THE TAYLOR SERIES FOR THE CONVECTIVE CORE IN A STELLAR MODEL

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Summary

The first seven terms of the Taylor expansion about the centre for the solutions of a convective core with radiation pressure have been calculated. This truncated series can be used to calculate the values of quantities in the core for cores up to 40% of the radius and for radiation pressure up to $1-\beta_c = 0 \cdot 2$.

I. Introduction

One method that has often been used for the construction of stellar models is to start numerical integrations simultaneously inwards from the surface and outwards from the centre of the star. The parameters (such as luminosity and radius) are then adjusted until the two integrations fit together at a suitable point, often the boundary between core and envelope.

In the case of a convective core (such as in a massive star), it has been shown (Henrich 1941) that the outward integrations can be made to depend on a single parameter, for instance y_c, the ratio of the radiation pressure to the gas pressure at the centre. It is then possible (e.g. Hayashi and Cameron 1962) to construct a table of integrations using a set of values of y_c and, whenever a convective core is required, to interpolate in this mesh of solutions.

Alternatively, See (1905) had considered the particular case of a convective model with no radiation pressure (a polytrope of index $1 \cdot 5$). He showed that the interior values could be obtained by taking sufficient terms in the Taylor expansion about the centre, thus eliminating the need for numerical integration in this case.

The present paper gives terms of the Taylor series, expanded about the centre, for a convective core in which radiation pressure is included. The series for the mass extends up to the fifteenth power and the series for y/y_c up to the twelfth power. These terms are sufficient to give values of variables in the core for stars of reasonable mass, even though the rate of convergence of the series decreases as radiation pressure gains in importance. Small cores can thus be fitted without numerical integration, apart from a quadrature for evaluation of the luminosity.

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II. Equations

The differential equations for a convective core with radiation pressure included are well known to be (Hayashi, Hoshi, and Sugimoto 1962)

$$\frac{\mathrm{d}M_r}{\mathrm{d}r} = 4\pi r^2 \rho\,,$$

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM_r\rho}{r^2}\,,$$

$$\frac{\mathrm{d}(\ln T)}{\mathrm{d}(\ln P)} = \frac{8-6\beta}{32-24\beta-3\beta^2}\,,$$

with $$(1-\beta)P = \tfrac{1}{3}aT^4, \qquad \beta P = \mathscr{R}\rho T/\mu\,. \tag{1}$$

The composition is homogeneous due to the convective mixing, so that μ is constant. At the centre $r = 0$, $M_r = 0$. Introducing as a variable $y = (1-\beta)/\beta$, the ratio of the radiation pressure to the gas pressure, one of the differential equations can be eliminated, leading to

$$\frac{\mathrm{d}M_r}{\mathrm{d}r} = \left(\frac{4\pi a\mu T_\mathrm{c}^3}{3\mathscr{R}y_\mathrm{c}^2}\right) r^2 y \exp\{8(y-y_\mathrm{c})\}\,,$$

$$\frac{\mathrm{d}y}{\mathrm{d}r} = -\left(\frac{3G\mu y_\mathrm{c}}{\mathscr{R}T_\mathrm{c}}\right)\left(\frac{M_r}{r^2}\right)\left(\frac{(y/y_\mathrm{c})^{\frac{1}{3}}}{5+40y+32y^2}\right)\exp\{-8(y-y_\mathrm{c})/3\}\,.$$

Finally, using the transformation

$$\xi = r\left(\frac{4\pi aG\mu^2T_\mathrm{c}^2}{3\mathscr{R}^2y_\mathrm{c}}\right)^{\frac{1}{2}} \doteqdot \frac{\mu r T_\mathrm{c}/R_\odot}{2\cdot 6\times 10^7\, y_\mathrm{c}^{\frac{1}{2}}}\,,$$

$$\zeta = M_r\left(\frac{4\pi aG^3\mu^4}{3\mathscr{R}^4y_\mathrm{c}}\right)^{\frac{1}{2}} \doteqdot \frac{\mu^2 M_r/M_\odot}{1\cdot 135\, y_\mathrm{c}^{\frac{1}{2}}}$$

to remove the dimensioned quantities, the equations become

$$\frac{\mathrm{d}\zeta}{\mathrm{d}\xi} = \left(\frac{y}{y_\mathrm{c}}\right)\xi^2 \exp\{8(y-y_\mathrm{c})\}\,, \tag{2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}\xi} = -\left(\frac{3(y/y_\mathrm{c})^{\frac{1}{3}}}{5+40y+32y^2}\right)\left(\frac{y_\mathrm{c}\zeta}{\xi^2}\right)\exp\{-8(y-y_\mathrm{c})/3\}\,, \tag{3}$$

with the boundary conditions $y = y_\mathrm{c}$, $\zeta = 0$ at the centre $\xi = 0$. The solutions depend on a single parameter y_c.

