NOTE ON COWLING'S METHOD IN THE THEORY OF NON-RADIAL OSCILLATIONS OF MASSIVE STARS*

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The study of non-radial adiabatic oscillations of stars requires the solution of a complicated system of differential equations and the extensive use of numerical methods.

Recently, the basic equations of the problem have been solved numerically in the case of massive stars by Smeyers (1963) and by Van der Borght and Wan (1965), and results are now available to test the accuracy of a perturbation method proposed by Cowling (1941).

The comparison carried out in the present paper supplements the work done by Kopal (1949) in the case of polytropes and by Sauvenier-Goffin (1951) in the case of homogeneous stars.

The results quoted here are based on a model of a massive star of mass $10M_{\odot}/\mu^2$ derived by Van der Borght (1964), $M_{\odot}$ being the mass of the Sun and μ the molecular weight of the star material.

The principal characteristics $\bar{p}(x)$, $\bar{t}(x)$, $\bar{m}(x)$, and $\beta(x)$ of the model are functions of $x = r/R$, these variables being defined by

$$\bar{p} = \frac{R^4\mu^4 p}{M_{\odot}^2},$$

$$\bar{t} = \frac{R\mu T}{M_{\odot}},$$

$$\bar{m} = \frac{\mu^2 M_r}{M_{\odot}},$$

where

T = temperature, r = distance to centre,

p = pressure, M_r = mass inside a sphere of radius r,

R = radius, β = ratio of gas pressure to total pressure.

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The basic differential equations of the problem can be written (Ledoux and Walraven 1958)

$$\frac{\mathrm{d}\xi}{\mathrm{d}x}+\frac{(l+1)\xi}{x} = \frac{G\beta\bar{m}}{\gamma\mathscr{R}\bar{t}x^2}\xi+\left(\frac{l(l+1)}{\omega^2}-\frac{x^2\beta\mathscr{M}G}{\gamma\mathscr{R}\bar{t}}\right)\frac{\eta}{x}+\frac{l(l+1)}{\omega^2}\frac{\Phi}{x},$$

$$\frac{\mathrm{d}\eta}{\mathrm{d}x}+\frac{l\eta}{x} = -A\eta+\left(\omega^2+\frac{\bar{m}A}{\mathscr{M}x^2}\right)\frac{\xi}{x}-\frac{\mathrm{d}\Phi}{\mathrm{d}x}-l\frac{\Phi}{x},$$

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}x^2}+\frac{2(l+1)}{x}\frac{\mathrm{d}\Phi}{\mathrm{d}x} = \frac{4\pi\beta\bar{p}G}{\mathscr{R}\bar{t}}\left(\frac{\beta\eta}{\mathscr{R}\gamma\bar{t}}-\frac{A}{\mathscr{M}Gx}\xi\right),$$

where

$$\xi = \frac{r^2\,\delta r}{x^{l+1}}, \qquad \eta = \frac{\mu^2R^4}{\mathscr{M}GM_\odot x^l}\frac{p'}{\rho}, \qquad \Phi = \frac{\mu^2R^4}{\mathscr{M}GM_\odot x^l}\phi', \qquad \mathscr{M} = \frac{\mu^2M}{M_\odot},$$

M being the mass of the star. In these equations, δr is the radial component of the displacement, p' and ϕ' are the Euler variations of the pressure and gravitational potential respectively, and also

(i) $\omega^2 = \dfrac{R^3\sigma^2}{GM}$, where $2\pi/\sigma$ is the period of the oscillation;

(ii) $\gamma = \beta+\dfrac{2(4-3\beta)^2}{3\beta+24(1-\beta)}$;

(iii) l is the order of the spherical harmonic used;

(iv) $A = \dfrac{1}{\beta}\dfrac{\mathrm{d}\beta}{\mathrm{d}x}+\dfrac{1}{\bar{p}}\dfrac{\mathrm{d}\bar{p}}{\mathrm{d}x}-\dfrac{1}{\bar{t}}\dfrac{\mathrm{d}\bar{t}}{\mathrm{d}x}-\dfrac{1}{\gamma\bar{p}}\dfrac{\mathrm{d}\bar{p}}{\mathrm{d}x}$, which vanishes in the case of adiabatic equilibrium, i.e. in the core of the star.

