PROPAGATION OF HELICON WAVES IN A NON-UNIFORM PLASMA

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[*Manuscript received April 22, 1966*]

Summary

Theoretical results are obtained for the dispersion of helicon waves in a non-uniform cylindrical plasma. Although a direct comparison with the experimental data of Lehane and Thonemann cannot be made (due to simplifying assumptions in the present theory), a qualitative explanation is given for some of the discrepancies between their experimental results and the uniform plasma theory.

I. Introduction

An experimental study of the propagation of helicon waves in a cylindrical plasma column with an axial magnetic field has been made by Lehane and Thonemann (1965) and their results were in general agreement with the theory given by Klozenberg, McNamara, and Thonemann (1965). The features of the experimental results that differed most from the theory were the amplitude and phasing of the components of the wave field near the plasma boundary. It was also found that the measured average electron density had to be multiplied by a factor greater than unity in order to obtain agreement with the theoretical dispersion curves. Lehane and Thonemann thought that the non-uniform electron density in their experiments could account for these discrepancies with the theory derived for a uniform plasma. In the present paper we consider the effects of a radial density gradient on the dispersion of helicon waves.

II. Theory

The generalized Ohm's law for a plasma can be written (Spitzer 1962)

$$\frac{m_e}{ne^2}\frac{\partial \mathbf{J}}{\partial t} = \mathbf{E}_1+\mathbf{V}\times\mathbf{B}+\frac{1}{ne}\nabla p_e-\frac{1}{ne}\mathbf{J}\times\mathbf{B}-\eta\mathbf{J}\,,$$

where m_e is the electron mass, n the electron density, e the electronic charge, $\mathbf{E}_1$ the electric field, $\mathbf{J}$ the current density, $\mathbf{B}$ the magnetic induction, $\mathbf{V}$ the plasma velocity, p_e the electron partial pressure, and η the plasma resistivity (CGS electromagnetic units are used). More rigorously, the term ∇p_e in this equation should be replaced by $\nabla\,.\,\check{\mathbf{p}}$, where $\check{\mathbf{p}}$ is the pressure tensor. However, it will be shown later that this term is unimportant in the situations discussed in the present treatment. The above equation must be used in conjunction with Maxwell's equations to determine the propagation characteristics of an electromagnetic perturbation in the medium.

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A general theory of wave propagation in non-uniform bounded plasmas is very complex and the following assumptions are made in order to simplify the problem.

(i) The plasma is cylindrical with a radial density gradient only. A uniform magnetic field $\mathbf{B}_0$ is applied in the axial direction, and in the steady state no axial currents flow in the plasma.

(ii) The term $\eta\mathbf{J}$ in Ohm's law is neglected in comparison with the term $(1/ne)\mathbf{J}\times\mathbf{B}$. Noting that $m\nu/ne^2 = \eta$ (Spitzer 1962), this approximation is valid when the electron collision frequency for momentum transfer, ν, is much less than the electron cyclotron frequency Be/m_e.

(iii) The terms $(m_e/ne^2)\partial\mathbf{J}/\partial t$ and $(\mathbf{V}\times\mathbf{B})$ are neglected. This is valid when the following conditions are fulfilled (Klozenberg, McNamara, and Thonemann 1965):

(1) the wave frequency ω is much less than the electron cyclotron frequency but much greater than the ion cyclotron frequency,

(2) the Alfven velocity is much less than the phase velocity of the wave.

(iv) The plasma is a "temperate" one in which the electron temperature is sufficiently low that ∇p_e can be neglected, but not so low that the contribution to the plasma resistivity by Coulomb interactions becomes important. This restriction is discussed in greater detail in the Appendix.

(v) The currents and fields associated with the wave are assumed to be small quantities so that the equations can be linearized.

(vi) The phase velocity of the wave is much less than the velocity of light so that displacement currents can be neglected (Klozenberg, McNamara, and Thonemann 1965).

(vii) The plasma column is bounded by a perfectly conducting wall. For this case it is possible to avoid the difficulties that arise in determining boundary conditions for an infinitely conducting plasma enclosed by an insulating medium (Klozenberg, McNamara, and Thonemann 1965).