The first seven terms of the power series for y/y_c and ζ were obtained on a CDC 3600 computer, a program being specifically written in Fortran for the purpose. The method used was the following iterative one: given the expansion for y/y_c as far as the power ξ^n, this polynomial was substituted in the right-hand side of

TABLE 1

COEFFICIENTS IN THE SERIES FOR ζ

p	m	a_k	b_k	k
3	0	1	3	0
5	1	−1	10	0
		−4	5	1
7	3	1	12	0
		47	21	1
		96	5	2
		6656	105	3
		4864	105	4
9	5	−25	486	0
		−4435	1701	1
		−26164	567	2
		−721408	1701	3
		−17801344	8505	4
		−15337472	2835	5
		−51613696	8505	6
		−20283392	8505	7
11	7	95	3564	0
		36515	16632	1
		372278	6237	2
		526868	567	3
		271817248	31185	4
		1635031424	31185	5
		890764288	4455	6
		1328226304	2835	7
		96979877888	155925	8
		22025076736	51975	9
		17948475392	155925	10
13	9	−13595	1111968	0
		−250325	162162	1
		−27391733	486486	2
		−103779496	81081	3
		−3111924704	173745	4
		−10012093952	57915	5
		−2353079962624	2027025	6
		−1590252163072	289575	7
		−36620267831296	2027025	8
		−1400966742016	34749	9
		−592265608167424	10135125	10
		−1578894872281088	30405375	11
		−86351676440576	3378375	12
		−162174341218304	30405375	13
15	11	103105	20015424	0
		22610195	23351328	1
		18528551	428652	2
		20821163516	15324309	3
		1931663470768	76621545	4
		26128164559136	76621545	5
		254364989556992	76621545	6
		838885745437696	34827975	7
		83391815485751296	638512875	8
		1013113089163526144	1915538625	9
		1010239672836161536	638512875	10
		52596172527763456	15324309	11
		50310836976544120832	9577693125	12
		52351506534442729472	9577693125	13
		35017599219008536576	9577693125	14
		13542207561335308288	9577693125	15
		209211988746698752	870699375	16

TABLE 2

COEFFICIENTS IN THE SERIES FOR y/y_c

p	m	a_k	b_k	k
0	0	1	1	0
2	1	−1	2	0
4	3	7	12	0
		1	1	1
		32	5	2
		128	15	3
6	5	−25	54	0
		25	7	1
		−1504	63	2
		−24064	189	3
		−11392	35	4
		−131072	315	5
		−212992	945	6
8	7	95	324	0
		−46835	4536	1
		56593	567	2
		141668	567	3
		2944448	2835	4
		2038144	405	5
		40073216	2835	6
		316669952	14175	7
		258998272	14175	8
		92274688	14175	9
10	9	−13595	85536	0
		1088015	74844	1
		−11405893	37422	2
		14985064	18711	3
		−288128	4455	4
		−88542208	4455	5
		−5181129728	51975	6
		−52558508032	155925	7
		−24998674432	31185	8
		−54644965376	42525	9
		−3069068705792	2338875	10
		−1785632653312	2338875	11
		−470567354368	2338875	12
12	11	515525	6671808	0
		−8806625	598752	1
		480907165	785862	2
		−10402287095	1702701	3
		21032110496	1702701	4
		460548009536	5108103	5
		1350492898816	5108103	6
		3575168003072	2837835	7
		240595609133056	42567525	8
		218243374514176	11609325	9
		13446443696128	297675	10
		9782380394971136	127702575	11
		191917660831744	2149875	12
		14533168602284032	212837625	13
		3949720644878336	127702575	14
		4133613964623872	638512875	15

(2) and the expression expanded in powers of ξ; one integration gave a polynomial for ζ; substitution in (3) and a second integration gave the expansion for y/y_c up to the term in ξ^{n+2}. The alternative method of substituting a series with undetermined coefficients, and then solving successively for these coefficients, was not used because of the prohibitive amount of storage that would have been required. The series for T/T_c was obtained from the series for y/y_c by substitution in Henrich's expression

$$T/T_c = (y/y_c)^{\frac{2}{3}} \exp\{8(y-y_c)/3\} \tag{4}$$

followed again by expansion in powers. Rational arithmetic was used throughout. The rational numbers were kept in their reduced form by using the Euclidean algorithm to calculate the greatest common divisor of numerator and denominator.

