THE DAMPING OF A FLUCTUATING CONCENTRATION BY CONTINUOUS SAMPLING THROUGH A TUBE

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Summary

Concentrations in turbulent fields may be measured by sampling through a tube leading to a continuous analyser. This technique has important applications in geophysics and chemical engineering.

This paper examines the question of how dispersal by molecular diffusion and convection in the sampling tube damps out the fluctuations in concentration occurring in the turbulent field.

The underlying mathematical theory has been developed by the author in a recent, separate, series of papers. It is here applied specifically to the tube sampling technique.

By appropriate choice of tube diameter and flow velocity, the technique may be used to realize different purposes. If it is desired to use the tube as a high frequency filter, a suitable criterion is

$$a \approx 5(D/\omega_1)^{\frac{1}{2}},$$

where a is tube diameter, D molecular diffusivity, and ω_1 the frequency (radians per unit time) separating "high" and "low" frequencies. If, on the other hand, it is desired to limit attenuation to a maximum of 10% throughout the whole frequency range, an appropriate criterion is

$$Ua^2 \geqslant 100DX,$$

where U is the mean flow velocity and X is the tube length. One or other of these criteria must then be considered in conjunction with the limit on the Reynolds number set by the requirement that the tube flow be laminar.

Two numerical examples illustrate the application to (i) gas sampling and (ii) solute sampling in water.

I. Introduction

It is well known that the properties of the fluid at a point in a turbulent field undergo fluctuations of an essentially random character. The statistics of the fluctuations are related to the structure of the turbulent field and to the density of the flux (if any) associated with the property under consideration. Some of these fluctuating properties (e.g. velocity, pressure) are intimately connected with the dynamics of the field, whereas others are, at least to a first approximation, dynamically neutral. Examples of the latter type of property include: temperature (in many circumstances); concentration of dissolved material (in a liquid); concentration of a constituent (of a gaseous mixture); concentration of suspended material.

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One method of measuring a concentration* in a turbulent field is to sample the field by continuous slow withdrawal of fluid by means of a tube with its inlet exposed at the required fixed point in the turbulent field.† The sampling tube is led into an analyser which may be required to be at some distance from the sampling point. The analyser (which may be, for example, an infrared gas analyser, a recording conductivity meter, an optical densitometer, or a particle counter) then yields a continuous measure of concentration in the fluid passing through it.

The particular application which prompted the present study is the measurement of CO_2 concentration in the lower atmosphere. My colleague, Dr. O. T. Denmead, began observations of this type in 1960; and he is continuing further work. Similar studies have been made at Cornell University by Dr. E. R. Lemon (personal communication), who observed both CO_2 and water vapour concentrations. Tanner and Suomi (1956) measured water vapour only by the same method. Inoue (personal communication) has also observed CO_2 concentrations in this way near Tokyo. Monteith and Szeicz (1960) observed CO_2 concentration differentials somewhat similarly.

The question to be considered here is the manner in which dispersal by molecular diffusion and convection in the sampling tube damps out the fluctuations in concentration occurring in the turbulent field. It is evident that, for certain purposes, the damping out of high frequency fluctuations is a desideratum; and, indeed, special steps‡ have often been taken to ensure this. Other types of study require the preservation of the fluctuations present at the sampling tube inlet; or, when this is not possible, that means be available of recovering the "concentration spectrum" for the turbulent field from that at the tube outlet.

Micrometeorology is, of course, not the only field in which this class of problem may arise. There are evidently applications to oceanography, hydrology, and chemical engineering, which need not be elaborated here.

II. Formulation of the Problem

The sampling tube is taken to be circular and straight. The fluid flow within it is taken to be steady, laminar, and longitudinally uniform (i.e. any disturbance to the flow pattern associated with the tube inlet or outlet is neglected). We assume the molecular diffusivity of the substance which is fluctuating in concentration to be independent of concentration.

We introduce the following symbolism:

a, tube radius,

C, concentration, averaged over tube cross section,

* The treatment which follows is applicable to suspended material only if the settling velocity is negligible.

† In principle the method might be used for measuring temperature also; in practice more efficient means are available.

‡ We note that the present study indicates that devices such as mixing tanks (cf. Tanner and Suomi 1956) are not essential. It would appear that dispersion in the sampling tube has frequently provided all the "smoothing" required. Criteria to enable any desired damping out of high frequency fluctuations are given later in the paper.

