OPTIMUM THERMAL EFFICIENCY IN COUNTERCURRENT HEAT EXCHANGERS

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Summary

When a heat exchanger operates on fixed hot and cold fluid streams and is adjusted to give optimum efficiency, the measure of irreversibility produced by the exchange process is a sum of two terms, one of which is proportional to the pivotal temperature difference and the other to the terminal temperature difference.

I. Introduction

In the preceding paper, the authors (Bosworth and Groden 1964, hereafter referred to as paper I) discussed the value of Keenan's (1951) *availability* in considering the Kelvin efficiency of countercurrent heat exchangers and have derived a number of thermodynamical properties of such interacting systems measured at the so-called pivotal temperature of the exchanger or at the point at which the temperature difference becomes stationary. It is the aim of the present paper to devise an expression for the optimum attainable efficiency, subject to the fixed properties of the interacting fluids, and, for that purpose, it will first be necessary to find conditions for the stationary value of a quantity closely related to the efficiency and which may be expressed as a function of stream variables, which in this case are the stream flow rates.

II. The Irreversibility

The quantity most suitable for this purpose is the *irreversibility*, which has been given a definite metrical significance by Keenan (1932) and used extensively by Bruges (1959). The irreversibility ΔI, measured in units of energy, is the product of the sink temperature by the entropy increment due to irreversible change (ΔS_{irr}). In the system considered above, $\Delta S^{\mathrm{irr}} = \Delta S_2 - \Delta S_1$, where ΔS_1 is the entropy decrease of the donor system and ΔS_2 the corresponding entropy increase of the acceptor system, so that the net irreversibility ΔI becomes

$$\begin{aligned}\Delta I &= T_0 \Delta S_{\mathrm{irr}} \\ &= T_0(\Delta S_2 - \Delta S_1),\end{aligned} \tag{1}$$

with T_0 the sink temperature assumed constant while the irreversibility entropy is passed to it. The total irreversibility I is then

$$I = \int T_0 \, \mathrm{d}S$$

taken over the adiabatic path so that $\mathrm{d}S$ refers only to the internal entropy equation.

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If the entropies in equation (1) are total entropies, the quantity ΔI is the measure of the net irreversibility. If, as is usual in engineering practice, the ΔS's are specific entropies, ΔI becomes a specific irreversibility measured in units of energy per unit mass, but if the ΔS's, as it is the common practice in chemical applications, are molar entropies, ΔI becomes a molar irreversibility measured in units of energy per mole.

For an inexhaustible sink at a temperature T_0, the admission at pressure p and temperature T of an element of enthalpy $\mathrm{d}H$ gives rise to the development of irreversibility $\mathrm{d}I$,

$$\mathrm{d}I = T_0\,\mathrm{d}S,$$

and if $\mathrm{d}B = \mathrm{d}H - T_0\,\mathrm{d}S$, where B is "availability" (see paper I), then

$$\mathrm{d}I = \mathrm{d}H - \mathrm{d}B,$$

and on integration

$$\Delta I = \Delta H - \Delta B \tag{2}$$

for a finite temperature change of the source.

For an exhaustible source, the admission of enthalpy $\mathrm{d}H$ takes place at a temperature T and a sink temperature $T_0 + \alpha T$, a linear function of T, where α is the ratio of the water equivalents of the source and the sink and may be either positive or negative. The increment of irreversibility then becomes

$$\Delta I = \int (T_0 + \alpha T)\,\frac{\mathrm{d}H}{T},$$

$$\Delta I = (1+\alpha)\Delta H - \Delta B, \tag{3}$$

There are equivalent expressions for the measured irreversibility in flow and no-flow systems given in Bruges (Chapter 4, loc. cit.).

In each case, however, minimum values of the irreversibility change at constant enthalpy change are associated with maximum values in the availability change. The performance of heat exchangers, treated largely qualitatively by Bruges (Chapter 6, loc. cit.), is considered in terms of what is called the "effectiveness" of the exchange process. Effectiveness in this content is defined by

$$\text{Effectiveness} = \Delta B_2 / \Delta B_1,$$

and approaches unity when the availability ratio of the donor stream tends to that of the acceptor stream. Effectiveness thus takes a maximum value when irreversibility takes a minimum value.

