# HEAT FLOW IN AN INFINITE REGION BOUNDED INTERNALLY BY A CIRCULAR CYLINDER WITH FORCED CONVECTION AT THE SURFACE

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*Summary*

Numerical values are given for the surface temperature of an infinite region bounded internally by a circular cylinder of radius $a$, which is initially at constant temperature and has heat transfer with constant coefficient $H$ at its surface. The surface temperature is tabulated for $\tau = 0\cdot01(0\cdot01)0\cdot1(0\cdot1)1(1)10(10)100(100)1000(1000)10\,000$ and $\beta = 0\cdot1(0\cdot1)1(1)10(10)100$, and at a few intermediate values where $\tau = \alpha t/a^2$ and $\beta = aH/k$, with $k$, $\alpha$, and $t$ being the thermal conductivity, the diffusivity of the material, and the time respectively. The total heat loss from the surface up to time $t$ is also tabulated. Approximations for the functions tabulated are discussed, as well as the temperature at any point in the region. A brief account of the history of the problem and its applications is given.

## Introduction

The problem of radial diffusion or flow of heat in an infinite region bounded internally by a circular cylinder arises in many practical contexts, typical of which are heat transfer from a buried pipe (Ingersoll *et al.* 1950), flow of water or petroleum to a well (van Everdingen and Hurst 1949), and diffusion from a cylindrical structure such as a capillary or a nerve fibre (Hill 1965). A most important practical application is to the problem of extraction of heat from tunnels and drives in mines.

Unlike the corresponding problems in linear flow or radial flow from a sphere, the results for the cylindrical case are expressed as rather sophisticated integrals and the lack of tables of these has led to the use of simple approximations with little prospect of checking their accuracy. In the thermal case, the three simplest linear boundary conditions at the surface are: (1) prescribed temperature, (2) prescribed flux of heat, and (3) forced convection or a thin boundary layer at the surface. Of these, numerical information about (1) and (2) has been available for some time, but often neither corresponds to the practical situation, which is much better described by (3). In hydrology and petroleum technology, the simplest result for the problem of flow to a well is the exponential integral solution for the continuous line source (Carslaw and Jaeger 1959, Section 10.4 (5)), which has been used by many authors from Theis (1935) onwards. When it is desired to generalize this to finite radius for a well, van Everdingen and Hurst (1949) give extensive tables for the boundary conditions (1) and (2) above, both of which represent limiting conditions that may be far from the actual ones. In physiology, the importance of the boundary layer has been recognized since Weber's idealization of the kernleiter (see Weinberg 1941). The situation is perhaps most acute in the mining case, in which the boundary

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condition is definitely one of forced convection at the surface; for this reason, the problem will be stated below in the form in which it occurs in mine ventilation.

For the boundary condition (1), the problem in which the initial temperature of the solid is constant and the surface is subsequently maintained at a different constant temperature was first solved by Nicholson (1921) (see Carslaw and Jaeger 1959). Numerical values of this solution are available: Jaeger and Clarke (1942) tabulated the heat flux at the surface; Goch and Patterson (1940) and van Everdingen and Hurst (1949) tabulated the heat flux and the total heat flow from the surface; Jaeger (1956) tabulated the radial temperature distribution.

The problem of the region bounded internally by a circular cylinder with forced convection at its surface and constant initial temperature was first solved by Goldstein (1932); a few numerical values for the surface temperature were given by Jaeger (1942). These values were reproduced by Carslaw and Jaeger (1959) and were extended by Hitchcock and Jones (1958). Additional values were published by Boldizsár (1960) and approximations to them were discussed by Jaeger and le Marne (1963). In view of the importance and difficulty of the problem, Jones (1961) went so far as to determine temperature by measurement under accurately controlled conditions in a large concrete cylinder.

The object of the present paper is to tabulate values of the surface temperature and heat flux and to discuss the validity of various approximations for them and for the radial temperature distribution.

## Mathematical Solution

The problem is stated in the form in which it arises in mine ventilation and the notation used is as follows.

$\theta$, temperature
$\theta_0$, constant initial temperature of solid
$\theta_1$, temperature (constant) of material into which forced convection takes place
$\theta_s$, surface temperature of cylindrical opening
$f_s$, surface flux of heat
$Q_s$, total flow across unit area of the surface up to time $t$
$a$, radius of cylindrical opening
$r$, distance from axis of cylindrical opening
$\alpha$, thermal diffusivity of solid
$k$, thermal conductivity of solid
$t$, time during which convection has occurred
$H$, surface heat transfer coefficient

$$\tau = \alpha t/a^2, \tag{1}$$

$$R = r/a, \tag{2}$$

$$\beta = aH/k. \tag{3}$$

The region $r > a$ is supposed to be initially at constant temperature $\theta_0$ and for times $t > 0$ there is supposed to be heat transfer by forced convection at the surface to a medium at constant temperature $\theta_1$, so that the boundary condition at $r = a$ is

$$k\,\partial\theta/\partial r = H(\theta_s - \theta_1), \tag{4}$$

where $\theta_s$ is the surface temperature at time $t$, $H$ is the heat transfer coefficient, and $k$ is the thermal conductivity of the rock. $H$ and $k$ are assumed to be independent of temperature. The equation of conduction of heat has to be solved with this boundary condition and with $\theta = \theta_0$ when $t = 0$. The solution (Carslaw and Jaeger 1959, Section 13.5 (15)) is that the temperature $\theta$ at distance $r$ from the axis and time $t$ is given by

$$\frac{\theta-\theta_1}{\theta_0-\theta_1} = \frac{2}{\pi}\int_0^\infty \frac{Y_0(uR)[(u/\beta)J_1(u)+J_0(u)]-J_0(uR)[(u/\beta)Y_1(u)+Y_0(u)]}{u\{[(u/\beta)J_1(u)+J_0(u)]^2+[(u/\beta)Y_1(u)+Y_0(u)]^2\}} \times \exp(-\tau u^2)\,\mathrm{d}u, \tag{5}$$

where $\tau$, $R$, and $\beta$ are the dimensionless quantities defined in (1), (2), and (3).

It follows from (5) that the surface temperature $\theta_s$ is given by

$$(\theta_s-\theta_1)/(\theta_0-\theta_1) = \phi(\beta;\tau), \tag{6}$$

where

$$\phi(\beta;\tau) = \frac{4}{\beta\pi^2}\int_0^\infty \frac{\exp(-\tau u^2)\,\mathrm{d}u}{u\{[(u/\beta)J_1(u)+J_0(u)]^2+[(u/\beta)Y_1(u)+Y_0(u)]^2\}}. \tag{7}$$

The heat flux $f_s$ from the solid at the surface is

$$f_s = H(\theta_s-\theta_1) = \{k(\theta_0-\theta_1)/a\}\beta\,\phi(\beta;\tau). \tag{8}$$

The total heat flow $Q_s$ across unit area of the surface up to time $t$ is

$$Q_s = \int_0^t f_s\,\mathrm{d}t = \{ka(\theta_0-\theta_1)/a\}\beta\int_0^\tau \phi(\beta;\tau)\,\mathrm{d}\tau. \tag{9}$$

The function $\phi(\beta;\tau)$ defined in (7) will be regarded as fundamental. It is easy to tabulate since it lies between 0 and 1. However, since it appears multiplied by $\beta$ in (8) and (9), it has been tabulated to four significant figures to give four-place accuracy for the larger values of $\beta$.

To evaluate the integral in (7), it is first transformed by integration by parts into the form

$$\phi(\beta;\tau) = \frac{4\beta}{\pi}\int_0^\infty \exp(-\tau u^2)\left\{\frac{\tau}{u^2+\beta^2}+\frac{1}{(u^2+\beta^2)^2}\right\}\psi(u)\,u\,\mathrm{d}u, \tag{10}$$

where

$$\psi(u) = \tfrac{1}{2}\pi+\arg[\{u\mathrm{H}_1^{(1)}(u)+\beta\mathrm{H}_0^{(1)}(u)\}/\beta] \tag{11}$$

and $\mathrm{H}_1^{(1)}(u)$ and $\mathrm{H}_0^{(1)}(u)$ are Hankel functions of the first kind. Since $\psi(0) = 0$ and $\psi(u) \sim u$ as $u \to \infty$, the integral (10) is non-singular. It was evaluated on the CSIRO C.D.C. 3600 computer. In all cases, six significant figures were used, being rounded off to four in the final tabulation. Weddle's rule for numerical integration was used, an interval being chosen and halved until there was no change in the sixth significant figure; if agreement was obtained at the first halving, the interval was doubled. Integration at the upper limit was terminated when the value of the integral over a unit range was $< 10^{-6}$.

Results are shown in Table 1, and four significant figures are given so that these results may be used for subsequent numerical work; for example, Starfield (1965) has made extensive use of tabulated functions in calculations of this sort.