D, molecular diffusivity,
t, time,
U, mean flow velocity in tube,
x, axial coordinate with $x = 0$ at tube inlet,
ω, frequency in radians per unit time.

We shall have occasion to use the following dimensionless quantities:

$$\left.\begin{aligned} \xi &= x/a, \\ \Upsilon &= Ua/D, \\ \Omega &= \omega a^2/D. \end{aligned}\right\} \tag{2.1}$$

Υ is the "diffusion Péclet number" (Philip 1963*a*).

A boundary condition of the problem is

$$x = 0; \qquad C = C_0(t), \tag{2.2}$$

where $C_0(t)$ is a stationary random continuous time series. Usually a will be small compared with the characteristic length scales of the turbulent field, so that the concentration will not differ much from C_0 at any point across the inlet; and the spectrum of fluctuations of C_0 will be very nearly the spectrum of fluctations of concentration at a point (of zero magnitude) in the field. The analysis to be given here is also valid when a is not small compared with the characteristic length scales of the field. Physical interpretation of the "inlet spectrum" must, however, then take into account the fact that the sampling inlet area is non-zero.

III. The Spectrum of Concentration Fluctuations

The value of C at any point x along the tube forms a stationary random continuous time series, which we denote by $C_x(t)$. We denote the associated concentration fluctuation by $c_x(t)$, where

$$c_x(t) = C_x(t) - \bar{C}_x, \tag{3.1}$$

and

$$\bar{C}_x = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} C_x(t)\, \mathrm{d}t. \tag{3.2}$$

The total variance of the concentration fluctuation at $x = X$ is

$$\bar{c}_x^2 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} c_x^2(t)\, \mathrm{d}t. \tag{3.3}$$

The formal harmonic decomposition of $c_x(t)$ may be performed in the usual way. We denote the resulting spectral density function by $N_x(\omega)$. The quantity

$$N_x(\omega)\, \mathrm{d}\omega$$

is then the contribution to the total variance due to frequencies in the spectral range ω to $\omega + \mathrm{d}\omega$; and it is clear that

$$\int_0^\infty N_x(\omega)\, \mathrm{d}\omega = \bar{c}_x^2. \tag{3.4}$$

As we have noted above, the "inlet spectrum" $N_0(\omega)$ is subject to modification along the tube by the operation in combination of convection and molecular diffusion. Our problem thus reduces to the study of how $N_x(\omega)$ varies with x (and with a, U, and D). The problem is greatly simplified by the fact that the partial differential equation governing dispersal in the tube (cf. Philip 1963*a*) is linear. We can, in consequence of this, introduce the quantity β, a function of ω and x, such that

$$\beta(\omega, x) = N_x(\omega)/N_0(\omega). \tag{3.5}$$

$\beta(\omega, x)$ may then be established by considering the dispersal of a single harmonic fluctuation of frequency ω.

IV. Solution of the Problem

The necessary mathematical analysis has been carried out in a separate series of papers (Philip 1963*a*, 1963*b*, 1963*c*). Subject to the requirement that

$$\xi \gg \Upsilon/10, \tag{4.1}*$$

the solution we seek is

$$\beta = \exp\{\mathscr{R}[a_0(\Omega, \Upsilon)].\ \xi/\Upsilon\}. \tag{4.2}$$

$\mathscr{R}[\ ]$ denotes "the real part of".

The function $a_0(\Omega, \Upsilon)$ is established for various ranges of Ω and Υ in Philip (1963*b*, 1963*c*). In many practical applications the Péclet number is so large that we may adopt for a_0 the function $a_0(\Omega, \infty)$, which we shall write, for simplicity, as $a_0(\Omega)$. This function, which is established in Philip (1963*b*), will be used in the subsequent developments here. The parallel treatment for small Péclet number need not be elaborated.

We observe that (4.2) is of the form

$$\beta = \exp\{-\gamma(\omega, a, D, U)\ .\ x\}. \tag{4.3}$$

It is evident that the attenuation function, $\gamma(\omega, a, D, U)$, specifying the exponential rate of decrease of variance with x, provides a convenient means of characterizing the damping of the fluctuations.

V. Some Characteristics of the Solution

In general, $a_0(\Omega)$ is not expressible readily as an explicit function of Ω, but the extreme cases of Ω small and Ω large yield simple results of some practical significance.