Using these assumptions the generalized Ohm's law then becomes

$$ne\mathbf{E}_1 = \mathbf{J}\times\mathbf{B}. \tag{1}$$

If $\mathbf{E}_0$, $\mathbf{j}_0$, $\mathbf{B}_0$ are the values of $\mathbf{E}_1$, $\mathbf{J}$, $\mathbf{B}$ in the unperturbed plasma and $\mathbf{E}$, $\mathbf{j}$, $\mathbf{b}$ are the perturbations due to the wave, then equation (1) can be written

$$ne(\mathbf{E}_0+\mathbf{E}) = (\mathbf{j}_0+\mathbf{j})\times(\mathbf{B}_0+\mathbf{b}), \tag{2}$$

and in the unperturbed plasma

$$ne\mathbf{E}_0 = \mathbf{j}_0\times\mathbf{B}_0. \tag{3}$$

Subtracting (3) from (2) and neglecting second-order terms gives

$$ne\mathbf{E} = \mathbf{j}\times\mathbf{B}_0+\mathbf{j}_0\times\mathbf{b}. \tag{4}$$

The equation of motion for the plasma in the unperturbed state is

$$\nabla p = \mathbf{j}_0\times\mathbf{B}_0.$$

Since the plasma-pressure gradients are neglected in this treatment and it has been specified that no axial currents flow in the steady state plasma, $\mathbf{j}_0$ can be put equal to zero. When the plasma pressure is retained, it can be shown by dimensional arguments that the term $\mathbf{j}_0 \times \mathbf{b}$ occuring in equation (4) is of second order when $aB_0^2/2\lambda \gg p$, where a is the radius of the plasma column and λ is the wavelength of the perturbation. Equation (4) then becomes

$$ne\mathbf{E} = \mathbf{j} \times \mathbf{B}_0. \tag{5}$$

Neglecting displacement currents, Maxwell's equations give

$$\nabla \times \mathbf{E} = -\partial \mathbf{b}/\partial t, \tag{6}$$

$$\nabla \times \mathbf{b} = 4\pi \mathbf{j}. \tag{7}$$

Equations (5), (6), and (7) give nine independent equations in the components of **E**, **b**, and **j** enabling a differential equation to be obtained for any one of these components.

Using cylindrical coordinates, we will consider solutions of the form $f(r)\exp \mathrm{i}(\omega t - m\theta - kz)$. Taking the curl of equation (5) and using (6),

$$ne\mathrm{i}\omega \mathbf{b} - e\nabla n \times \mathbf{E} = B_0 \mathrm{i}k\mathbf{j}. \tag{8}$$

From equation (5), $E_z = 0$, and since ∇n has an r component only, the r and θ components of equation (8) give

$$j_r = (ne\omega/B_0 k)b_r$$

and

$$j_\theta = (ne\omega/B_0 k)b_\theta.$$

Substituting these values for j_r and j_θ in the r and θ components of (7),

$$\left.\begin{aligned} -(\mathrm{i}m/r)b_z + \mathrm{i}kb_\theta &= anb_r, \\ -\mathrm{i}kb_r - \partial b_z/\partial r &= anb_\theta, \end{aligned}\right\} \tag{9}$$

where $a = 4\pi e\omega/B_0 k$.

Eliminating b_θ and b_r in turn from the set of equations (9), then

$$b_r = \frac{\mathrm{i}amn}{\beta}\frac{b_z}{r} + \frac{\mathrm{i}k}{\beta}\frac{\partial b_z}{\partial r}$$

and

$$\left. b_\theta = \frac{mk}{\beta}\frac{b_z}{r} + \frac{an}{\beta}\frac{\partial b_z}{\partial r}, \right\} \tag{10}$$

where $\beta = k^2 - a^2 n^2$.

Using equations (10) in $\nabla . \mathbf{b} = 0$, it follows that

$$\nabla^2 b_z + \frac{2a^2 n}{\beta}\frac{\mathrm{d}n}{\mathrm{d}r}\frac{\partial b_z}{\partial r} + \left\{a^2 n^2 + \left(\frac{2k^2}{\beta} - 1\right)\frac{am}{kr}\frac{\mathrm{d}n}{\mathrm{d}r}\right\} b_z = 0, \tag{11}$$

where the Laplacian of b_z has the form

$$\nabla^2 b_z = \frac{\partial^2 b_z}{\partial r^2} + \frac{1}{r}\frac{\partial b_z}{\partial r} - \frac{m^2}{r^2}b_z - k^2 b_z.$$