TABLE 1

VALUES OF $\phi(\beta;\tau)$

| $\beta$ | $\tau = 0{\cdot}01$ | 0·015 | 0·02 | 0·025 | 0·03 | 0·035 | 0·04 | 0·05 | 0·06 | 0·07 | 0·08 | 0·09 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0·01 | 0·9989 | 0·9987 | 0·9985 | 0·9983 | 0·9982 | 0·9981 | 0·9979 | 0·9977 | 0·9975 | 0·9973 | 0·9972 | 0·9970 |
| 0·03 | 0·9968 | 0·9961 | 0·9955 | 0·9950 | 0·9946 | 0·9942 | 0·9938 | 0·9931 | 0·9925 | 0·9920 | 0·9915 | 0·9911 |
| 0·1 | 0·9893 | 0·9870 | 0·9851 | 0·9835 | 0·9821 | 0·9808 | 0·9796 | 0·9774 | 0·9755 | 0·9738 | 0·9722 | 0·9707 |
| 0·15 | 0·9840 | 0·9806 | 0·9779 | 0·9755 | 0·9733 | 0·9714 | 0·9696 | 0·9664 | 0·9636 | 0·9611 | 0·9588 | 0·9566 |
| 0·2 | 0·9787 | 0·9743 | 0·9706 | 0·9675 | 0·9647 | 0·9621 | 0·9598 | 0·9556 | 0·9520 | 0·9486 | 0·9456 | 0·9428 |
| 0·25 | 0·9735 | 0·9680 | 0·9635 | 0·9596 | 0·9562 | 0·9530 | 0·9502 | 0·9450 | 0·9405 | 0·9365 | 0·9328 | 0·9294 |
| 0·3 | 0·9684 | 0·9618 | 0·9565 | 0·9519 | 0·9478 | 0·9441 | 0·9407 | 0·9346 | 0·9293 | 0·9246 | 0·9202 | 0·9162 |
| 0·4 | 0·9582 | 0·9496 | 0·9427 | 0·9366 | 0·9313 | 0·9265 | 0·9222 | 0·9144 | 0·9076 | 0·9015 | 0·8959 | 0·8908 |
| 0·5 | 0·9482 | 0·9377 | 0·9291 | 0·9218 | 0·9154 | 0·9095 | 0·9043 | 0·8948 | 0·8866 | 0·8793 | 0·8726 | 0·8666 |
| 0·6 | 0·9383 | 0·9260 | 0·9159 | 0·9074 | 0·8998 | 0·8930 | 0·8869 | 0·8760 | 0·8665 | 0·8580 | 0·8504 | 0·8434 |
| 0·7 | 0·9287 | 0·9145 | 0·9030 | 0·8933 | 0·8847 | 0·8770 | 0·8701 | 0·8577 | 0·8470 | 0·8376 | 0·8290 | 0·8212 |
| 0·8 | 0·9191 | 0·9032 | 0·8904 | 0·8795 | 0·8700 | 0·8615 | 0·8538 | 0·8401 | 0·8283 | 0·8179 | 0·8085 | 0·8000 |
| 0·9 | 0·9098 | 0·8922 | 0·8781 | 0·8661 | 0·8557 | 0·8464 | 0·8379 | 0·8231 | 0·8103 | 0·7990 | 0·7889 | 0·7797 |
| 1·0 | 0·9006 | 0·8814 | 0·8660 | 0·8531 | 0·8418 | 0·8317 | 0·8226 | 0·8067 | 0·7929 | 0·7808 | 0·7700 | 0·7603 |
| 1·5 | 0·8567 | 0·8304 | 0·8097 | 0·7923 | 0·7774 | 0·7643 | 0·7525 | 0·7320 | 0·7147 | 0·6996 | 0·6862 | 0·6743 |
| 2·0 | 0·8161 | 0·7841 | 0·7591 | 0·7385 | 0·7209 | 0·7055 | 0·6918 | 0·6684 | 0·6487 | 0·6318 | 0·6169 | 0·6037 |
| 2·5 | 0·7787 | 0·7419 | 0·7135 | 0·6904 | 0·6709 | 0·6539 | 0·6390 | 0·6136 | 0·5925 | 0·5745 | 0·5589 | 0·5451 |
| 3 | 0·7440 | 0·7033 | 0·6724 | 0·6474 | 0·6265 | 0·6085 | 0·5928 | 0·5662 | 0·5443 | 0·5258 | 0·5099 | 0·4959 |
| 4 | 0·6818 | 0·6356 | 0·6012 | 0·5740 | 0·5515 | 0·5325 | 0·5159 | 0·4884 | 0·4662 | 0·4477 | 0·4319 | 0·4183 |
| 5 | 0·6279 | 0·5782 | 0·5421 | 0·5139 | 0·4909 | 0·4716 | 0·4551 | 0·4279 | 0·4062 | 0·3883 | 0·3732 | 0·3603 |
| 6 | 0·5808 | 0·5293 | 0·4924 | 0·4641 | 0·4412 | 0·4222 | 0·4060 | 0·3797 | 0·3589 | 0·3420 | 0·3277 | 0·3156 |
| 7 | 0·5395 | 0·4871 | 0·4502 | 0·4222 | 0·3998 | 0·3814 | 0·3658 | 0·3406 | 0·3209 | 0·3049 | 0·2916 | 0·2803 |
| 8 | 0·5031 | 0·4505 | 0·4141 | 0·3867 | 0·3650 | 0·3472 | 0·3323 | 0·3083 | 0·2897 | 0·2748 | 0·2623 | 0·2518 |
| 9 | 0·4707 | 0·4186 | 0·3829 | 0·3563 | 0·3354 | 0·3183 | 0·3041 | 0·2814 | 0·2639 | 0·2498 | 0·2382 | 0·2284 |
| 10 | 0·4419 | 0·3905 | 0·3557 | 0·3300 | 0·3099 | 0·2936 | 0·2801 | 0·2585 | 0·2420 | 0·2289 | 0·2180 | 0·2088 |
| 15 | 0·3354 | 0·2899 | 0·2604 | 0·2392 | 0·2230 | 0·2101 | 0·1995 | 0·1829 | 0·1704 | 0·1605 | 0·1524 | 0·1457 |
| 20 | 0·2682 | 0·2289 | 0·2041 | 0·1865 | 0·1732 | 0·1628 | 0·1542 | 0·1409 | 0·1310 | 0·1232 | 0·1168 | 0·1115 |
| 25 | 0·2224 | 0·1884 | 0·1672 | 0·1524 | 0·1413 | 0·1325 | 0·1254 | 0·1144 | 0·1062 | 0·09981 | 0·09460 | 0·09026 |
| 30 | 0·1896 | 0·1598 | 0·1415 | 0·1287 | 0·1192 | 0·1117 | 0·1056 | 0·09625 | 0·08929 | 0·08385 | 0·07944 | 0·07578 |
| 40 | 0·1459 | 0·1223 | 0·1079 | 0·09797 | 0·09059 | 0·08482 | 0·08015 | 0·07297 | 0·06764 | 0·06349 | 0·06013 | 0·05733 |
| 50 | 0·1183 | 0·09885 | 0·08710 | 0·07901 | 0·07301 | 0·06833 | 0·06454 | 0·05873 | 0·05442 | 0·05106 | 0·04835 | 0·04610 |
| 60 | 0·09937 | 0·08290 | 0·07298 | 0·06617 | 0·06112 | 0·05719 | 0·05401 | 0·04913 | 0·04552 | 0·04270 | 0·04043 | 0·03854 |
| 70 | 0·08563 | 0·07136 | 0·06279 | 0·05691 | 0·05256 | 0·04916 | 0·04642 | 0·04222 | 0·03911 | 0·03669 | 0·03473 | 0·03311 |
| 80 | 0·07521 | 0·06263 | 0·05508 | 0·04992 | 0·04609 | 0·04311 | 0·04070 | 0·03701 | 0·03428 | 0·03216 | 0·03044 | 0·02902 |
| 90 | 0·06703 | 0·05579 | 0·04906 | 0·04445 | 0·04104 | 0·03838 | 0·03623 | 0·03295 | 0·03052 | 0·02862 | 0·02709 | 0·02583 |
| 100 | 0·06045 | 0·05030 | 0·04422 | 0·04006 | 0·03698 | 0·03459 | 0·03265 | 0·02969 | 0·02750 | 0·02579 | 0·02441 | 0·02327 |

TABLE 1 (*Continued*)