For $\Omega<10$, we may adopt the approximation

$$\mathscr{R}[a_0] = -0{\cdot}021\Omega^2. \tag{5.1}$$

Compare (2.4.2) of Philip (1963*b*). Using (5.1) in (4.2) (4.3), we have the approximation

$$\Omega<10; \qquad \gamma x = 0{\cdot}021\Omega^2\xi/\Upsilon. \tag{5.2}$$

* The numerical constant 10 in (4·1) is the value appropriate to large Υ, and holds for the applications to be discussed here. The dependence of this constant on Υ for small Υ will be treated in Philip (1963*c*).

Using (2.1) in (5.2), we obtain

$$\gamma = 0{\cdot}021\omega^2 a^2/DU. \tag{5.3}$$

In this case γ increases rapidly with ω. The sampling tube acts as a high frequency filter. Passage down the tube distorts the spectrum, high frequency fluctuations being attenuated more rapidly than low frequency ones.

For $\Omega > 100$, we may adopt the approximation

$$\mathscr{R}[a_0] = -11. \tag{5.4}$$

Compare (2.4.9) of Philip (1963*b*). Using (5.4) in (4.2), (4.3), we obtain the approximation

$$\Omega > 100; \qquad \gamma x = 11\xi/Y. \tag{5.5}$$

Using (2.1) in (5.5), we then have

$$\gamma = 11D/Ua^2. \tag{5.6}$$

In this case γ is independent of ω. Here there is no distortion of the spectrum, the rate of attenuation being the same at all frequencies.

The foregoing interpretations of (5.3) and (5.5) are, of course, strictly valid only if the whole of the frequency band of interest satisfies one or other of the inequalities given for the reduced frequency Ω. However, as we shall see, they provide a useful basis for understanding the behaviour in other cases also.

The value $\Omega = 25$ is a convenient, though arbitrary, reduced frequency, which may be taken as separating the low values of Ω for which γ increases rapidly with ω from the high values for which γ varies only mildly with ω. A criterion for the effective filtering out of frequencies higher than any given value follows and is developed below.

VI. Criteria governing Physical Parameters of Sampling Method

It is useful to bring together the various criteria governing the physical parameters of the sampling method. In general, the spectral band of interest, the tube length, X cm, and the value of D will be fixed for the experimenter. The quantities which may be varied, then, are U and a. Normally the experimenter will aim either (i) to suppress high frequency fluctuations or (ii) to minimize distortion of the spectrum. In either case, appropriate criteria on U and a may be found.

To begin with, we must state the criterion that the flow in the tube be laminar. We take the "lower critical Reynolds number" (based on tube diameter) for the stability of laminar flow as 1600. In the present symbols, this reduces to

$$Ua/\nu < 800. \tag{6.1}$$

ν is the kinematic viscosity. We shall also make use of the "diffusion Prandtl number",

$$P = \nu/D. \tag{6.2}$$

(a) Criteria to Suppress High Frequency Fluctuations

An appropriate criterion to ensure the effective filtering out of frequencies higher than ω_1, is that

$$\Omega(\omega_1) \approx 25, \tag{6.3}$$

Compare Section V above. Putting (2.1) in (6.3) yields the more convenient form

$$a \approx 5(D/\omega_1)^{\frac{1}{2}}. \tag{6.4}$$

Using this value of a in (6.1), we obtain the associated criterion on U,

$$U < 160(\nu P \omega_1)^{\frac{1}{2}}. \tag{6.5}$$

Gas sampling.—We take, as typical of gas sampling, the sampling of CO_2 in air, with $D = 0{\cdot}14$ cm^2/s, $\nu = 0{\cdot}15$ cm^2/s. P is then $1{\cdot}07$. (6.4) then reduces to

$$a \approx 1{\cdot}9\omega_1^{-\frac{1}{2}} \text{ cm}.$$

Taking $\omega_1 = 0{\cdot}3$, we obtain the convenient value of $3{\cdot}4$ cm for a.