The above system of differential equations was solved in the case where $l = 2$, subject to the boundary conditions

$$\xi = \eta, \qquad \frac{\mathrm{d}\Phi}{\mathrm{d}x}+(2l+1)\Phi = 0.$$

at the boundary where $x = 1$.

The results of the integration are given in Table 1. Column 2 contains the values of the eigenvalues ω_{m} obtained under the assumption that $\phi' = 0$, i.e. that the variations in the gravitational potential can be neglected, and column 3 gives the values of the eigenvalues ω_{e} obtained by the numerical integration of the complete system of differential equations.

Cowling's method consists in applying a correction to the eigenvalues obtained under the assumption that $\phi' = 0$ in order to take into account the perturbations of the gravitational potential. This method leads to the following expression for the

TABLE 1

COMPARISON BETWEEN EXACT AND CORRECTED EIGENVALUES FOR VARIOUS MODES OF NON-RADIAL OSCILLATION OF A STAR OF MASS $10M_{\odot}/\mu^2$

Mode	ω_m	ω_e	ω_c	$\dfrac{\omega_c - \omega_e}{\omega_e} \times 100$
f	2·6120	2·2530	2·3610	4·793
p_1	3·8847	3·6017	3·6464	1·243
p_2	5·1481	4·8667	4·9039	0·766
p_3	6·3911	6·1141	6·1448	0·502
p_4	7·6217	7·3547	7·3784	0·323
g_1	0·7811	0·7759	0·7767	0·098
g_2	0·5009	0·4998	0·4999	0·019
g_3	0·3708	0·3704	0·3704	0·006
g_4	0·2952	0·2950	0·2950	0·003