III. Properties of a Countercurrent Exchanger

Paper I discussed the thermodynamical properties of two systems, one, the donor stream, entering at a temperature T_∞ and leaving at a temperature $T_0 + \theta_0$ at a rate g_1 units of mass per unit time and having a specific heat c_1 which was a function of stream temperature $(T+\theta)$. The second, or acceptor stream entered at a lower temperature T_0 and left at a temperature $T_\infty-\theta_\infty$. The flow rate of this stream was g_2 units of mass per unit time and had a specific heat c_2 which was a function of the stream temperature T. The temperature difference θ (greater than zero) between the streams could pass one or more stationary values θ^* at a lower stream temperature T^*, the so-called pivotal temperature. Paper I showed that

$$\frac{g_2}{g_1} = \frac{c_1^*}{c_2^*} = \frac{\Delta H_1}{\Delta H_2} = \frac{\Delta H_1^*}{\Delta H_2^*}, \tag{4}$$

where

c_1^* = specific heat of the upper stream at temperature $(T^*+\theta^*)$;

c_2^* = specific heat of the lower stream at temperature T^*;

$\Delta H_1 = \int_{T_0+\theta_0}^{T\infty} c_1 \mathrm{d}(T+\theta)$ = total specific enthalpy change of the upper stream;

$\Delta H_1^* = \int_{T_0+\theta_0}^{T^*+\theta^*} c_1 \mathrm{d}(T+\theta)$ = specific enthalpy change of upper stream in passing from the sink temperature to the pivotal temperature.

ΔH_2 and ΔH_2^* were the corresponding enthalpy changes for the lower stream.

It was also found that

$$\theta = \frac{g_2}{g_1}\int_{T^*}^{T} \frac{c_2}{c_1}\,\mathrm{d}T - T + Y^* \geqslant 0, \tag{5}$$

and

$$\theta = Y - T^* - \frac{g_1}{g_2}\int_{Y^*}^{Y} \frac{c_1}{c_2}\,\mathrm{d}Y \geqslant 0, \tag{6}$$

with Y $(= T+\theta)$ and Y^* $(= T^*+\theta^*)$ representing temperatures in the upper stream.

Now, if we set the upper limits at the sink temperature, equations (5) and (6) yield respectively

$$\theta_0 = \frac{g_2}{g_1}\int_{T^*}^{T_0} \frac{c_2}{c_1}\,\mathrm{d}T - T_0 + Y^*, \tag{7}$$

and

$$0 = T_0 - T^* - \frac{g_1}{g_2}\int_{Y^*}^{Y_0} \frac{c_1}{c_2}\,\mathrm{d}Y. \tag{8}$$

If a total enthalpy element $g_1\,\mathrm{d}H_1$ is lost in unit time from the upper stream at a temperature $Y(= T+\theta)$, then the total enthalpy gain per unit time by the cooler stream at a temperature T amounts to $g_2\,\mathrm{d}H_2$; the irreversibility element $\mathrm{d}J$ produced *in unit time* is therefore

$$\mathrm{d}J = T\left(g_2\frac{\mathrm{d}H_2}{T} - g_1\frac{\mathrm{d}H_1}{T+\theta}\right),$$

where the g_1 and g_2 are the mass velocities of the stream.

The total irreversibility *per unit mass* of the upper stream then becomes

$$I = \frac{1}{g_1}\int \mathrm{d}J = \frac{g_2}{g_1}\int_{T_0}^{T\infty-\theta\infty} \frac{\mathrm{d}H_2}{\mathrm{d}T}\,\mathrm{d}T - \int_{T_0+\theta_0}^{T\infty} \frac{T}{Y}\frac{\mathrm{d}H_1}{\mathrm{d}Y}\,\mathrm{d}Y$$

$$= \frac{g_2}{g_1}\,\Delta H_2 - \Delta H_1 + \int_{T_0+\theta_0}^{T\infty} \frac{\theta}{Y}\frac{\mathrm{d}H_1}{\mathrm{d}Y}\,\mathrm{d}Y.$$

In view of (4), the first two terms are equal and

$$I = \int_{T_0+\theta_0}^{T\infty} \frac{\theta}{Y}\,c_1\,\mathrm{d}Y. \tag{9}$$