We observe further that (6.5) reduces to the not very stringent condition

$$U < 35 \text{ cm/s}.$$

Water sampling.—For the case of sampling dissolved salts in water, we may take $D = 10^{-5}$ cm^2/s, $\nu = 10^{-2}$ cm^2/s. P is then 10^3. (6.4) becomes, in this case,

$$a \approx 0{\cdot}016\omega_1^{-\frac{1}{2}}.$$

Taking $\omega_1 = 0{\cdot}3$, we obtain $a \approx 0{\cdot}03$ cm, which is, again, a feasible value. In this case also (6.5) sets no practical limit to the method, yielding

$$U < 290 \text{ cm/s}.$$

(b) Criteria to Minimize Distortion of the Spectrum

A possible criterion to set a limit on distortion of the spectrum is that attenuation at the highest frequency should not exceed 10%. It follows from (5.6) that this condition reduces to

$$Ua^2 \geqslant 100DX. \tag{6.6}$$

Combining this with (6.1) yields

$$a > DX/8\nu, \tag{6.7}$$

which may be rewritten as

$$a/X > 1/8P. \tag{6.8}$$

We also have, from (4.1), that the validity of the analysis requires that

$$a/X \ll 10/Y. \tag{6.9}$$

It follows from (6.8) and (6.9) that

$$Y \ll 80P. \tag{6.10}$$

This is equivalent to

$$Ua/\nu \ll 80. \tag{6.11}$$

We observe that (6.11) is a more stringent condition on the Reynolds number than is (6.1).

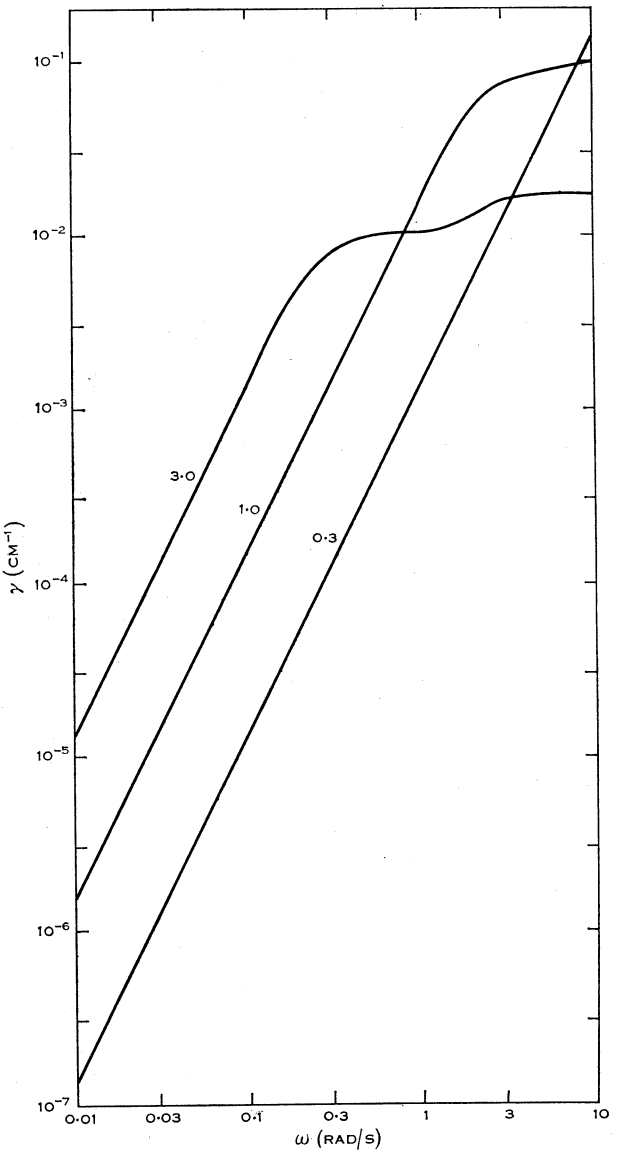

Fig. 1.—Example 1. Logarithmic plot of attenuation function γ cm^{-1} against frequency ω rad/s. Numerals on the three curves indicate sampling tube radius in centimetres.

Adopting the minimum value of a indicated by (6.8) and using this in (6.11), we obtain the following requirements on a and U:

$$a = X/8P; \qquad U \ll 640\nu P/X. \tag{6.12}$$

We now consider specific results for the numerical values of D and ν we have used above as typical of gas and water sampling.

Gas sampling.—(6.12) becomes

$$a = X/8{\cdot}6 \text{ cm}; \qquad U \ll 100/X \text{ cm/s}.$$

This value of a will be, in general, impractically large. In practice, the sampling tube is often required to be at least hundreds of diameters long. The condition on U is also rather stringent.

Water sampling.— (6.12) becomes

$$a = X/8000 \text{ cm}; \qquad U \ll 6400/X \text{ cm/s}.$$

Here the diffusion Prandtl number is much larger. In this case, the design of a sampling system with small spectral distortion is possible even for quite lengthy sampling tubes and fairly large flow velocities.