| $\beta$ | $\tau = 0{\cdot}1$ | 0·15 | 0·2 | 0·25 | 0·3 | 0·35 | 0·4 | 0·5 | 0·6 | 0·7 | 0·8 | 0·9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0·01 | 0·9969 | 0·9963 | 0·9958 | 0·9954 | 0·9950 | 0·9947 | 0·9944 | 0·9939 | 0·9934 | 0·9930 | 0·9927 | 0·9923 |
| 0·03 | 0·9906 | 0·9888 | 0·9874 | 0·9862 | 0·9851 | 0·9842 | 0·9833 | 0·9818 | 0·9805 | 0·9793 | 0·9782 | 0·9773 |
| 0·1 | 0·9694 | 0·9636 | 0·9590 | 0·9551 | 0·9518 | 0·9488 | 0·9461 | 0·9414 | 0·9373 | 0·9337 | 0·9305 | 0·9276 |
| 0·15 | 0·9546 | 0·9462 | 0·9396 | 0·9340 | 0·9291 | 0·9248 | 0·9209 | 0·9142 | 0·9084 | 0·9033 | 0·8988 | 0·8947 |
| 0·2 | 0·9402 | 0·9293 | 0·9207 | 0·9135 | 0·9073 | 0·9018 | 0·8969 | 0·8883 | 0·8810 | 0·8746 | 0·8689 | 0·8638 |
| 0·25 | 0·9262 | 0·9130 | 0·9025 | 0·8939 | 0·8864 | 0·8798 | 0·8739 | 0·8637 | 0·8550 | 0·8474 | 0·8407 | 0·8347 |
| 0·3 | 0·9125 | 0·8971 | 0·8849 | 0·8749 | 0·8662 | 0·8587 | 0·8519 | 0·8402 | 0·8303 | 0·8217 | 0·8140 | 0·8072 |
| 0·4 | 0·8861 | 0·8666 | 0·8514 | 0·8389 | 0·8282 | 0·8189 | 0·8106 | 0·7964 | 0·7844 | 0·7740 | 0·7650 | 0·7568 |
| 0·5 | 0·8610 | 0·8378 | 0·8200 | 0·8054 | 0·7930 | 0·7822 | 0·7727 | 0·7564 | 0·7427 | 0·7311 | 0·7208 | 0·7118 |
| 0·6 | 0·8370 | 0·8106 | 0·7904 | 0·7740 | 0·7602 | 0·7482 | 0·7377 | 0·7197 | 0·7048 | 0·6921 | 0·6810 | 0·6713 |
| 0·7 | 0·8141 | 0·7849 | 0·7627 | 0·7448 | 0·7297 | 0·7167 | 0·7054 | 0·6861 | 0·6702 | 0·6567 | 0·6450 | 0·6347 |
| 0·8 | 0·7922 | 0·7605 | 0·7366 | 0·7173 | 0·7013 | 0·6875 | 0·6754 | 0·6552 | 0·6385 | 0·6244 | 0·6123 | 0·6016 |
| 0·9 | 0·7713 | 0·7373 | 0·7119 | 0·6916 | 0·6747 | 0·6603 | 0·6477 | 0·6266 | 0·6094 | 0·5949 | 0·5824 | 0·5715 |
| 1·0 | 0·7513 | 0·7154 | 0·6887 | 0·6675 | 0·6499 | 0·6349 | 0·6219 | 0·6002 | 0·5826 | 0·5678 | 0·5551 | 0·5441 |
| 1·5 | 0·6634 | 0·6208 | 0·5900 | 0·5661 | 0·5467 | 0·5305 | 0·5166 | 0·4937 | 0·4756 | 0·4606 | 0·4479 | 0·4369 |
| 2·0 | 0·5919 | 0·5461 | 0·5139 | 0·4893 | 0·4697 | 0·4535 | 0·4397 | 0·4175 | 0·4000 | 0·3857 | 0·3738 | 0·3636 |
| 2·5 | 0·5328 | 0·4860 | 0·4538 | 0·4296 | 0·4105 | 0·3949 | 0·3817 | 0·3606 | 0·3442 | 0·3310 | 0·3199 | 0·3106 |
| 3 | 0·4835 | 0·4369 | 0·4054 | 0·3821 | 0·3638 | 0·3489 | 0·3366 | 0·3168 | 0·3016 | 0·2894 | 0·2792 | 0·2707 |
| 4 | 0·4062 | 0·3619 | 0·3328 | 0·3116 | 0·2953 | 0·2822 | 0·2713 | 0·2542 | 0·2411 | 0·2307 | 0·2221 | 0·2149 |
| 5 | 0·3489 | 0·3078 | 0·2813 | 0·2623 | 0·2478 | 0·2362 | 0·2267 | 0·2117 | 0·2004 | 0·1915 | 0·1841 | 0·1779 |
| 6 | 0·3050 | 0·2672 | 0·2432 | 0·2261 | 0·2132 | 0·2029 | 0·1944 | 0·1813 | 0·1713 | 0·1635 | 0·1571 | 0·1517 |
| 7 | 0·2705 | 0·2358 | 0·2139 | 0·1985 | 0·1868 | 0·1776 | 0·1701 | 0·1584 | 0·1495 | 0·1426 | 0·1369 | 0·1322 |
| 8 | 0·2427 | 0·2107 | 0·1908 | 0·1768 | 0·1662 | 0·1579 | 0·1511 | 0·1405 | 0·1326 | 0·1264 | 0·1213 | 0·1171 |
| 9 | 0·2200 | 0·1904 | 0·1721 | 0·1593 | 0·1496 | 0·1420 | 0·1359 | 0·1263 | 0·1191 | 0·1135 | 0·1089 | 0·1050 |
| 10 | 0·2010 | 0·1736 | 0·1566 | 0·1449 | 0·1360 | 0·1291 | 0·1234 | 0·1146 | 0·1081 | 0·1029 | 0·09874 | 0·09524 |
| 15 | 0·1399 | 0·1200 | 0·1079 | 0·09959 | 0·09336 | 0·08847 | 0·08451 | 0·07839 | 0·07383 | 0·07026 | 0·06735 | 0·06492 |
| 20 | 0·1070 | 0·09158 | 0·08223 | 0·07579 | 0·07100 | 0·06725 | 0·06421 | 0·05953 | 0·05604 | 0·05330 | 0·05108 | 0·04923 |
| 25 | 0·08658 | 0·07398 | 0·06638 | 0·06115 | 0·05727 | 0·05423 | 0·05176 | 0·04797 | 0·04515 | 0·04294 | 0·04114 | 0·03964 |
| 30 | 0·07267 | 0·06204 | 0·05564 | 0·05125 | 0·04798 | 0·04542 | 0·04335 | 0·04017 | 0·03780 | 0·03594 | 0·03444 | 0·03318 |
| 40 | 0·05497 | 0·04689 | 0·04203 | 0·03870 | 0·03622 | 0·03428 | 0·03272 | 0·03031 | 0·02851 | 0·02711 | 0·02597 | 0·02502 |
| 50 | 0·04419 | 0·03768 | 0·03377 | 0·03108 | 0·02909 | 0·02753 | 0·02627 | 0·02433 | 0·02289 | 0·02176 | 0·02084 | 0·02008 |
| 60 | 0·03694 | 0·03149 | 0·02822 | 0·02597 | 0·02430 | 0·02300 | 0·02195 | 0·02032 | 0·01912 | 0·01817 | 0·01741 | 0·01677 |
| 70 | 0·03173 | 0·02704 | 0·02423 | 0·02230 | 0·02087 | 0·01975 | 0·01884 | 0·01745 | 0·01641 | 0·01560 | 0·01494 | 0·01440 |
| 80 | 0·02781 | 0·02370 | 0·02123 | 0·01954 | 0·01828 | 0·01730 | 0·01651 | 0·01529 | 0·01438 | 0·01367 | 0·01309 | 0·01261 |
| 90 | 0·02475 | 0·02109 | 0·01889 | 0·01739 | 0·01627 | 0·01540 | 0·01469 | 0·01360 | 0·01279 | 0·01216 | 0·01165 | 0·01122 |
| 100 | 0·02230 | 0·01900 | 0·01702 | 0·01566 | 0·01465 | 0·01387 | 0·01323 | 0·01225 | 0·01152 | 0·01095 | 0·01049 | 0·01010 |

TABLE 1 (*Continued*)