IV. Optimum Conditions for Operation

To find the optimum conditions for operation, it is necessary to substitute for the θ in equation (9) and to find the value of g_1/g_2 (the only independent variable left in the system) which makes the value of I a minimum. Two values of θ may be used: that in equation (5) and that in (6). The former gives

$$I = \frac{g_2}{g_1}\int_{T_0+\theta_0}^{T\infty} \frac{c_1}{Y}\left\{\int_{T^*}^{T}\frac{c_2}{c_1}\,\mathrm{d}T\right\}\mathrm{d}Y - \int_{T_0+\theta_0}^{T\infty}\frac{T}{Y}\,c_1\,\mathrm{d}Y + Y^*\int_{T_0+\theta_0}^{T\infty}\frac{c_1}{Y}\,\mathrm{d}Y;$$

the second term becomes the difference between the specific enthalpy and the irreversibility of the upper stream and the last integral becomes the specific entropy of the upper stream; hence

$$0 = \frac{g_2}{g_1}\int_{T_0+\theta_0}^{T\infty} \frac{c_1}{Y}\left\{\int_{T^*}^{T}\frac{c_2}{c_1}\,\mathrm{d}T\right\}\mathrm{d}Y - \Delta H_1 + Y^*\,\Delta S_1, \tag{10}$$

where

$$\Delta S_1 = \int_{T_0+\theta_0}^{T\infty}\frac{\mathrm{d}H_1}{Y}.$$

Substitution of equation (6) in equation (9) likewise gives

$$I = \Delta H_1 - T^*\,\Delta S_1 - \frac{g_1}{g_2}\int_{T_0+\theta_0}^{T\infty}\frac{c_1}{Y^1}\left\{\int_{Y^*}^{Y^1}\frac{c_1}{c_2}\,\mathrm{d}Y\right\}\mathrm{d}Y^1. \tag{11}$$

For the optimum conditions of heat exchange, the irreversibility I must be a minimum. To achieve this, we shall now differentiate equations (10) and (11) with respect to g_1, keeping g_2 constant. Putting $(\partial I/\partial g_1)_{g_2} = 0$, and recalling the facts that θ_0, T^*, and θ *but not* θ^* are functions of the g's, we first obtain from (10)

$$-Y^*\frac{\partial \Delta S_1}{\partial g_1} - \Delta S_1\frac{\partial T^*}{\partial g_1} + \frac{\partial \Delta H_1}{\partial g_1} = -\frac{g_2}{g_1^2}\int_{Y_0}^{T\infty}\frac{c_1}{Y}\left\{\int_{T^*}^{T}\frac{c_2}{c_1}\,\mathrm{d}T\right\}\mathrm{d}Y$$

$$+\frac{g_2}{g_1}\frac{\partial}{\partial g_1}\left[\int_{T_0+\theta_0(g_1)}^{T\infty}\frac{c_1}{Y}\left\{\int_{T^*(g_1)}^{T}\frac{c_2}{c_1}\,\mathrm{d}T\right\}\mathrm{d}Y\right],$$

where $Y_0 = T_0+\theta_0$,

or, using equations (7), (10), and the definitions of ΔH_1 and ΔS_1, we find that

$$\Delta H_1 = Y^* \Delta S_1, \tag{12}$$

which gives a measure of the pivotal temperature. Corresponding differentiation of equation (11) yields for $\partial I/\partial g_1 = 0$ the minimum value $I_{\text{min.}}$ as

$$I_{\text{min.}} = \Delta H_1 - T^* \Delta S_1 + g_1 c_{1,0} \frac{\theta_0}{Y_0} \left(\frac{\partial \theta_0}{\partial g_1}\right)_{g_2}, \tag{13}$$

or, in view of (12),

$$I_{\text{min.}} = \theta^* \Delta S_1 + g_1 c_{1,0} \frac{\theta_0}{Y_0} \left(\frac{\partial \theta_0}{\partial g_1}\right)_{g_2}, \tag{14}$$

where $c_{1,0}$ is the specific heat of the upper stream at the temperature Y_0. A definite value can be given to equation (14) for the minimum irreversibility and thus for the maximum effectiveness of the exchange process once a definite value of $(\partial \theta_0/\partial g_1)_{g_2}$ is assigned.

For this purpose, we use equation (4) and differentiate with respect to g_1, keeping g_2 constant. We have

$$\frac{\partial}{\partial g_1}(g_1 \Delta H_1^*) = \frac{\partial}{\partial g_1}(g_2 \Delta H_2^*),$$

giving

$$g_1 c_{1,0} \left(\frac{\partial \theta_0}{\partial g_1}\right)_{g_2} = \Delta H_1^*. \tag{15}$$

Substitution of equation (15) into (14) finally gives

$$I_{\text{min.}} = \theta^* \, \Delta S_1 + \frac{\theta_0}{T_0 + \theta_0} \Delta H_1^*, \tag{16}$$

and the minimum irreversibility is the sum of two contributions, one coming from the pivotal temperature difference and one from the initial temperature difference.