A partial check on the results was obtained by comparing the series for T/T_c in the limiting cases $y_c \to 0$ and $y_c \to \infty$ with the series for the Emden function. Letting y_c approach zero and replacing $2\xi^2/5$ by ξ^2 leads to the Emden function with index $n = 1\cdot5$, while replacing $\xi^2/4y_c$ by ξ^2 and letting y_c approach infinity leads to the Emden function with index 3. The Emden series for general n, up to the term in ξ^{20}, was derived separately on the computer. It was also found to agree in the case $n = 1\cdot5$ with the series given by See.

III. Numerical Values

Table 1 gives the exact rational coefficients of the series for ζ (the mass variable) in the form

$$\zeta = \tfrac{1}{3}\xi^3 - \left(\frac{\frac{1}{10}+\frac{4}{5}y_c}{5+40y_c+32y_c^2}\right)\xi^5 + \ldots + \left(\frac{\sum a_k y_c^k/b_k}{(5+40y_c+32y_c^2)^m}\right)\xi^p + \ldots .$$

In a similar manner, Table 2 gives coefficients in the series

$$\frac{y}{y_c} = 1 - \left(\frac{\frac{1}{2}}{5+40y_c+32y_c^2}\right)\xi^2 + \left(\frac{\frac{7}{12}+y_c+\frac{32}{5}y_c^2+\frac{128}{15}y_c^3}{(5+40y_c+32y_c^2)^3}\right)\xi^4 + \ldots ,$$

and Table 3 gives coefficients in the series for T/T_c. Given two central values, such as T_c and β_c, and a point r in the core, the series then give directly y/y_c and M_r. Once y is known, T is in fact most easily obtained from (4), and P and ρ from (1).

The rates of convergence of the series in Tables 1, 2, and 3 decrease both as the distance from the centre of the star increases and as the radiation pressure increases. However, due to the regular behaviour of the terms, convergence can be substantially accelerated by using the e_1 method described by Shanks (1955), which consists of applying the transformation

$$S_n^* = \frac{S_{n+1}^2 - S_{n+2}\,S_n}{2S_{n+1} - S_{n+2} - S_n}$$

to the sequence $\{S_n\}$ of partial sums. As an example, Table 4 compares the values of y and ζ, obtained from the series, with the values obtained by direct numerical integration of the differential equations (2) and (3). In each comparison, the first line gives the true value from the numerical integration, the second line gives the

Table 3

COEFFICIENTS IN THE SERIES FOR T/T_c

p	m	a_k	b_k	k
0	0	1	1	0
2	1	−1	3	0
		−4	3	1
4	3	1	4	0
		10	3	1
		424	15	2
		1088	15	3
		256	5	4
6	5	−5	36	0
		−460	189	1
		−11624	189	2
		−12064	27	3
		−1932032	945	4
		−128000	27	5
		−4726784	945	6
		−622592	315	7
8	7	125	1944	0
		5165	6804	1
		47464	567	2
		438152	567	3
		20995456	2835	4
		116415232	2835	5
		693517312	4725	6
		4591812608	14175	7
		5835882496	14175	8
		11718098944	42525	9
		649068544	8505	10
10	9	−95	3564	0
		43135	149688	1
		−214919	2079	2
		−10360388	18711	3
		−437654144	31185	4
		−743760128	6237	5
		−39260081152	51975	6
		−58821050368	17325	7
		−332048728064	31185	8
		−1184764788736	51975	9
		−10697471688704	334125	10
		−4359484080128	155925	11
		−32193699119104	2338875	12
		−2297404850176	779625	13
12	11	67975	6671808	0
		−453800	729729	1
		26789965	224532	2
		−981043270	1702701	3
		870515488	35721	4
		1018947275968	5108103	5
		48468218177536	25540515	6
		48451536705536	3648645	7
		587249080582144	8513505	8
		34324846021378048	127702575	9
		495667670422913024	638512875	10
		1042768401212112896	638512875	11
		4946707922026496	2027025	12
		45637075000623104	18243225	13
		1064163139886514176	638512875	14
		415458029845086208	638512875	15
		10379157837971456	91216125	16

sum of the first seven terms of the Taylor series, and the third and fourth lines give the result of applying the e_1 transform once and twice respectively to the series. The distance from the centre ξ was chosen to correspond approximately with the core boundary of the massive homogeneous stellar model with the same value of β_c (Van der Borght and Meggitt 1963). The table also quotes the values of r/R at the core boundary and $\mu^2 M/M_\odot$ for these models. Direct comparison with the tables of Henrich gave results of a similar order of accuracy.