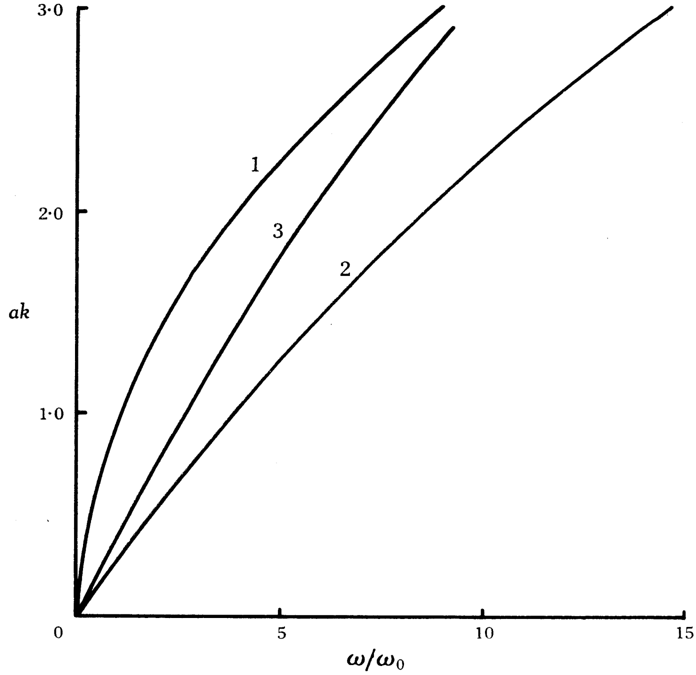

Fig. 1.—Dispersion curves for $m = 0$ mode. 1, plane wave; 2, uniform plasma; 3, $n = n_0 J_0(2 \cdot 4048 r/a)$.

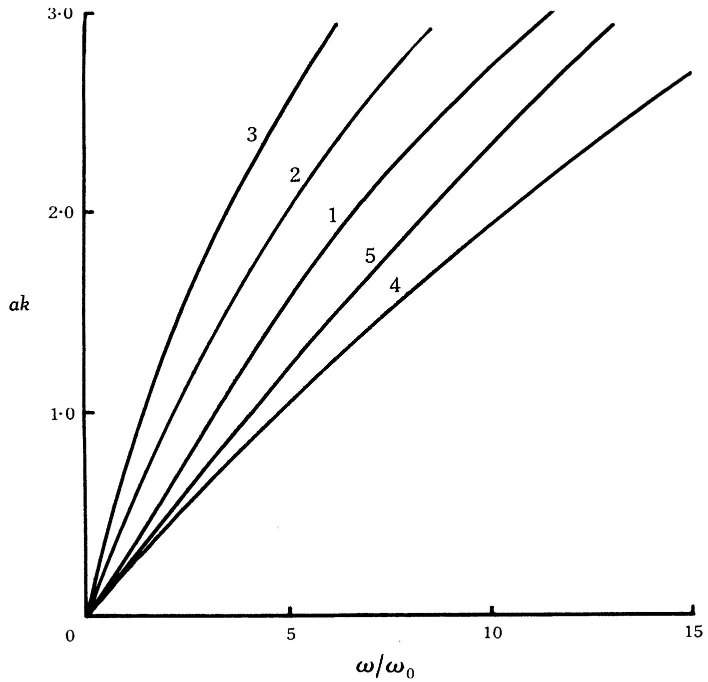

Fig. 2.—Dispersion curves for $m = 1, -1$ modes. 1, $m = 1$, uniform plasma; 2, $m = 1$, $n = n_0 J_0(1 \cdot 7r/a)$; 3, $m = 1$, $n = n_0 J_0(2 \cdot 4048 r/a)$; 4, $m = -1$, uniform plasma; 5, $m = -1$, $n = n_0 J_0(2 \cdot 4048 r/a)$.

For a perfectly conducting wall ($E_\theta = 0$, $E_z = 0$ when $r = a$) the boundary conditions for b_z and its derivatives are determined by equations (5), (6), and (7); that is, $b_z(r = a)$ = constant, the value of which is arbitrary and gives the wave amplitude, and

$$\frac{\partial b_z}{\partial r}(r = a) = -\frac{m}{a}\frac{\omega}{\omega_0}\frac{n(r = a)}{\bar{n}}\frac{b_z(r = a)}{(ak)^2},$$

where $\omega_0 = B_0/4\pi\bar{n}ea^2$ and $\bar{n}$ is the average electron density.