V. An Example without any Phase Change

As a simple example, consider the case in which both streams have isopiestic heat capacities which are linear in temperature throughout the range of the exchanger:

$$c_1 = c_{10}\{1 + a(Y - Y_0)\},$$

$$c_2 = c_{20}\{1 + b(T - T_0)\},$$

where a and b are small in comparison with the reciprocal of T.

Omitting powers and products of the small temperature differences (θ_0, θ^*, θ_∞) and of the temperature coefficients a and b we get

$$\Delta H_1 = c_{10}(T_\infty - T_0 - \theta_0) + \tfrac{1}{2}a(T_\infty - T_0)^2,$$

$$\Delta H_2 = c_{20}(T_\infty - T_0 - \theta_\infty) + \tfrac{1}{2}b(T_\infty - T_0)^2,$$

$$\Delta H_1^* = c_{10}(T^* - T_0 + \theta^* - \theta_0) + \tfrac{1}{2}a(T^* - T_0)^2,$$

$$\Delta H_2^* = c_{20}\,(T^*-T_0)+\tfrac{1}{2}b(T^*-T_0)^2,$$

$$\Delta S_1 = c_{10}\,(1-aT_0)\,\ln\{T_\infty/(T+\theta_0)\}+a(T_\infty-T_0).$$

From $\Delta H_1^*/\Delta H_2^* = c_1^*/c_2^* = \Delta H_1/\Delta H_2$ we get

$$\theta^*-\theta_0 = \tfrac{1}{2}(a-b)(T^*-T_0)^2, \tag{17}$$

and

$$\theta_\infty-\theta_0 = \tfrac{1}{2}(b-a)(T_\infty-T_0)(T_\infty+T_0-2T^*). \tag{18}$$

The value of T^* for this system may be obtained from equation (12) as

$$T^*+\theta^* = \frac{(T_\infty-T_0-\theta_0)+\frac{1}{2}a(T_\infty-T_0)^2}{\ln(T_\infty/T_0)-\theta_0/T_0+a\{T_\infty-T_0-T_0\ln(T_\infty/T_0)\}}$$

$$\simeq \frac{T_\infty-T_0}{\ln(T_\infty/T_0)}\left(1+\frac{\theta_0}{2T_0}\right), \tag{19}$$

or the pivotal temperature is slightly greater than the logarithmic mean temperature of the system, provided θ_0 has a finite value>0.

The attainable value of θ^* depends on the relative values of a and b. The value of θ attains a minimum which may be allowed to vanish at $T = T^*$ when the inequality $b>a$ holds.

Equations (17)–(19) then become

$$\theta_0 = \tfrac{1}{2}(b-a)(T^*-T_0)^2$$

$$\simeq \tfrac{1}{8}(b-a)(T_\infty-T_0)^2,$$

$$\theta_\infty = \tfrac{1}{2}(b-a)(T_\infty-T^*)^2$$

$$\simeq \tfrac{1}{8}(b-a)(T_\infty-T_0)^2.$$

Therefore

$$\theta_0 = \theta_\infty,$$

$$T^* = \frac{(T_\infty-T_0)}{\ln(T_\infty/T_0)}\left[1+\frac{\frac{1}{16}(b-a)(T_\infty-T_0)^2}{T_0}\right].$$

The minimum obtainable irreversibility per unit mass then becomes

$$I_{\text{min.}} = \frac{\theta_0}{T_0+\theta_0}\,\Delta H_1^*$$

$$\simeq \tfrac{1}{16}c_{10}(b-a)(T_\infty-T_0)^3/T_0, \tag{20}$$

which accordingly vanishes only when $b = a$, i.e. when the two interacting streams have specific heats with the same logarithmic temperature coefficients, or alternatively, when the overall temperature difference across the exchanger $(T_\infty-T_0)$ is vanishingly small.

Take now the opposite case in which

$$a > b.$$

The θ versus T curve then gives a maximum value θ^* at the pivotal temperature T^*. Optimum efficiency then demands that either θ_0 or θ_∞ should vanish. Detailed working shows that, in all normal cases in which a and b are small in comparison with the reciprocal of the temperature, the maximum efficiency is obtained when θ_∞ vanishes. Under these conditions equations (17)–(19) give

$$\theta_0 = \tfrac{1}{8}(a-b)^2(T_\infty - T_0)^3,\dagger$$

$$\theta^* = \tfrac{1}{8}(a-b)(T_\infty - T_0)^2,$$

$$T^* = \frac{T_\infty - T_0}{\ln(T_\infty/T_0)} - \tfrac{1}{8}(a-b)(T_\infty - T_0)^2,$$

in which now T^* is slightly less than the logarithmic mean of T and T_0.