VII. The $\gamma(\omega)$ Function. Two Numerical Examples

The basic information required to determine $\gamma(\omega, a, D, U)$, and hence the damping of the spectrum, is available in the earlier sections, the required values of $\mathscr{R}[a_0(\Omega, \Upsilon)]$ being given in Philip (1963*b*, 1963*c*). In the present section we consider some numerical examples illustrating the behaviour of $\gamma(\omega)$ and the way in which this function determines the characteristics of the sampling method.

(*a*) *Example 1: Gas Sampling*

We use the previous values of D and ν. We take $U = 10$ cm/s and limit our interest to the spectral band $\omega = 10^{-2}$ to 10 rad/s. We wish to find $\gamma(\omega)$ for tubes of radius 0·3, 1, and 3 cm.

First, we note that the three cases we consider yield $\Upsilon = 21$, 71, and 214. For $\Upsilon > 10$, we may safely use the large Péclet number solution (Philip 1963*b*), compare Section IV. The corresponding Reynolds numbers, based on tube diameter, are 40, 133, and 400. The flow is therefore laminar. We note also that (4.1) implies that our results are valid for tube lengths large compared with 0.64, 7, and 64 cm, respectively. Usually the sampling tubes will be much longer than this.

Figure 1 presents the variation of γ with ω for the three values of a. It will be observed that in all three cases the tube operates as a high frequency filter (cf. Section VI), the attenuation constant increasing markedly with ω; and that this effect becomes somewhat reduced as a increases.

(*b*) *Example 2: Water Sampling*

We use the previous values of D and ν. We take $U = 0{\cdot}1$ cm/s and the spectral band $\omega = 10^{-2}$ to 10 rad/s. We wish to find $\gamma(\omega)$ for tubes of radius 0·03, 0·1, and 0·3 cm.

The three cases we consider yield $\Upsilon = 300$, 1000, and 3000—values for which we may use the large Péclet number solution. The corresponding Reynolds numbers

are 0·6, 2, and 6, so the flow is laminar. (4.1) indicates that the results are valid for tube lengths large compared with 0·9, 10, and 90 cm respectively.

Figure 2 presents the variation of γ with ω for the three values of a. We observe that, in this case, the 0·03 cm diameter tube operates as a high frequency filter, and that increasing the tube diameter tends to eliminate spectral distortion, γ being essentially independent of ω when $a = 0{\cdot}3$ cm.

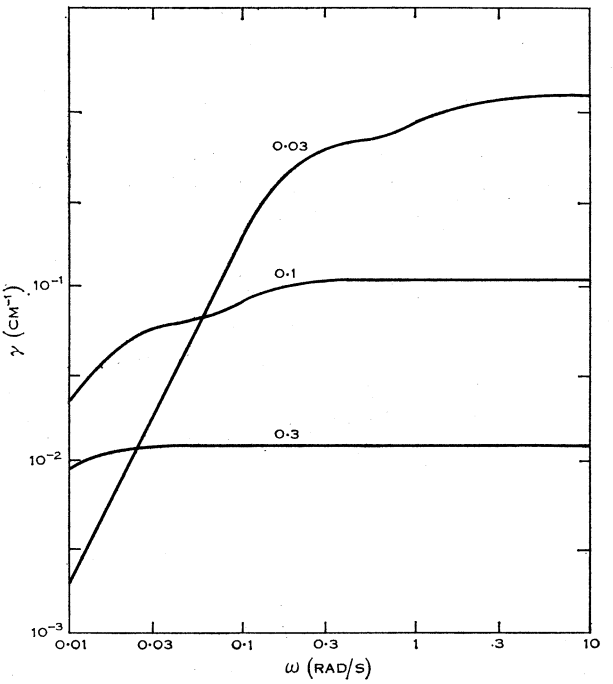

Fig. 2.—Example 2. Logarithmic plot of attenuation function γ cm^{-1} against frequency ω rad/s. Numerals on the three curves indicate sampling tube radius in centimetres.

VIII. Discussion

There are, clearly, considerable degrees of freedom in the use of continuous tube sampling. For both gas and water sampling, the method will operate as a high frequency filter under a wide range of conditions. The method can also be used fairly readily for water sampling where it is wished to keep spectral distortion to a minimum. However, gas sampling with small spectral distortion can be realized only under rather special conditions. In any case, the inlet spectrum can be recovered from the outlet spectrum, at least in principle, by the use of the analysis developed here.

IX. Acknowledgment

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