The conditions at $r = 0$ are derived from equation (7) as follows.

$$(\nabla\times\mathbf{b})_\theta = -ikb_r - \partial b_z/\partial r = 4\pi j_\theta,$$

and, since b_r and $j_\theta = 0$ when $r = 0$ and $m = 0$,

$$\partial b_z/\partial r = 0, \qquad \text{at } r = 0.$$

Similarly for $m = 1$, the r component of equation (7) gives

$$b_z = 0, \qquad \text{at } r = 0.$$

When $n(r)$ is specified, equation (11) can be integrated numerically. Using the Runge–Kutta method (Tenenbaum and Pollard 1963), the integration is carried out from the wall, where the boundary conditions are known, to the axis of the plasma cylinder. To establish a point on the dispersion curve, the value of ak is kept fixed during integrations but the value of ω/ω_0 is varied until the correct terminal condition is obtained at the axis.

This procedure is repeated for several values of ak, enabling dispersion curves to be obtained for any value of the mode number m. For each value of m there are an infinite number of dispersion curves corresponding to different numbers of radial nodes in the field components. In the present paper we discuss only the results for the lowest value of the radial node number for each value of m.

In order to illustrate the effects of a non-uniform plasma on the dispersion curves, we have assumed a variation of electron density of the form $n = n_0 \mathrm{J}_0(\gamma r/a)$, where J_0 is the zero-order Bessel function and γ is an arbitrary parameter equal to or less than the first root of J_0. Figure 1 shows the $m = 0$ dispersion curves plotted as ak versus ω/ω_0, for $\gamma = 0$ (uniform plasma) and $\gamma = 2\cdot4048$ (n decreasing to zero at the walls). The dispersion curve for plane waves in a uniform plasma is also given in this figure. Dispersion curves for $m = 1, -1$ are shown in Figure 2 for $\gamma = 0, 2\cdot4048$, and, in the case of $m = 1$, for $\gamma = 1\cdot7$.

Lehane and Thonemann (1965) found that their experimental points could be fitted to the theoretical dispersion curves for a uniform plasma by increasing the measured values of the average electron density by a factor that depended on the electron density distribution. Although the present analysis is not directly applicable to their experiments (since different boundary conditions are assumed and collisions are neglected), it is seen from Figures 1 and 2 that the dispersion curves for non-uniform and uniform plasmas can be brought into reasonable agreement in this way. However, to obtain exact agreement, this correction factor must be a function of ak, m, and the degree of non-uniformity of the plasma. This is illustrated by the values of the correction factor shown in Table 1.

Table 1

DENSITY CORRECTION FACTORS IN NON-UNIFORM PLASMAS

$m = 0$	$\gamma = 2\cdot4048$	$m = 1$	$\gamma = 2\cdot4048$	$m = 1$	$ak = 2\cdot3$
ak	Correction Factor	ak	Correction Factor	γ	Correction Factor
0·7	1·48	0·7	2·72	0	1·00
1·5	1·53	1·5	2·06	1·7	1·31
2·3	1·56	2·3	1·91	2·4048	1·91
2·9	1·58	2·9	1·90		

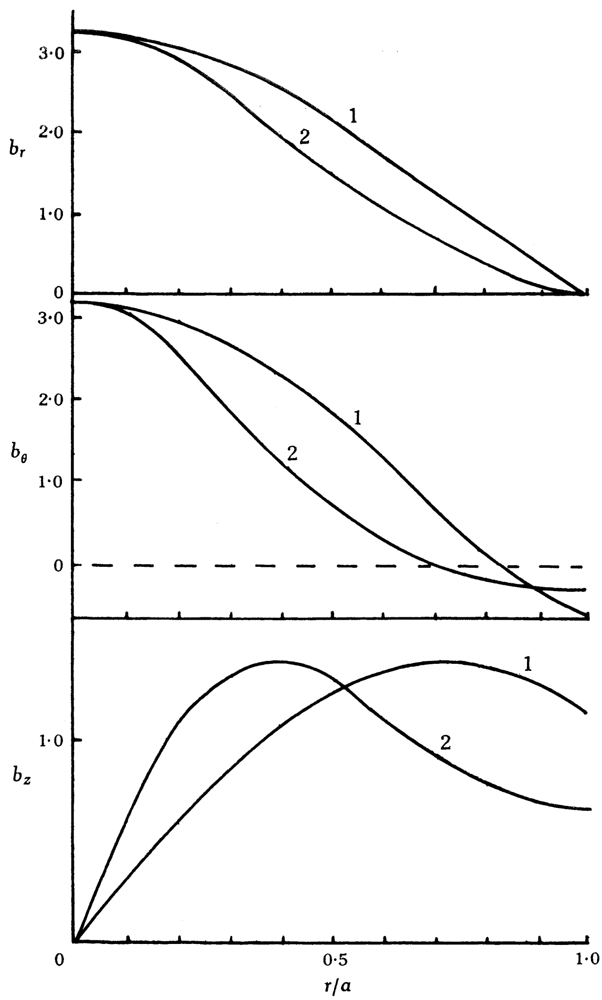

Fig. 3.—Radial variation of the magnetic field components for $m = 1$, $ak = 2\cdot3$. 1, uniform plasma; 2, $n = n_0 J_0(2\cdot4048r/a)$.