| $\beta$ | $\tau = 1$ | 1·5 | 2·0 | 2·5 | 3·0 | 3·5 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0·01 | 0·9920 | 0·9908 | 0·9899 | 0·9891 | 0·9885 | 0·9879 | 0·9874 | 0·9865 | 0·9858 | 0·9852 | 0·9847 | 0·9842 |
| 0·03 | 0·9764 | 0·9729 | 0·9701 | 0·9679 | 0·9661 | 0·9644 | 0·9630 | 0·9606 | 0·9585 | 0·9568 | 0·9552 | 0·9539 |
| 0·1 | 0·9250 | 0·9142 | 0·9062 | 0·8997 | 0·8943 | 0·8896 | 0·8856 | 0·8786 | 0·8729 | 0·8680 | 0·8638 | 0·8600 |
| 0·15 | 0·8910 | 0·8760 | 0·8649 | 0·8560 | 0·8486 | 0·8423 | 0·8368 | 0·8275 | 0·8199 | 0·8134 | 0·8078 | 0·8028 |
| 0·2 | 0·8591 | 0·8405 | 0·8268 | 0·8160 | 0·8070 | 0·7994 | 0·7927 | 0·7815 | 0·7724 | 0·7647 | 0·7581 | 0·7523 |
| 0·25 | 0·8292 | 0·8075 | 0·7917 | 0·7792 | 0·7689 | 0·7602 | 0·7527 | 0·7401 | 0·7298 | 0·7212 | 0·7138 | 0·7074 |
| 0·3 | 0·8011 | 0·7767 | 0·7591 | 0·7453 | 0·7340 | 0·7244 | 0·7162 | 0·7025 | 0·6914 | 0·6821 | 0·6742 | 0·6672 |
| 0·4 | 0·7495 | 0·7210 | 0·7007 | 0·6849 | 0·6721 | 0·6614 | 0·6522 | 0·6371 | 0·6249 | 0·6148 | 0·6062 | 0·5987 |
| 0·5 | 0·7036 | 0·6722 | 0·6500 | 0·6330 | 0·6193 | 0·6079 | 0·5981 | 0·5822 | 0·5695 | 0·5590 | 0·5501 | 0·5424 |
| 0·6 | 0·6625 | 0·6290 | 0·6056 | 0·5879 | 0·5737 | 0·5619 | 0·5519 | 0·5356 | 0·5227 | 0·5121 | 0·5031 | 0·4954 |
| 0·7 | 0·6255 | 0·5906 | 0·5665 | 0·5484 | 0·5339 | 0·5220 | 0·5120 | 0·4956 | 0·4827 | 0·4722 | 0·4633 | 0·4557 |
| 0·8 | 0·5921 | 0·5564 | 0·5319 | 0·5136 | 0·4991 | 0·4872 | 0·4772 | 0·4610 | 0·4483 | 0·4379 | 0·4292 | 0·4217 |
| 0·9 | 0·5618 | 0·5256 | 0·5010 | 0·4828 | 0·4683 | 0·4566 | 0·4466 | 0·4307 | 0·4182 | 0·4081 | 0·3996 | 0·3923 |
| 1·0 | 0·5343 | 0·4979 | 0·4734 | 0·4552 | 0·4410 | 0·4294 | 0·4197 | 0·4040 | 0·3919 | 0·3820 | 0·3737 | 0·3667 |
| 1·5 | 0·4274 | 0·3925 | 0·3697 | 0·3531 | 0·3403 | 0·3300 | 0·3214 | 0·3078 | 0·2973 | 0·2888 | 0·2818 | 0·2759 |
| 2·0 | 0·3547 | 0·3228 | 0·3023 | 0·2876 | 0·2764 | 0·2673 | 0·2599 | 0·2481 | 0·2391 | 0·2318 | 0·2259 | 0·2208 |
| 2·5 | 0·3024 | 0·2736 | 0·2553 | 0·2423 | 0·2324 | 0·2244 | 0·2179 | 0·2076 | 0·1998 | 0·1935 | 0·1883 | 0·1839 |
| 3 | 0·2633 | 0·2372 | 0·2208 | 0·2091 | 0·2003 | 0·1933 | 0·1875 | 0·1784 | 0·1715 | 0·1660 | 0·1614 | 0·1576 |
| 4 | 0·2087 | 0·1870 | 0·1735 | 0·1640 | 0·1568 | 0·1511 | 0·1464 | 0·1391 | 0·1336 | 0·1291 | 0·1255 | 0·1224 |
| 5 | 0·1726 | 0·1542 | 0·1428 | 0·1348 | 0·1288 | 0·1240 | 0·1201 | 0·1140 | 0·1093 | 0·1057 | 0·1026 | 0·1001 |
| 6 | 0·1471 | 0·1312 | 0·1213 | 0·1144 | 0·1092 | 0·1051 | 0·1018 | 0·09651 | 0·09254 | 0·08938 | 0·08679 | 0·08461 |
| 7 | 0·1281 | 0·1141 | 0·1054 | 0·09937 | 0·09482 | 0·09122 | 0·08827 | 0·08368 | 0·08021 | 0·07745 | 0·07519 | 0·07328 |
| 8 | 0·1134 | 0·1009 | 0·09318 | 0·08780 | 0·08376 | 0·08056 | 0·07794 | 0·07386 | 0·07077 | 0·06833 | 0·06632 | 0·06463 |
| 9 | 0·1018 | 0·09043 | 0·08349 | 0·07865 | 0·07500 | 0·07212 | 0·06977 | 0·06609 | 0·06332 | 0·06112 | 0·05932 | 0·05780 |
| 10 | 0·09226 | 0·08194 | 0·07562 | 0·07121 | 0·06790 | 0·06528 | 0·06314 | 0·05981 | 0·05729 | 0·05529 | 0·05365 | 0·05228 |
| 15 | 0·06286 | 0·05574 | 0·05139 | 0·04835 | 0·04608 | 0·04428 | 0·04281 | 0·04053 | 0·03880 | 0·03743 | 0·03631 | 0·03537 |
| 20 | 0·04765 | 0·04222 | 0·03891 | 0·03660 | 0·03487 | 0·03350 | 0·03238 | 0·03064 | 0·02933 | 0·02829 | 0·02744 | 0·02673 |
| 25 | 0·03837 | 0·03398 | 0·03130 | 0·02944 | 0·02804 | 0·02694 | 0·02604 | 0·02464 | 0·02358 | 0·02274 | 0·02205 | 0·02148 |
| 30 | 0·03211 | 0·02843 | 0·02619 | 0·02462 | 0·02345 | 0·02253 | 0·02177 | 0·02060 | 0·01971 | 0·01901 | 0·01843 | 0·01795 |
| 40 | 0·02421 | 0·02143 | 0·01973 | 0·01855 | 0·01767 | 0·01697 | 0·01640 | 0·01551 | 0·01484 | 0·01431 | 0·01388 | 0·01351 |
| 50 | 0·01943 | 0·01719 | 0·01583 | 0·01488 | 0·01417 | 0·01361 | 0·01315 | 0·01244 | 0·01190 | 0·01148 | 0·01113 | 0·01084 |
| 60 | 0·01623 | 0·01436 | 0·01322 | 0·01243 | 0·01183 | 0·01136 | 0·01098 | 0·01038 | 0·009935 | 0·009579 | 0·009288 | 0·009043 |
| 70 | 0·01393 | 0·01232 | 0·01134 | 0·01066 | 0·01015 | 0·009751 | 0·009422 | 0·008911 | 0·008525 | 0·008220 | 0·007970 | 0·007760 |
| 80 | 0·01220 | 0·01079 | 0·009937 | 0·009340 | 0·008893 | 0·008540 | 0·008252 | 0·007804 | 0·007466 | 0·007199 | 0·006979 | 0·006795 |
| 90 | 0·01086 | 0·009603 | 0·008840 | 0·008309 | 0·007911 | 0·007597 | 0·007340 | 0·006942 | 0·006641 | 0·006403 | 0·006208 | 0·006044 |
| 100 | 0·009776 | 0·008648 | 0·007961 | 0·007483 | 0·007124 | 0·006841 | 0·006610 | 0·006251 | 0·005980 | 0·005766 | 0·005590 | 0·005442 |

TABLE 1 (*Continued*)

| $\beta$ | $\tau = 10$ | 15 | 20 | 25 | 30 | 35 | 40 | 50 | 60 | 70 | 80 | 90 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0·01 | 0·9837 | 0·9820 | 0·9808 | 0·9798 | 0·9790 | 0·9783 | 0·9777 | 0·9766 | 0·9758 | 0·9751 | 0·9745 | 0·9739 |
| 0·03 | 0·9526 | 0·9478 | 0·9443 | 0·9415 | 0·9392 | 0·9373 | 0·9357 | 0·9329 | 0·9306 | 0·9287 | 0·9270 | 0·9255 |
| 0·1 | 0·8566 | 0·8436 | 0·8343 | 0·8272 | 0·8214 | 0·8165 | 0·8123 | 0·8053 | 0·7997 | 0·7949 | 0·7908 | 0·7873 |
| 0·15 | 0·7984 | 0·7814 | 0·7695 | 0·7604 | 0·7531 | 0·7469 | 0·7416 | 0·7329 | 0·7259 | 0·7201 | 0·7151 | 0·7107 |
| 0·2 | 0·7471 | 0·7273 | 0·7136 | 0·7032 | 0·6948 | 0·6879 | 0·6819 | 0·6721 | 0·6643 | 0·6578 | 0·6522 | 0·6474 |
| 0·25 | 0·7016 | 0·6799 | 0·6650 | 0·6537 | 0·6446 | 0·6371 | 0·6308 | 0·6203 | 0·6120 | 0·6051 | 0·5992 | 0·5942 |
| 0·3 | 0·6611 | 0·6380 | 0·6223 | 0·6104 | 0·6010 | 0·5932 | 0·5866 | 0·5758 | 0·5672 | 0·5601 | 0·5541 | 0·5489 |
| 0·4 | 0·5921 | 0·5675 | 0·5510 | 0·5387 | 0·5289 | 0·5209 | 0·5141 | 0·5031 | 0·4944 | 0·4872 | 0·4812 | 0·4760 |
| 0·5 | 0·5356 | 0·5106 | 0·4940 | 0·4816 | 0·4719 | 0·4640 | 0·4573 | 0·4464 | 0·4379 | 0·4309 | 0·4251 | 0·4200 |
| 0·6 | 0·4886 | 0·4638 | 0·4474 | 0·4353 | 0·4258 | 0·4181 | 0·4116 | 0·4011 | 0·3929 | 0·3862 | 0·3805 | 0·3757 |
| 0·7 | 0·4490 | 0·4247 | 0·4087 | 0·3970 | 0·3878 | 0·3803 | 0·3741 | 0·3640 | 0·3561 | 0·3497 | 0·3443 | 0·3397 |
| 0·8 | 0·4152 | 0·3915 | 0·3761 | 0·3647 | 0·3559 | 0·3488 | 0·3428 | 0·3331 | 0·3256 | 0·3195 | 0·3144 | 0·3100 |
| 0·9 | 0·3860 | 0·3631 | 0·3482 | 0·3373 | 0·3288 | 0·3220 | 0·3162 | 0·3070 | 0·2999 | 0·2941 | 0·2892 | 0·2850 |
| 1·0 | 0·3606 | 0·3384 | 0·3241 | 0·3136 | 0·3055 | 0·2990 | 0·2935 | 0·2847 | 0·2779 | 0·2724 | 0·2677 | 0·2638 |
| 1·5 | 0·2707 | 0·2523 | 0·2405 | 0·2320 | 0·2254 | 0·2201 | 0·2157 | 0·2087 | 0·2032 | 0·1988 | 0·1952 | 0·1920 |
| 2·0 | 0·2164 | 0·2009 | 0·1910 | 0·1839 | 0·1784 | 0·1740 | 0·1704 | 0·1646 | 0·1601 | 0·1565 | 0·1535 | 0·1509 |
| 2·5 | 0·1802 | 0·1668 | 0·1583 | 0·1523 | 0·1476 | 0·1439 | 0·1408 | 0·1358 | 0·1321 | 0·1290 | 0·1264 | 0·1243 |
| 3 | 0·1543 | 0·1426 | 0·1352 | 0·1299 | 0·1259 | 0·1226 | 0·1199 | 0·1156 | 0·1124 | 0·1097 | 0·1075 | 0·1056 |
| 4 | 0·1198 | 0·1105 | 0·1046 | 0·1004 | 0·09720 | 0·09463 | 0·09250 | 0·08912 | 0·08653 | 0·08444 | 0·08270 | 0·08123 |
| 5 | 0·09788 | 0·09015 | 0·08528 | 0·08181 | 0·07916 | 0·07704 | 0·07528 | 0·07249 | 0·07035 | 0·06863 | 0·06720 | 0·06599 |
| 6 | 0·08273 | 0·07613 | 0·07198 | 0·06903 | 0·06677 | 0·06496 | 0·06346 | 0·06109 | 0·05927 | 0·05781 | 0·05659 | 0·05556 |
| 7 | 0·07164 | 0·06589 | 0·06227 | 0·05969 | 0·05773 | 0·05615 | 0·05485 | 0·05278 | 0·05120 | 0·04993 | 0·04887 | 0·04798 |
| 8 | 0·06317 | 0·05807 | 0·05486 | 0·05258 | 0·05084 | 0·04945 | 0·04829 | 0·04647 | 0·04507 | 0·04394 | 0·04301 | 0·04222 |
| 9 | 0·05649 | 0·05191 | 0·04903 | 0·04699 | 0·04542 | 0·04417 | 0·04314 | 0·04150 | 0·04025 | 0·03924 | 0·03840 | 0·03769 |
| 10 | 0·05109 | 0·04693 | 0·04432 | 0·04247 | 0·04105 | 0·03991 | 0·03898 | 0·03749 | 0·03636 | 0·03544 | 0·03468 | 0·03404 |
| 15 | 0·03456 | 0·03172 | 0·02994 | 0·02867 | 0·02771 | 0·02693 | 0·02629 | 0·02528 | 0·02451 | 0·02389 | 0·02337 | 0·02293 |
| 20 | 0·02611 | 0·02395 | 0·02260 | 0·02164 | 0·02091 | 0·02032 | 0·01984 | 0·01907 | 0·01849 | 0·01802 | 0·01762 | 0·01729 |
| 25 | 0·02098 | 0·01924 | 0·01815 | 0·01738 | 0·01679 | 0·01632 | 0·01593 | 0·01531 | 0·01484 | 0·01446 | 0·01415 | 0·01388 |
| 30 | 0·01754 | 0·01608 | 0·01517 | 0·01452 | 0·01403 | 0·01363 | 0·01330 | 0·01279 | 0·01239 | 0·01208 | 0·01181 | 0·01159 |
| 40 | 0·01320 | 0·01210 | 0·01141 | 0·01093 | 0·01055 | 0·01025 | 0·01001 | 0·009620 | 0·009322 | 0·009083 | 0·008885 | 0·008716 |
| 50 | 0·01058 | 0·009702 | 0·009149 | 0·008757 | 0·008458 | 0·008219 | 0·008021 | 0·007709 | 0·007470 | 0·007278 | 0·007119 | 0·006984 |
| 60 | 0·008833 | 0·008096 | 0·007635 | 0·007307 | 0·007057 | 0·006857 | 0·006692 | 0·006432 | 0·006232 | 0·006072 | 0·005939 | 0·005826 |
| 70 | 0·007579 | 0·006946 | 0·006550 | 0·006269 | 0·006054 | 0·005883 | 0·005741 | 0·005518 | 0·005346 | 0·005209 | 0·005095 | 0·004998 |
| 80 | 0·006637 | 0·006083 | 0·005735 | 0·005489 | 0·005301 | 0·005151 | 0·005027 | 0·004831 | 0·004681 | 0·004560 | 0·004461 | 0·004376 |
| 90 | 0·005903 | 0·005410 | 0·005101 | 0·004882 | 0·004715 | 0·004581 | 0·004471 | 0·004296 | 0·004163 | 0·004056 | 0·003967 | 0·003891 |
| 100 | 0·005315 | 0·004871 | 0·004593 | 0·004396 | 0·004245 | 0·004125 | 0·004025 | 0·003868 | 0·003748 | 0·003651 | 0·003571 | 0·003503 |