From these figures, the minimum irreversibility per unit mass may be derived: from equation (16)

$$I_{\text{min.}} = \theta^* \, \Delta S_1 + \frac{\theta_0}{T_0+\theta_0} \, \Delta H_1^*,$$

and, since the second term is negligible in comparison with the first one, we have

$$\begin{aligned} I_{\text{min.}} &\simeq \tfrac{1}{8}c_{10}(a-b)(T_\infty - T_0)^2 \ln(T_\infty/T_0) \\ &\simeq \tfrac{1}{4}c_{10}(a-b)\,(T_\infty - T_0)^3/(T_\infty + T_0), \end{aligned} \tag{20a}$$

which is always positive and tends to zero only when $a \to b$ or $T_0 \to T_\infty$, namely, when the temperature coefficients of the two streams take the same value or when the overall temperature difference $(T_\infty - T_0)$ is an infinitesimal quantity.

To gain perhaps a clearer interpretation of the meaning of the finite positive irreversibilities in heat exchange processes let us allow a gas A (with logarithmic temperature coefficient of heat capacity as a) interact with a different gas B with a corresponding property b. Later let the heater gas B interact in another heat exchanger with a different sample of gas A. Then, no matter how efficient the exchange processes may be made, the second sample of gas A will never be heated to the input temperature of the first sample of gas A and will, in fact, always be cooler by at least $\tfrac{1}{8}(b-a)(T_\infty - T_0)^2$, while the maximum discharge temperature of the first sample must exceed the source temperature of the second sample by at least a like amount.

VI. Examples with Phase Changes

In any examples in which first-order changes in phase occur, discontinuities appear in the specific heat curves. As discussed in paper I this means that the ratio c_1^*/c_2^* passes through a wide range at a steady value of T or $(T+\theta)$ depending on whether the upper or lower stream is subject to a phase change. Consequently, provided only the enthalpy change associated with the phase change is not trivial in comparison with the total enthalpy change in the exchanger, the pivotal temperature T^* is equal to the temperature of the phase change, say T_b, and completely independent of the flow rate.

† A small quantity of a lower order than that of θ^*.

The measure of the minimal irreversibility is accordingly given by

$$I_{\text{min.}} = \frac{\theta_0}{T_0+\theta_0}\,\Delta H_1^*, \tag{21}$$

as, for reasons advanced in paper I, θ^* in these examples can be made to vanish.

Suppose now we take a simple example in which the upper stream enters as a vapour at T_∞ having a constant specific heat c_1 and let the vapour condense at T_b and be discharged mainly as liquid at the temperature T_b. Let L be the latent heat of condensation at $T = T_b$.

Let the lower stream enter at $T_0 < T_b$ and let this stream have a constant specific heat c_2. Then the appropriate functions for calculating the minimal irreversibility are as follows:

$$\begin{aligned}
\Delta H_1^* &= L,\\
\Delta H_1 &= L+c_1(T_\infty - T_b),\\
\Delta S_1 &= L/T_b + c_1 \ln(T_\infty/T_b),\\
\Delta H_2 &= c_2(T_\infty - \theta_\infty - T_0),\\
\Delta H_2^* &= c_2(T_b - T_0),\\
\theta_0 &= T_b - T_0.
\end{aligned}$$

Equation (21) now reduces to

$$I_{\text{min.}} = (1-T_0/T_b)\,\Delta H_1^*, \tag{22}$$

while the heat balance condition gives

$$1+c_1(T_\infty - T_b)/L = (T_\infty - \theta_\infty - T_0)/(T_b - T_0),$$

or

$$\theta_\infty = T_\infty - T_b - (c_1/L)(T_\infty - T_b)(T_b - T_0), \tag{23}$$

which now may be used to express θ_∞ in terms of the terminal temperature and the transition temperature. This means that, if we wish to remove, under optimum conditions, all the heat of transition, then there is a prescribed low temperature T_0 for any entrance high temperature T_∞, the relative flow rates of the stream being prescribed by the requirements for optimum thermal efficiency.

VII. References

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