psi}{\sqrt{\{x_0^2+(2mcw/e\lambda)\sin\psi\}}}. \quad \ldots\ldots\ldots\ldots \quad (10)$$

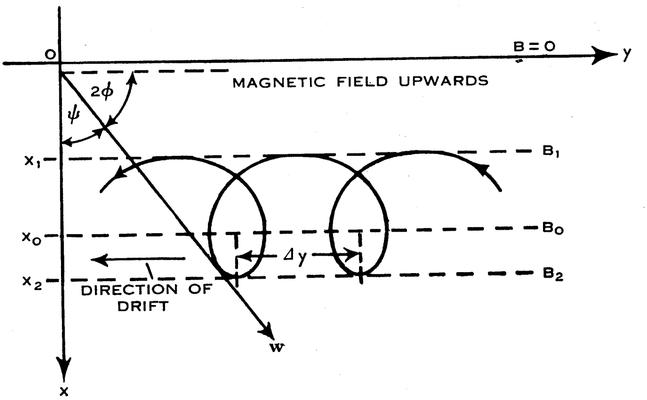

Fig. 1.—Electron drift in a magnetic field $B_z=\lambda x$: Case 1, motion does not cross the line $B=0$.

II. Comparison with Alfvén's Result

For motion which does not cross the line $B=0$, as in Figure 1, (10) gives the y-drift per cycle exactly as

$$\Delta y=\rho_0\int_0^{2\pi}\frac{\sin\psi\,d\psi}{\sqrt{\{1+2(\rho_0/x_0)\sin\psi\}}}, \quad \ldots\ldots\ldots\ldots \quad (11)$$

where $B_0=\lambda x_0$, and $\rho_0=mcw/eB_0$ is the orbit's radius of curvature for field strength B_0.

When $\rho_0\ll x_0$, which implies electron orbital motion far from the y-axis, a first-order result for the drift velocity in the y-direction can be readily obtained. After expansion of the denominator in (11), integration gives approximately

$$\Delta y=-\pi\rho_0^2/x_0=-\pi\lambda\rho_0^2/B_0.$$

Similarly, from (8) the periodic time in this case is $T=2\pi\rho_0/w$, and so the corresponding drift velocity in the y-direction is given by

$$w_D=\Delta y/T=-\tfrac{1}{2}\rho_0\lambda w/B_0. \quad \ldots\ldots\ldots\ldots\ldots \quad (12)$$

This is the result obtained by Alfvén (1940, 1950), and discussed by Post (1956) and Spitzer (1956).

When B_0 has a fixed value and λ approaches zero, we obtain the circular orbit result for a homogeneous magnetic field, i.e. $w_D=0$ and $T=2\pi\rho_0/w=2\pi/\omega_c$, where $\omega_c=eB_0/mc$ is the so-called cyclotron angular frequency.

III. Exact Solution in Terms of Elliptic Integrals

To complete the exact solution, two cases require consideration.

Case 1. *Electron does not enter Region of Reversed Magnetic Field*

For an electron motion which does not cross the line $B=0$, as shown in Figure 1, the limits of x are, from (7),

$$\left.\begin{aligned} x_1&=\sqrt{(x_0^2-2mcw/e\lambda)} \text{ for } \psi=3\pi/2,\\ x_2&=\sqrt{(x_0^2+2mcw/e\lambda)} \text{ for } \psi=\pi/2, \end{aligned}\right\} \qquad \ldots\ldots\ldots \quad (13)$$

so that $0\leqslant x_1<x_0<x_2$.

Using the substitution $\psi=\frac{1}{2}\pi-2\varphi$, we obtain from (10) the drift per cycle in the y-direction as

$$\Delta y=\frac{2mcw}{eB_2}\int_0^\pi \frac{(1-2\sin^2\varphi)\mathrm{d}\varphi}{\sqrt{(1-k_1^2\sin^2\varphi)}}, \qquad \ldots\ldots\ldots\ldots \quad (14)$$

where $\lambda x_2=B_2$ and $k_1^2=4mcw/e\lambda x_2^2=4\rho_2/x_2$, if ρ_2 is the orbit's radius of curvature for field strength B_2.

When $x_1\geqslant 0$, (13) gives $x_2\geqslant\sqrt{(4mcw/e\lambda)}\geqslant\sqrt{(x_2^2k_1^2)}$, so that the upper limit of k_1^2 is unity.

Reduction of (14) to standard form for elliptic integrals results in

$$\begin{aligned}\Delta y&=x_2\left\{2\int_0^{\pi/2}\sqrt{(1-k_1^2\sin^2\varphi)}\mathrm{d}\varphi-[2-k_1^2]\int_0^{\pi/2}\frac{\mathrm{d}\varphi}{\sqrt{(1-k_1^2\sin^2\varphi)}}\right\}\\ &=-x_2\{2[K-E]-k_1^2K\}, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (15)\end{aligned}$$

where K and E are complete elliptic integrals of the first and second kind respectively, of modulus k_1 (Dwight 1947).