TABLE 1 (*Continued*)

| $\beta$ | $\tau$ = 100 | 150 | 200 | 250 | 300 | 350 | 400 | 500 | 600 | 700 | 800 | 900 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0·01 | 0·9735 | 0·9716 | 0·9702 | 0·9692 | 0·9684 | 0·9677 | 0·9670 | 0·9660 | 0·9652 | 0·9644 | 0·9638 | 0·9633 |
| 0·03 | 0·9242 | 0·9192 | 0·9156 | 0·9128 | 0·9106 | 0·9087 | 0·9070 | 0·9043 | 0·9021 | 0·9002 | 0·8986 | 0·8972 |
| 0·1 | 0·7841 | 0·7721 | 0·7637 | 0·7574 | 0·7522 | 0·7479 | 0·7443 | 0·7382 | 0·7333 | 0·7292 | 0·7257 | 0·7228 |
| 0·15 | 0·7068 | 0·6923 | 0·6823 | 0·6747 | 0·6686 | 0·6635 | 0·6592 | 0·6520 | 0·6463 | 0·6416 | 0·6375 | 0·6340 |
| 0·2 | 0·6431 | 0·6271 | 0·6162 | 0·6080 | 0·6014 | 0·5959 | 0·5913 | 0·5837 | 0·5776 | 0·5725 | 0·5682 | 0·5645 |
| 0·25 | 0·5897 | 0·5729 | 0·5616 | 0·5531 | 0·5463 | 0·5407 | 0·5359 | 0·5281 | 0·5219 | 0·5168 | 0·5124 | 0·5086 |
| 0·3 | 0·5443 | 0·5273 | 0·5158 | 0·5072 | 0·5004 | 0·4947 | 0·4899 | 0·4822 | 0·4760 | 0·4708 | 0·4665 | 0·4627 |
| 0·4 | 0·4714 | 0·4545 | 0·4432 | 0·4348 | 0·4281 | 0·4227 | 0·4180 | 0·4105 | 0·4046 | 0·3996 | 0·3955 | 0·3919 |
| 0·5 | 0·4155 | 0·3992 | 0·3884 | 0·3804 | 0·3740 | 0·3688 | 0·3644 | 0·3573 | 0·3517 | 0·3471 | 0·3432 | 0·3398 |
| 0·6 | 0·3714 | 0·3559 | 0·3456 | 0·3380 | 0·3320 | 0·3271 | 0·3229 | 0·3163 | 0·3110 | 0·3067 | 0·3030 | 0·2999 |
| 0·7 | 0·3357 | 0·3209 | 0·3112 | 0·3040 | 0·2984 | 0·2938 | 0·2899 | 0·2836 | 0·2787 | 0·2747 | 0·2713 | 0·2683 |
| 0·8 | 0·3062 | 0·2922 | 0·2830 | 0·2762 | 0·2709 | 0·2666 | 0·2630 | 0·2571 | 0·2525 | 0·2487 | 0·2455 | 0·2428 |
| 0·9 | 0·2814 | 0·2682 | 0·2595 | 0·2531 | 0·2481 | 0·2440 | 0·2406 | 0·2351 | 0·2307 | 0·2272 | 0·2242 | 0·2216 |
| 1·0 | 0·2603 | 0·2478 | 0·2396 | 0·2335 | 0·2288 | 0·2250 | 0·2217 | 0·2165 | 0·2125 | 0·2091 | 0·2063 | 0·2039 |
| 1·5 | 0·1893 | 0·1795 | 0·1730 | 0·1684 | 0·1647 | 0·1617 | 0·1593 | 0·1552 | 0·1521 | 0·1496 | 0·1474 | 0·1456 |
| 2·0 | 0·1487 | 0·1406 | 0·1354 | 0·1316 | 0·1286 | 0·1262 | 0·1242 | 0·1210 | 0·1185 | 0·1164 | 0·1147 | 0·1132 |
| 2·5 | 0·1224 | 0·1156 | 0·1112 | 0·1080 | 0·1055 | 0·1035 | 0·1018 | 0·09910 | 0·09699 | 0·09527 | 0·09382 | 0·09259 |
| 3 | 0·1040 | 0·09813 | 0·09434 | 0·09158 | 0·08944 | 0·08771 | 0·08625 | 0·08392 | 0·08211 | 0·08063 | 0·07939 | 0·07833 |
| 4 | 0·07995 | 0·07535 | 0·07238 | 0·07022 | 0·06855 | 0·06719 | 0·06606 | 0·06424 | 0·06282 | 0·06167 | 0·06071 | 0·05988 |
| 5 | 0·06493 | 0·06115 | 0·05871 | 0·05694 | 0·05557 | 0·05445 | 0·05352 | 0·05203 | 0·05087 | 0·04993 | 0·04914 | 0·04846 |
| 6 | 0·05466 | 0·05145 | 0·04938 | 0·04788 | 0·04672 | 0·04577 | 0·04499 | 0·04372 | 0·04274 | 0·04195 | 0·04128 | 0·04070 |
| 7 | 0·04720 | 0·04441 | 0·04261 | 0·04131 | 0·04030 | 0·03948 | 0·03880 | 0·03770 | 0·03685 | 0·03616 | 0·03558 | 0·03509 |
| 8 | 0·04153 | 0·03906 | 0·03748 | 0·03633 | 0·03543 | 0·03471 | 0·03411 | 0·03314 | 0·03239 | 0·03178 | 0·03127 | 0·03083 |
| 9 | 0·03707 | 0·03487 | 0·03344 | 0·03241 | 0·03161 | 0·03097 | 0·03043 | 0·02956 | 0·02889 | 0·02835 | 0·02789 | 0·02750 |
| 10 | 0·03348 | 0·03148 | 0·03019 | 0·02926 | 0·02854 | 0·02795 | 0·02746 | 0·02668 | 0·02607 | 0·02558 | 0·02517 | 0·02481 |
| 15 | 0·02256 | 0·02120 | 0·02032 | 0·01969 | 0·01920 | 0·01880 | 0·01847 | 0·01794 | 0·01753 | 0·01719 | 0·01691 | 0·01667 |
| 20 | 0·01701 | 0·01598 | 0·01532 | 0·01484 | 0·01447 | 0·01417 | 0·01391 | 0·01351 | 0·01320 | 0·01295 | 0·01274 | 0·01256 |
| 25 | 0·01365 | 0·01282 | 0·01229 | 0·01190 | 0·01160 | 0·01136 | 0·01116 | 0·01084 | 0·01059 | 0·01039 | 0·01022 | 0·01007 |
| 30 | 0·01140 | 0·01071 | 0·01026 | 0·009938 | 0·009688 | 0·009486 | 0·009318 | 0·009048 | 0·008839 | 0·008669 | 0·008527 | 0·008405 |
| 40 | 0·008570 | 0·008049 | 0·007714 | 0·007471 | 0·007283 | 0·007131 | 0·007004 | 0·006801 | 0·006643 | 0·006515 | 0·006408 | 0·006316 |
| 50 | 0·006867 | 0·006449 | 0·006180 | 0·005985 | 0·005834 | 0·005712 | 0·005610 | 0·005448 | 0·005321 | 0·005219 | 0·005133 | 0·005059 |
| 60 | 0·005729 | 0·005380 | 0·005155 | 0·004992 | 0·004866 | 0·004765 | 0·004680 | 0·004544 | 0·004438 | 0·004353 | 0·004281 | 0·004219 |
| 70 | 0·004914 | 0·004614 | 0·004421 | 0·004282 | 0·004174 | 0·004087 | 0·004014 | 0·003897 | 0·003806 | 0·003733 | 0·003671 | 0·003619 |
| 80 | 0·004302 | 0·004040 | 0·003871 | 0·003749 | 0·003654 | 0·003577 | 0·003514 | 0·003412 | 0·003332 | 0·003268 | 0·003214 | 0·003168 |
| 90 | 0·003826 | 0·003592 | 0·003442 | 0·003333 | 0·003249 | 0·003181 | 0·003124 | 0·003034 | 0·002963 | 0·002906 | 0·002858 | 0·002817 |
| 100 | 0·003445 | 0·003234 | 0·003099 | 0·003001 | 0·002925 | 0·002864 | 0·002813 | 0·002731 | 0·002667 | 0·002616 | 0·002573 | 0·002536 |