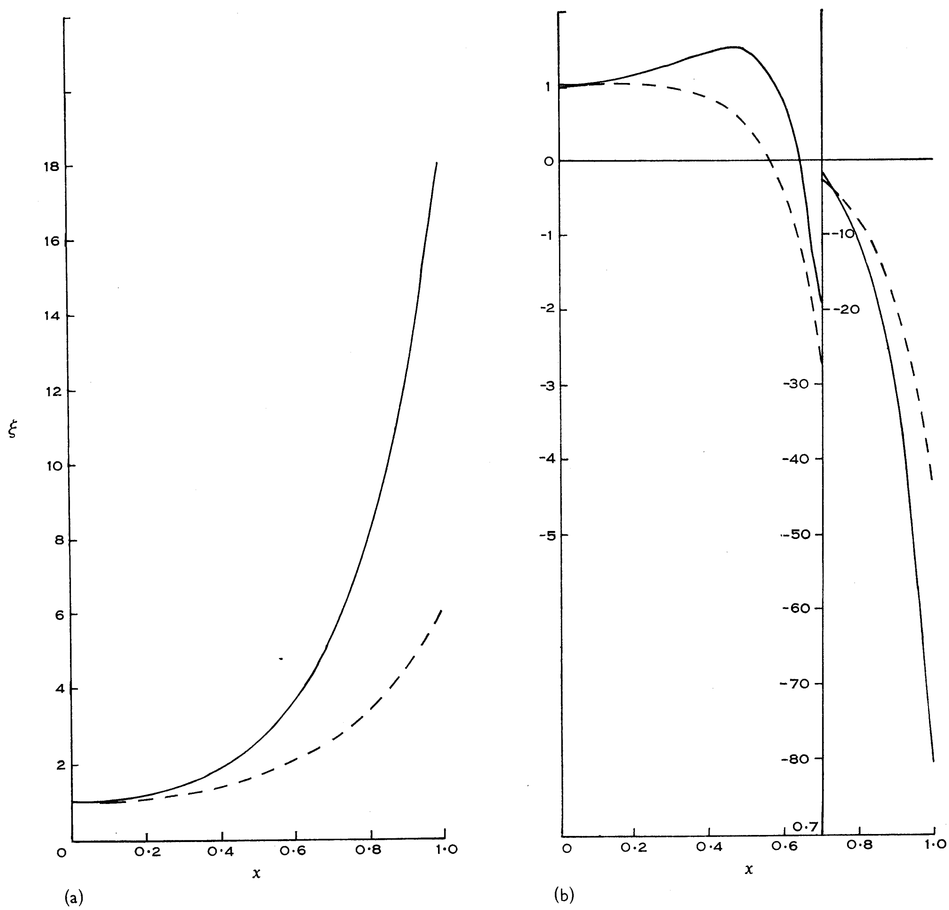

Fig. 1.—Values of the eigenfunction ξ, in the case where $l = 2$, for a star of mass $10M_{\odot}/\mu^2$ oscillating in (a) the fundamental mode (b) the first p-mode. ——— $\phi' = 0$; - - - $\phi' \neq 0$.

corrected eigenvalue ω_{c}:

$$\omega_{\mathrm{c}}^2 = \omega_{\mathrm{m}}^2 + \frac{B}{D},$$

where

$$B = \int_0^1 \phi \left\{ \frac{1}{\gamma} \frac{\beta \bar{p} x^2}{\mathscr{R} \bar{t}} \left(\frac{\beta \lambda}{\mathscr{R} \bar{t} M_{\odot}} + \frac{u}{x^2} \frac{1}{\bar{p}} \frac{\mathrm{d}\bar{p}}{\mathrm{d}x} \right) - \frac{u}{\mathscr{R}} \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\beta \bar{p}}{\bar{t}} \right) \right\} \mathrm{d}x,$$

$$D = G \mathscr{M} M_{\odot} \int_0^1 \left(\frac{\beta \bar{p}}{\mathscr{R} \bar{t}} \frac{u^2}{x^2} + \frac{l(l+1)}{G^2 \mathscr{M}^2 M_{\odot}^2} \frac{\beta \bar{p}}{\mathscr{R} \bar{t}} \frac{\lambda^2}{\omega_{\mathrm{m}}^4} \right) \mathrm{d}x,$$

$$\phi = -\frac{4\pi G}{(2l+1)} \left(x^l \int_x^{\infty} \rho_{\mathrm{m}}'(x)\, x^{1-l}\, \mathrm{d}x + x^{-1-l} \int_0^x \rho_{\mathrm{m}}'(x)\, x^{l+2}\, \mathrm{d}x \right),$$

$$\rho_{\mathrm{m}}' = \frac{\beta M_{\odot} \bar{p}}{\gamma \bar{t} \mathscr{R}} \left(\frac{\beta \lambda}{\mathscr{R} \bar{t} M_{\odot}} + \frac{u}{x^2} \frac{1}{\bar{p}} \frac{\mathrm{d}\bar{p}}{\mathrm{d}x} \right) - \frac{u}{x^2 \mathscr{R}} \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\beta \bar{p}}{\bar{t}} \right) M_{\odot}.$$

The values of $u = \xi x^{l+1}$ and $\lambda = \eta \mathscr{M} G M_{\odot} x^l$ are obtained from the integration of the basic differential equations under the assumption that $\phi' = 0$.

The corrected values ω_{c} obtained by this method are listed in column 4 of the table, and in column 5 a comparison is made with the exact values ω_{e}.

It is seen from the table that Cowling's method gives good approximations to the correct eigenvalues, except for the fundamental mode and the first p-mode of oscillation. This is due to the fact that, for these two modes, the corresponding eigenfunctions differ by a fairly large amount. This can be seen in Figure 1, in which we give the eigenfunctions ξ when $\phi' = 0$ and $\phi' \neq 0$.

For higher modes of oscillation, the Cowling method gives the various eigenvalues to a high degree of accuracy, in particular in the case of the g-modes.

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