The periodic time is derived from (8) as

$$T=\frac{4mc}{eB_2}\int_0^{\pi/2}\frac{\mathrm{d}\varphi}{\sqrt{(1-k_1^2\sin^2\varphi)}}=x_2\frac{k_1^2}{w}K. \qquad \ldots\ldots\ldots \quad (16)$$

The exact drift velocity is, therefore, from (15) and (16),

$$w_D=\frac{\Delta y}{T}=-w\left\{\frac{2}{k_1^2}\left[1-\frac{E}{K}\right]-1\right\}. \qquad \ldots\ldots\ldots\ldots \quad (17)$$

Since $\{(2/k_1^2)[1-E/K]-1\}\geqslant 0$ for $0\leqslant k_1^2\leqslant 1$, the drift velocity is always in the negative y-direction.

When $k_1^2\ll 1$, Alfvén's result $w_D=-\frac{1}{8}wk_1^2$ is again obtained.

When $k_1^2=1$, which occurs when $x_1=0$, we have $w_D=-w$, and the electron moves along the line $B=0$.

For $x<0$, the electron drift pattern is precisely the mirror image in the y-axis of that shown in Figure 1. Again the drift is always in the negative y-direction.

Case 2. Electron enters Region of Reversed Magnetic Field

Consider now a motion in which the electron crosses the line $B=0$, and thus enters a reversed magnetic field, as shown in Figure 2. Let ψ_0 be the value of ψ

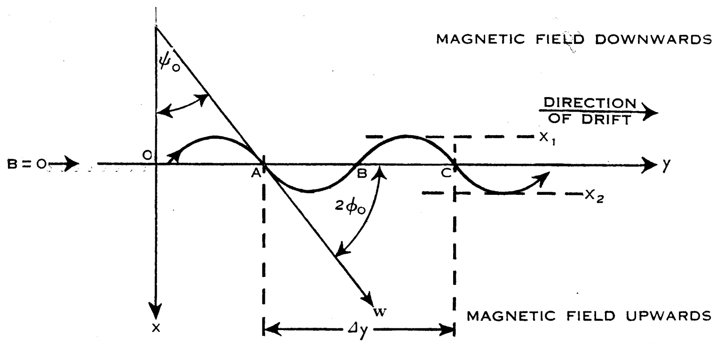

Fig. 2.—Electron drift in a magnetic field $B_z=\lambda x$: Case 2, motion crosses the line $B=0$.

when $x=0$ and $w_x>0$. Then in place of (7)

$$x=\pm\sqrt{\{(2mcw/e\lambda)(\sin\psi-\sin\psi_0)\}}, \qquad (18)$$

and so the limits of x are now

$$x_2=-x_1=\sqrt{\{(2mcw/e\lambda)(1-\sin\psi_0)\}} \quad \text{for } \psi=\tfrac{1}{2}\pi. \qquad (19)$$

In this case various drift patterns may occur, as shown in Figure 3, but each pattern possesses symmetry about the y-axis. It is therefore convenient to introduce the term "zero mean field" to describe the field condition at the y-axis.

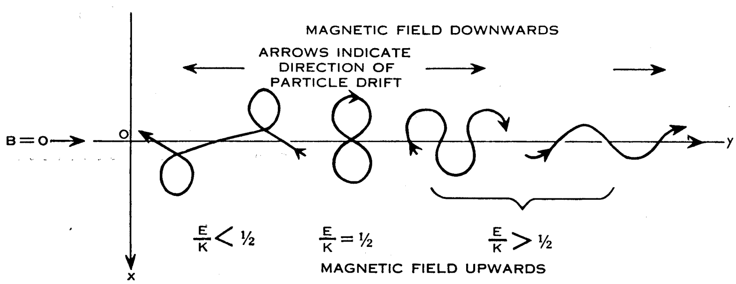

Fig. 3.—Typical electron drift patterns in terms of E/K for Case 2.

Introducing as before $\psi=\frac{1}{2}\pi-2\varphi$, then $\psi_0=\frac{1}{2}\pi-2\varphi_0$. From Figures 2 and 3, $-\frac{1}{2}\pi\leqslant\psi_0\leqslant+\frac{1}{2}\pi$, $0\leqslant\varphi_0\leqslant\frac{1}{2}\pi$, and the above limits thus reduce to

$$x_2=-x_1=2\sqrt{(mcw/e\lambda)}\sin\varphi_0. \qquad (20)$$

Utilizing the properties of symmetry exhibited by the drift pattern of, say, Figure 2, then for positive x in (**18**) we obtain from (**4**) and (**9**)

$$\Delta y = AC = 2AB = \frac{2mcw}{e\lambda}\int_{\psi_0}^{\pi-\psi_0} \frac{\sin\psi \mathrm{d}\psi}{\sqrt{\{(2mcw/e\lambda)(\sin\psi-\sin\psi_0)\}}}$$

$$= 8\sqrt{\left(\frac{mcw}{e\lambda}\right)}\left\{\int_0^{\varphi_0} \frac{\cos^2\varphi \mathrm{d}\varphi}{\sqrt{(\sin^2\varphi_0-\sin^2\varphi)}} - \tfrac{1}{2}\int_0^{\varphi_0} \frac{\mathrm{d}\varphi}{\sqrt{(\sin^2\varphi_0-\sin^2\varphi)}}\right\}. \quad \text{(21)}$$