TABLE 1 (*Continued*)

| $\beta$ | $\tau$ = 1000 | 1500 | 2000 | 2500 | 3000 | 3500 | 4000 | 5000 | 6000 | 7000 | 8000 | 9000 | 10 000 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0·01 | 0·9628 | 0·9609 | 0·9596 | 0·9586 | 0·9577 | 0·9570 | 0·9564 | 0·9554 | 0·9546 | 0·9539 | 0·9533 | 0·9527 | 0·9523 |
| 0·03 | 0·8959 | 0·8911 | 0·8877 | 0·8851 | 0·8829 | 0·8811 | 0·8796 | 0·8770 | 0·8749 | 0·8731 | 0·8716 | 0·8703 | 0·8691 |
| 0·1 | 0·7199 | 0·7096 | 0·7024 | 0·6970 | 0·6926 | 0·6889 | 0·6858 | 0·6806 | 0·6764 | 0·6729 | 0·6699 | 0·6673 | 0·6650 |
| 0·15 | 0·6308 | 0·6190 | 0·6109 | 0·6048 | 0·5998 | 0·5957 | 0·5922 | 0·5864 | 0·5818 | 0·5779 | 0·5746 | 0·5717 | 0·5691 |
| 0·2 | 0·5612 | 0·5488 | 0·5403 | 0·5339 | 0·5288 | 0·5246 | 0·5209 | 0·5150 | 0·5102 | 0·5063 | 0·5029 | 0·4999 | 0·4973 |
| 0·25 | 0·5053 | 0·4928 | 0·4843 | 0·4779 | 0·4727 | 0·4685 | 0·4649 | 0·4590 | 0·4543 | 0·4503 | 0·4470 | 0·4441 | 0·4415 |
| 0·3 | 0·4594 | 0·4471 | 0·4387 | 0·4324 | 0·4274 | 0·4232 | 0·4197 | 0·4139 | 0·4093 | 0·4055 | 0·4023 | 0·3995 | 0·3970 |
| 0·4 | 0·3887 | 0·3770 | 0·3691 | 0·3632 | 0·3585 | 0·3546 | 0·3513 | 0·3459 | 0·3417 | 0·3381 | 0·3351 | 0·3325 | 0·3303 |
| 0·5 | 0·3368 | 0·3259 | 0·3185 | 0·3130 | 0·3087 | 0·3051 | 0·3020 | 0·2971 | 0·2932 | 0·2899 | 0·2872 | 0·2848 | 0·2827 |
| 0·6 | 0·2971 | 0·2869 | 0·2801 | 0·2750 | 0·2710 | 0·2677 | 0·2649 | 0·2603 | 0·2567 | 0·2537 | 0·2512 | 0·2490 | 0·2471 |
| 0·7 | 0·2657 | 0·2562 | 0·2499 | 0·2452 | 0·2415 | 0·2384 | 0·2358 | 0·2316 | 0·2283 | 0·2256 | 0·2232 | 0·2212 | 0·2195 |
| 0·8 | 0·2403 | 0·2315 | 0·2256 | 0·2212 | 0·2178 | 0·2149 | 0·2125 | 0·2086 | 0·2055 | 0·2030 | 0·2009 | 0·1990 | 0·1974 |
| 0·9 | 0·2194 | 0·2111 | 0·2056 | 0·2015 | 0·1983 | 0·1956 | 0·1934 | 0·1898 | 0·1869 | 0·1846 | 0·1826 | 0·1808 | 0·1793 |
| 1·0 | 0·2018 | 0·1940 | 0·1888 | 0·1850 | 0·1820 | 0·1795 | 0·1774 | 0·1741 | 0·1714 | 0·1692 | 0·1673 | 0·1657 | 0·1643 |
| 1·5 | 0·1440 | 0·1381 | 0·1342 | 0·1313 | 0·1290 | 0·1272 | 0·1256 | 0·1231 | 0·1211 | 0·1194 | 0·1180 | 0·1168 | 0·1158 |
| 2·0 | 0·1119 | 0·1072 | 0·1040 | 0·1017 | 0·09992 | 0·09844 | 0·09719 | 0·09516 | 0·09357 | 0·09227 | 0·09116 | 0·09021 | 0·08938 |
| 2·5 | 0·09150 | 0·08756 | 0·08495 | 0·08303 | 0·08153 | 0·08030 | 0·07926 | 0·07758 | 0·07626 | 0·07517 | 0·07426 | 0·07347 | 0·07278 |
| 3 | 0·07740 | 0·07402 | 0·07179 | 0·07014 | 0·06885 | 0·06780 | 0·06691 | 0·06548 | 0·06435 | 0·06342 | 0·06264 | 0·06197 | 0·06138 |
| 4 | 0·05916 | 0·05653 | 0·05480 | 0·05352 | 0·05252 | 0·05171 | 0·05102 | 0·04991 | 0·04903 | 0·04832 | 0·04772 | 0·04720 | 0·04674 |
| 5 | 0·04787 | 0·04572 | 0·04431 | 0·04327 | 0·04245 | 0·04179 | 0·04123 | 0·04032 | 0·03961 | 0·03902 | 0·03853 | 0·03811 | 0·03774 |
| 6 | 0·04020 | 0·03839 | 0·03719 | 0·03631 | 0·03562 | 0·03506 | 0·03459 | 0·03382 | 0·03322 | 0·03273 | 0·03231 | 0·03196 | 0·03164 |
| 7 | 0·03465 | 0·03308 | 0·03204 | 0·03128 | 0·03069 | 0·03020 | 0·02979 | 0·02913 | 0·02861 | 0·02818 | 0·02782 | 0·02751 | 0·02724 |
| 8 | 0·03045 | 0·02906 | 0·02815 | 0·02748 | 0·02695 | 0·02652 | 0·02616 | 0·02558 | 0·02512 | 0·02474 | 0·02443 | 0·02416 | 0·02392 |
| 9 | 0·02715 | 0·02591 | 0·02510 | 0·02450 | 0·02403 | 0·02364 | 0·02332 | 0·02280 | 0·02239 | 0·02205 | 0·02177 | 0·02153 | 0·02132 |
| 10 | 0·02450 | 0·02338 | 0·02264 | 0·02210 | 0·02167 | 0·02133 | 0·02104 | 0·02056 | 0·02019 | 0·01989 | 0·01964 | 0·01942 | 0·01922 |
| 15 | 0·01647 | 0·01571 | 0·01521 | 0·01484 | 0·01455 | 0·01432 | 0·01412 | 0·01380 | 0·01355 | 0·01335 | 0·01318 | 0·01303 | 0·01290 |
| 20 | 0·01240 | 0·01182 | 0·01145 | 0·01117 | 0·01095 | 0·01078 | 0·01063 | 0·01039 | 0·01020 | 0·01004 | 0·009914 | 0·009802 | 0·009704 |
| 25 | 0·009942 | 0·009481 | 0·009178 | 0·008955 | 0·008781 | 0·008639 | 0·008519 | 0·008326 | 0·008175 | 0·008051 | 0·007946 | 0·007856 | 0·007778 |
| 30 | 0·008299 | 0·007913 | 0·007660 | 0·007474 | 0·007328 | 0·007209 | 0·007109 | 0·006948 | 0·006821 | 0·006718 | 0·006631 | 0·006555 | 0·006490 |
| 40 | 0·006236 | 0·005946 | 0·005755 | 0·005615 | 0·005506 | 0·005416 | 0·005341 | 0·005220 | 0·005125 | 0·005047 | 0·004981 | 0·004924 | 0·004875 |
| 50 | 0·004995 | 0·004762 | 0·004609 | 0·004497 | 0·004409 | 0·004338 | 0·004277 | 0·004180 | 0·004104 | 0·004041 | 0·003989 | 0·003943 | 0·003904 |
| 60 | 0·004166 | 0·003972 | 0·003844 | 0·003750 | 0·003677 | 0·003617 | 0·003567 | 0·003486 | 0·003422 | 0·003370 | 0·003326 | 0·003288 | 0·003255 |
| 70 | 0·003573 | 0·003406 | 0·003297 | 0·003216 | 0·003153 | 0·003102 | 0·003059 | 0·002989 | 0·002935 | 0·002890 | 0·002852 | 0·002820 | 0·002791 |
| 80 | 0·003128 | 0·002982 | 0·002886 | 0·002815 | 0·002760 | 0·002715 | 0·002677 | 0·002617 | 0·002569 | 0·002530 | 0·002497 | 0·002468 | 0·002443 |
| 90 | 0·002781 | 0·002651 | 0·002566 | 0·002503 | 0·002454 | 0·002414 | 0·002381 | 0·002326 | 0·002284 | 0·002249 | 0·002220 | 0·002195 | 0·002172 |
| 100 | 0·002504 | 0·002387 | 0·002310 | 0·002254 | 0·002209 | 0·002173 | 0·002143 | 0·002094 | 0·002056 | 0·002025 | 0·001998 | 0·001975 | 0·001956 |