The effect of a non-uniform plasma on the magnitudes of the field components as a function of radius is shown in Figure 3 for the particular case $m = 1$, $ak = 2\cdot 3$. The field distributions have been calculated from equations (10) and normalized to give the same maximum value for the components of $\mathbf{b}$ as in the uniform plasma case. The decrease in the gradient of b_z near the wall and the shift in the b_z maximum towards the axis are also apparent in the experimental results of Lehane and Thonemann for this mode (Fig. 11 of their paper).

III. Conclusions

Since plasma resistivity has been neglected in the present treatment, it is not possible to comment on the effect of plasma non-uniformity on wave damping or on the discrepancies noted by Lehane and Thonemann (1965) with respect to the relative phasing of the field components. However, the discrepancies that were observed in the variation of the field components with radius and the effective electron density required for agreement with the uniform plasma theory are qualitatively explained by the present results.

IV. Acknowledgment

This work forms part of a program that has been supported by the Australian Institute of Nuclear Science and Engineering.

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Appendix

Conditions for the Neglect of ∇p_e *in the Generalized Ohm's Law*

We require $\nabla p_e \ll \mathbf{J}\times\mathbf{B}_0$. Putting $\nabla p_e \approx \bar{n}kT_e/a$, where $\bar{n}$ is the average electron density, and $\mathbf{J}\times\mathbf{B}_0 = (1/4\pi)(\nabla\times\mathbf{b})\times\mathbf{B}_0 \approx kbB_0/4\pi$, where $\mathbf{b}$ is the magnetic field associated with the wave, this requirement becomes

$$akbB_0/4\pi\bar{n}kT_e \gg 1. \tag{A1}$$

Further, $b/B_0 \ll 1$ for linearization of the equations to be valid.

As shown in Section II, the plasma resistivity term can be neglected when the collision frequency for momentum transfer $\nu \ll B_0 e/m_e$. Now ν can be written as $\nu_n + \nu_i$, where ν_n is the appropriate collision frequency for electron–neutral collisions and ν_i accounts for the Coulomb interaction between charged particles. For a hydrogen plasma (Blevin and Miller 1965)

$$\nu_{\mathrm{n}} \approx 2\times 10^{-7}\, n_{\mathrm{n}}\,,$$

$$\nu_{\mathrm{i}} \approx 2\cdot 6 n T_{\mathrm{e}}^{-3/2} \ln \Lambda\,,$$

where n_{n} is the neutral gas density and $\ln \Lambda$ is a slowly varying function of n and T_{e}. The restriction on collision frequencies becomes

$$(2\times 10^{-7}\, n_{\mathrm{n}} + 2\cdot 6 \bar{n} T_{\mathrm{e}}^{-3/2} \ln \Lambda) \ll B_0 e/m_{\mathrm{e}}. \tag{A2}$$

For the electron inertial term to be neglected we require $\omega \ll B_0 e/m_{\mathrm{e}}$, or, introducing the characteristic frequency ω_0,

$$\omega/\omega_0 \ll 4\pi \bar{n} e^2 a^2/m_{\mathrm{e}}. \tag{A3}$$

Since ak is a monotonically increasing function of ω/ω_0, this condition implies an upper limit for ak, whereas (A1) gives a lower limit when the discharge parameters are given.

As a specific example take $B_0 = 200$ G, $\bar{n} = 10^{12}$ electrons/cm^3, $n_{\mathrm{n}} = 10^{14}$ molecules/cm^3, $T_{\mathrm{e}} = 10^4\,{}^\circ$K, $a = 5$ cm, and $b = 2$ G. Then, taking the $m = 1$ mode for example, all conditions (A1), (A2), and (A3) are satisfied when $1 \leqslant ak \leqslant 3$. Other ranges of values for ak can be obtained by slight modification of the discharge parameters.