If another variable of integration θ is defined by $\sin\varphi = \sin\varphi_0 \sin\theta$, (**21**) may be transformed to

$$\Delta y = 8\sqrt{\left(\frac{mcw}{e\lambda}\right)}\left\{\int_0^{\pi/2} \sqrt{(1-k_2^2\sin^2\theta)}\mathrm{d}\theta - \tfrac{1}{2}\int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{(1-k_2^2\sin^2\theta)}}\right\}$$

$$= 8\sqrt{(mcw/e\lambda)}(E-\tfrac{1}{2}K). \quad \text{(22)}$$

where $k_2 = \sin\varphi_0 = x_2\sqrt{(e\lambda/4mcw)}$, using (**20**).
Similarly

$$T = 4\sqrt{(mc/e\lambda w)}K, \quad \text{(23)}$$

and the drift velocity in this case is

$$w_D = w(2E/K - 1). \quad \text{(24)}$$

When $k_2 \ll 1$, $E/K \approx 1-\frac{1}{2}k_2^2$ and

$$w_D \approx w(1-k_2^2) \sim w, \quad \text{(25)}$$

as expected from Figure 2.

For $E/K = \frac{1}{2}$, corresponding to $\varphi_0 \approx 65°$, the drift velocity becomes zero. As φ_0 increases beyond 65° ($E/K < \frac{1}{2}$) the drift velocity becomes negative, as in Case 1. Figure 3 gives typical drift patterns for $E/K < \frac{1}{2}$, $E/K = \frac{1}{2}$, and $E/K > \frac{1}{2}$. Since in this Case w_D becomes $+w$ and $-w$ for $k_2=0$ and $k_2=1$ respectively, it follows that at the limit $\varphi_0=0$ the electron moves along the line $B=0$ in the positive y-direction, while at the limit $\varphi_0 = \frac{1}{2}\pi$ it moves along the same line in the negative y-direction. Thus the drift patterns of Cases 1 and 2 coincide in the limits $k_1=1$ and $k_2=1$.

For positively charged particles, the principal results are that the formulae for k_1 and k_2 remain unchanged, if the particle charge is $+e$, whereas the w_D are changed only in sign.

Since $\mathrm{d}x/\mathrm{d}s = \cos\psi$, it can be readily ascertained that the drift velocity in the x-direction is zero in all cases.

IV. Discussion of Results

The drift velocity results obtained in Cases 1 and 2 for an electron are shown plotted against the parameter $x_2\sqrt{(e\lambda/4mcw)}$ in Figure 4. This parameter is k_2 in the region of Case 2, and $1/k_1$ in the region of Case 1. For $1/k_1 \to \infty$, we have the well-known region of Alfvén drift velocities. In spite of the smallness of Alfvén drift velocities relative to particle velocities, it has been considered that charge separation effects could be obtained in a plasma and that these might lead to motion of the plasma towards the region of weakest magnetic field (Post 1956). Larger drift velocities, of the order of the particle velocities, are expected, however, in the region $x_2\sqrt{(e\lambda/4mcw)}\lesssim 1$. Thus, if the magnetic field within a plasma varies in a direction normal to the field and somewhere changes sign, the motion of the charged particles in the neighbourhood where it changes sign may lead to larger charge separation effects than in the Alfvén region.

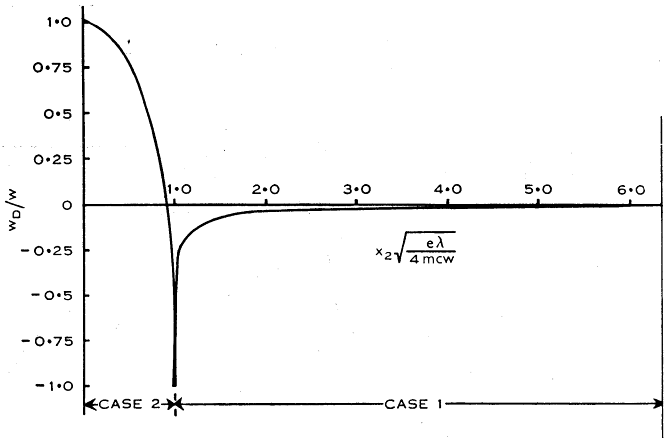

Fig. 4.—Variation of w_D/w with $x_2\sqrt{(e\lambda/4mcw)}$ for Cases 1 and 2.

V. Acknowledgments

The author is indebted to Professor K. J. Le Couteur, who suggested the problem and gave guidance during the course of the investigation, and to Dr. F. C. Barker for helpful discussions.

VI. References

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