TABLE 2

VALUES OF $\int_0^\tau \phi(\beta;\tau)\,d\tau$

| $\tau$ | $\beta = 0\cdot01$ | 0·03 | 0·1 | 0·3 | 1 | 3 | 10 | 30 | 100 | $\infty$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0·01 | 0·0000999 | 0·000299 | 0·000993 | 0·00294 | 0·00932 | 0·0245 | 0·0565 | 0·0880 | 0·1074 | 0·1177 |
| 0·02 | 0·000200 | 0·000598 | 0·00198 | 0·00582 | 0·0181 | 0·0457 | 0·0958 | 0·1365 | 0·1584 | 0·1693 |
| 0·03 | 0·000300 | 0·000897 | 0·00296 | 0·00868 | 0·0267 | 0·0651 | 0·1289 | 0·1752 | 0·1986 | 0·2100 |
| 0·04 | 0·000399 | 0·00119 | 0·00394 | 0·0115 | 0·0350 | 0·0834 | 0·1583 | 0·2088 | 0·2333 | 0·2450 |
| 0·06 | 0·000599 | 0·00179 | 0·00590 | 0·0171 | 0·0511 | 0·1174 | 0·2102 | 0·2668 | 0·2929 | 0·3052 |
| 0·08 | 0·000798 | 0·00239 | 0·00785 | 0·0227 | 0·0668 | 0·1490 | 0·2561 | 0·3172 | 0·3446 | 0·3574 |
| 0·1 | 0·000998 | 0·00298 | 0·00979 | 0·0282 | 0·0820 | 0·1787 | 0·2979 | 0·3627 | 0·3912 | 0·4043 |
| 0·2 | 0·00199 | 0·00595 | 0·0194 | 0·0551 | 0·1537 | 0·3106 | 0·4732 | 0·5510 | 0·5834 | 0·5980 |
| 0·3 | 0·00299 | 0·00891 | 0·0290 | 0·0813 | 0·2205 | 0·4254 | 0·6185 | 0·7053 | 0·7406 | 0·7564 |
| 0·4 | 0·00398 | 0·0119 | 0·0385 | 0·1071 | 0·2840 | 0·5302 | 0·7478 | 0·8418 | 0·8795 | 0·8963 |
| 0·6 | 0·00597 | 0·0177 | 0·0573 | 0·1575 | 0·4042 | 0·7208 | 0·9778 | 1·084 | 1·125 | 1·144 |
| 0·8 | 0·00796 | 0·0236 | 0·0760 | 0·2068 | 0·5178 | 0·8946 | 1·184 | 1·300 | 1·345 | 1·365 |
| 1 | 0·00994 | 0·0295 | 0·0945 | 0·2553 | 0·6266 | 1·057 | 1·375 | 1·499 | 1·547 | 1·568 |
| 2 | 0·0199 | 0·0587 | 0·1860 | 0·4886 | 1·126 | 1·773 | 2·201 | 2·359 | 2·419 | 2·445 |
| 3 | 0·0297 | 0·0877 | 0·2760 | 0·7123 | 1·582 | 2·402 | 2·915 | 3·100 | 3·169 | 3·200 |
| 4 | 0·0396 | 0·1167 | 0·3650 | 0·9297 | 2·012 | 2·983 | 3·568 | 3·776 | 3·854 | 3·888 |
| 6 | 0·0594 | 0·1743 | 0·5407 | 1·351 | 2·821 | 4·055 | 4·767 | 5·015 | 5·108 | 5·148 |
| 8 | 0·0791 | 0·2317 | 0·7144 | 1·761 | 3·586 | 5·052 | 5·874 | 6·157 | 6·262 | 6·308 |
| 10 | 0·0987 | 0·2889 | 0·8864 | 2·161 | 4·319 | 5·998 | 6·920 | 7·235 | 7·351 | 7·402 |
| 20 | 0·1969 | 0·5733 | 1·731 | 4·079 | 7·717 | 10·30 | 11·64 | 12·09 | 12·25 | 12·32 |
| 30 | 0·2949 | 0·8558 | 2·558 | 5·912 | 10·86 | 14·20 | 15·89 | 16·45 | 16·65 | 16·74 |
| 40 | 0·3928 | 1·137 | 3·375 | 7·692 | 13·85 | 17·88 | 19·89 | 20·54 | 20·78 | 20·89 |
| 60 | 0·5881 | 1·697 | 4·986 | 11·15 | 19·55 | 24·83 | 27·40 | 28·23 | 28·53 | 28·66 |
| 80 | 0·7831 | 2·254 | 6·576 | 14·51 | 25·00 | 31·42 | 34·49 | 35·48 | 35·84 | 36·00 |
| 100 | 0·9779 | 2·809 | 8·150 | 17·80 | 30·28 | 37·76 | 41·30 | 42·44 | 42·85 | 43·03 |
| 200 | 1·950 | 5·568 | 15·88 | 33·65 | 55·13 | 67·30 | 72·90 | 74·68 | 75·32 | 75·59 |
| 300 | 2·919 | 8·306 | 23·45 | 48·87 | 78·50 | 94·81 | 102·2 | 104·5 | 105·4 | 105·7 |
| 400 | 3·886 | 11·03 | 30·93 | 63·72 | 101·0 | 121·1 | 130·2 | 133·0 | 134·0 | 134·5 |
| 600 | 5·819 | 16·46 | 45·70 | 92·67 | 144·4 | 171·5 | 183·6 | 187·3 | 188·7 | 189·3 |
| 800 | 7·747 | 21·86 | 60·29 | 120·9 | 186·2 | 219·9 | 234·8 | 239·4 | 241·0 | 241·8 |
| 1000 | 9·674 | 27·24 | 74·74 | 148·7 | 227·0 | 266·9 | 284·4 | 289·8 | 291·8 | 292·6 |
| 2000 | 19·28 | 53·98 | 145·8 | 283·0 | 421·4 | 489·6 | 518·9 | 527·9 | 531·1 | 531·5 |
| 3000 | 28·87 | 80·54 | 215·5 | 412·8 | 606·6 | 700·2 | 740·0 | 752·3 | 756·7 | 759·0 |
| 4000 | 38·44 | 107·0 | 284·4 | 539·8 | 786·2 | 903·7 | 953·4 | 968·7 | 974·1 | 975·7 |
| 6000 | 57·55 | 159·6 | 420·5 | 788·3 | 1135 | 1297 | 1365 | 1386 | 1393 | 1395 |
| 8000 | 76·63 | 212·0 | 555·1 | 1032 | 1473 | 1678 | 1763 | 1789 | 1798 | 1800 |
| 10 000 | 95·68 | 264·2 | 688·6 | 1271 | 1805 | 2049 | 2151 | 2183 | 2194 | 2196 |

The integral required in (9) has been calculated from the values in Table 1 by using Simpson's rule for $\tau \geqslant 0\cdot 01$ and adding the value for $\tau = 0\cdot 01$ calculated from equation (26) (Appendix). Some values of it are given in Table 2, in which the limiting values for $\beta = \infty$ are taken from the tables of Goch and Patterson (1940) and van Everdingen and Hurst (1949) for comparison.

## Approximations

For large values of $\tau$, Jaeger (1942) has given the asymptotic expansion

$$\phi(\beta;\tau) \sim \frac{2}{\beta}\left\{\frac{1}{y} - \frac{\gamma}{y^2} - \frac{(\frac{1}{6}\pi^2)-\gamma^2}{y^3}\ldots\right\}, \tag{12}$$

where $\gamma = 0\cdot 5772$ is Euler's constant and

$$y = (2/\beta)-2\gamma+\ln 4\tau. \tag{13}$$

For $\tau > 100$, values calculated from (12) disagree with those of Table 1 only in the third significant figure at worst.

For small values of $\tau$ the usual Laplace transformation procedure gives (Jaeger 1942)

$$\phi(\beta;\tau) = 1-2\beta\pi^{-\frac{1}{2}}\tau^{\frac{1}{2}}+\tfrac{1}{2}\beta(2\beta+1)\tau+\ldots. \tag{14}$$

Unfortunately this series is only of use for small values of $\beta$. However, a modification of the procedure gives the approximate formula

$$\phi(\beta;\tau) \sim \frac{1}{2\beta+1}+\frac{2\beta}{2\beta+1}\exp\{(\beta+\tfrac{1}{2})^2\tau\}\operatorname{erfc}\{(\beta+\tfrac{1}{2})\tau^{\frac{1}{2}}\}, \tag{15}$$

which gives agreement with the values of Table 1 to the fourth significant figure for $\tau = 0\cdot 01$ and is only a few percent in error when $\tau = 1$. This formula is derived in the Appendix.

## Temperatures in the Solid

Knowing the surface temperature as a function of time, the temperature at any point and time may be written down by Duhamel's theorem (Carslaw and Jaeger 1959, Section 1.14 (10)). If $F(r,\tau)$ is the temperature at radius $r$ and time $a^2\tau/a$ in the region $r > a$ with zero initial temperature and a unit change of surface temperature (tabulated by Jaeger 1956), the temperature $\theta$ at radius $r$ at time $a^2\tau/a$ in the present case, in which the surface temperature is given by (6), is

$$\begin{aligned}\frac{\theta_0-\theta}{\theta_0-\theta_1} &= -\int_0^\tau\{1-\phi(\beta;\lambda)\}\frac{\partial}{\partial\lambda}F(r,\tau-\lambda)\,\mathrm{d}\lambda\\ &= \int_0^\tau\frac{\partial\phi(\beta;\lambda)}{\partial\lambda}F(r,\tau-\lambda)\,\mathrm{d}\lambda.\end{aligned} \tag{16}$$

Some values calculated in this way for the case $\beta = 1$ are shown in Figure 1.

Apart from this accurate numerical calculation there are two simple approximations that are useful.

Firstly, there is an approximation for small $\tau$

$$\frac{\theta-\theta_1}{\theta_0-\theta_1} = 1-\frac{\beta}{BR^{\frac{1}{2}}}\left\{\operatorname{erfc}[\tfrac{1}{2}(R-1)\tau^{-\frac{1}{2}}]-\exp[B(R-1)+\tau B^2]\operatorname{erfc}[\tfrac{1}{2}(R-1)\tau^{-\frac{1}{2}}+B\tau^{\frac{1}{2}}]\right\}, \tag{17}$$

where $R = r/a$ and $B = \beta+\frac{3}{8}+(1/8R)$. This result, which is derived in the Appendix, reduces to (15) for $R = 1$. When $\beta \to \infty$ it becomes

$$\frac{\theta-\theta_1}{\theta_0-\theta_1} = 1-R^{-\frac{1}{2}}\operatorname{erfc}[\tfrac{1}{2}(R-1)\tau^{-\frac{1}{2}}], \tag{18}$$

which corresponds to the first term of the series (Carslaw and Jaeger 1959, Section 13.5 (7)) for the temperature in the case in which the surface temperature is held constant. While this is only the first term of a series and should be expected to be accurate only for small $\tau$, it is in fact surprisingly accurate for moderate values of $\tau$, being less than 5% low for $\tau = 1$ and less than 15% low for $\tau = 10$. Some workers, e.g. Saunders and Topping (1950), have used it for unrestricted $\tau$.

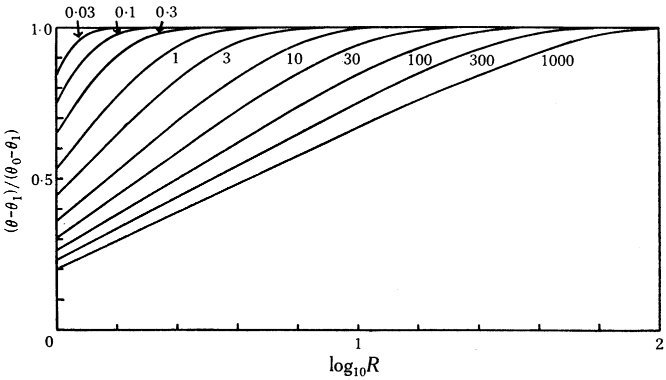


Fig. 1.—Temperature distribution for the region $R = r/a > 1$ for the case $\beta = aH/k = 1$. Numbers on the curves are values of $\tau = at/a^2$.

Secondly, for large values of $\tau$, the cylinder functions in (5) may be expanded in an ascending series as in Jaeger (1956). This gives an expansion in powers of $\ln R$ for the temperature of which the first terms are

$$\frac{\theta-\theta_1}{\theta_0-\theta_1} = \phi(\beta;\tau)+\beta\,\phi(\beta;\tau)\ln R + \ldots . \tag{19}$$

As in the case of constant surface temperature, it appears from Figure 1 that the plot of $\theta$ against $\ln R$ is nearly linear over a wide range. The slope of this line is

$$\mathrm{d}\theta/\mathrm{d}(\ln R) = \beta\,\phi(\beta;\tau) \tag{20}$$

and its intercept at $R = 1$ is $\theta_s$.

It follows that, in principle, measurements of the slope and intercept of the line give $\beta$ and $\phi(\beta;\tau)$ and hence $\tau$ from Table 1. In this way, by a series of measurements in a drill hole, the value of the diffusivity $a$ for rock in bulk may be determined. It is this bulk diffusivity which controls the transfer of heat from the rock and it may be very different from that from individual rock samples. The linearity of the plot of $\theta$ against $\ln R$ is a valuable indication that heat conduction is proceeding radially to the hole and is not disturbed by other circumstances, e.g. water flow or the presence of other excavations. These questions are discussed fully by Jaeger and le Marne (1963).

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## Appendix

The results (15) and (17) may be readily obtained by the use of the Laplace transformation. Using the notation of Carslaw and Jaeger (1959, Section 13.5 (13)), the Laplace transform $\bar{\theta}$ of the temperature $\theta$ at radius $r$ is

$$\bar{\theta} = \frac{\theta_0}{p} - \frac{(\theta_0-\theta_1)\beta K_0(qr)}{p\{\beta K_0(qa)+qaK_1(qa)\}}. \tag{21}$$

The routine method (Carslaw and Jaeger 1959, Section 13.3; Goldstein 1932) for finding expressions for the solutions useful for small values of the time is to introduce the asymptotic expansions of the Bessel functions into (21). This gives

$$\bar{\theta} = \frac{\theta_0}{p} - \frac{a^{\frac{1}{2}}(\theta_0-\theta_1)\beta[1-(1/8qr)+\ldots]}{r^{\frac{1}{2}}p[qa+\beta+\frac{3}{8}-\{(16\beta+15)/128qa\}+\ldots]}\exp[-q(r-a)]. \tag{22}$$

The usual procedure is to expand the fraction in (22) in ascending powers of $1/q$, and this leads to (14) for $r = a$. The present method uses the fact that the inverse Laplace transforms of expressions of the form (22) with the infinite series replaced by polynomials are known. Thus (22) may be written

$$\bar{\theta} = \frac{\theta_0}{p} - \frac{a^{\frac{1}{2}}(\theta_0-\theta_1)\beta}{r^{\frac{1}{2}}p[qa+B+\boldsymbol{O}(q^{-1})]}\exp[-q(r-a)], \tag{23}$$

where $$B = \beta+\tfrac{3}{8}+a/8r. \tag{24}$$

If the terms in $q^{-1}$ are neglected this gives

$$\theta = \theta_0 - \frac{(\theta_0-\theta_1)\beta}{BR^{\frac{1}{2}}}\left\{\operatorname{erfc}[\tfrac{1}{2}(R-1)\tau^{-\frac{1}{2}}] - \exp[B(R-1)+B^2\tau]\operatorname{erfc}[\tfrac{1}{2}(R-1)\tau^{-\frac{1}{2}}+B\tau^{\frac{1}{2}}]\right\}, \quad (25)$$

which is (17). If terms in $(aq)^{-1}$ are retained in the denominator of (23), a second approximation is obtained. In the same way the first approximation for the surface flow $Q_s$ per unit area up to time $t$ is found to be

$$Q_s = \frac{ka\beta(\theta_0-\theta_1)}{a}\left[\frac{\tau}{2\beta+1} - \frac{\beta}{(\beta+\frac{1}{2})^3} + \frac{2\beta\tau^{\frac{1}{2}}}{(\beta+\frac{1}{2})^2\pi^{\frac{1}{2}}} + \frac{\beta}{(\beta+\frac{1}{2})^3}\exp\{\tau(\beta+\tfrac{1}{2})^2\}\operatorname{erfc}\{\tau^{\frac{1}{2}}(\beta+\tfrac{1}{2})\}\right]. \quad (26)$$