TABLE 4

COMPARISON BETWEEN TRUE VALUES OF y AND ζ AT POINT ξ AND VALUES OBTAINED FROM THE SERIES

y_c	$\frac{1}{4}$	$\frac{2}{3}$
ξ	3·94	6·07
y: True value	0·15011	0·43896
y: Sum of seven terms of series	0·15012	0·44008
y: e_1 method	0·15011	0·43901
y: e_1^2 method	0·15011	0·43901
ζ: True value	9·7985	22·343
ζ: Sum of seven terms of series	9·8612	43·817
ζ: e_1 method	9·7974	22·119
ζ: e_1^2 method	9·7985	22·340
β_c	0·8	0·6
$\mu^2 M/M_\odot$	9·8	26·0
r/R	0·3885	0·4965

IV. Total Luminosity

The total energy produced in the star is obtained from the quadrature

$$L = \int_0^R 4\pi r^2 \rho\epsilon \, dr. \tag{5}$$

In the particular case in which ϵ can be approximated by $\epsilon_0 \rho^k T^s$ and the model is a polytrope of index n, this integral has been evaluated by Hayakawa *et al.* (1956) in the form of the rapidly convergent series

$$L = \epsilon_0 \rho_c^{k+1} T_c^s \left(\frac{3n+3}{n(1+k)+s} \cdot \frac{\mathscr{R} T_c}{2G\mu\rho_c}\right)^{3/2} \left\{1 + \frac{15}{8}\left(\frac{3n}{5} - 1\right)\left(\frac{1}{n(1+k)+s}\right) + \ldots\right\}.$$

Their method is readily extended to the convective core with radiation pressure. Using

$$\rho/\rho_c = (y/y_c)\exp\{8(y-y_c)\}$$

and formula (4), there follows

$$\begin{aligned}\epsilon\rho &= \epsilon_0 \rho^{k+1} T^s \\ &= \epsilon_0 \rho_c^{k+1} T_c^s (y/y_c)^{k+1+2s/3} \exp\{8(y-y_c)(k+1+\tfrac{1}{3}s)\}.\end{aligned}$$

The integrand of (5) can now be expanded in a series of even powers of ξ. The second term, the term in ξ^4, is next incorporated in an exponential which is factored out from the series. This finally leads to

$$\int_0^\infty 4\pi r^2\,\rho\epsilon\,\mathrm{d}r = 4\pi\epsilon_c\,\rho_c\left(\frac{3\mathscr{R}^2 y_c}{4\pi a G\mu^2 T_c^2}\right)^{3/2}\int_0^\infty \{\exp(-\lambda\xi^2)\}(1+a_4\,\xi^4+a_6\,\xi^6+\ldots)\xi^2\,\mathrm{d}\xi$$

$$= C\lambda^{-3/2}\left(1+\tfrac{15}{4}a_4\,\lambda^{-2}+\tfrac{105}{8}a_6\,\lambda^{-3}+\ldots\right),$$

where

$$C = \epsilon_c(3\mathscr{R}^4 y_c/256\,aG^3\mu^4)^{\frac{1}{2}},$$

$$\lambda = \tfrac{1}{2}\{k+1+\tfrac{2}{3}s+8y_c(k+1+\tfrac{1}{3}s)\}(5+40y_c+32y_c^2)^{-1},$$

$$(5+40y_c+32y_c^2)^3 a_4 = (k+1)\left(-\tfrac{1}{24}+\tfrac{2}{3}y_c+\tfrac{52}{5}y_c^2+\tfrac{896}{15}y_c^3+\tfrac{1024}{15}y_c^4\right)$$
$$+s\left(-\tfrac{1}{36}-\tfrac{10}{9}y_c+\tfrac{64}{15}y_c^2+\tfrac{1024}{45}y_c^3+\tfrac{1024}{45}y_c^4\right),$$

$$(5+40y_c+32y_c^2)^5 a_6 = (k+1)\left(-\tfrac{5}{108}-\tfrac{995}{378}y_c-\tfrac{1888}{63}y_c^2-\tfrac{49072}{189}y_c^3\right.$$
$$\left.-\tfrac{1052416}{945}y_c^4-\tfrac{908288}{315}y_c^5-\tfrac{671744}{189}y_c^6-\tfrac{1703936}{945}y_c^7\right)$$
$$+s\left(-\tfrac{5}{162}-\tfrac{295}{567}y_c-\tfrac{5576}{189}y_c^2-\tfrac{8864}{81}y_c^3-\tfrac{1142272}{2835}y_c^4\right.$$
$$\left.-\tfrac{142336}{135}y_c^5-\tfrac{3571712}{2835}y_c^6-\tfrac{1703936}{2835}y_c^7\right).$$

V. References

Hayakawa, S., Hayashi, C., Imoto, M., and Kikuchi, K. (1956).—*Progr. theor. Phys., Osaka* **16**, 507.

Hayashi, C., and Cameron, R. C. (1962).—*Astrophys. J.* **136**, 166.

Hayashi, C., Hoshi, R., and Sugimoto, D. (1962).—*Progr. theor. Phys., Osaka, Suppl.* No. 22.

Henrich, L. R. (1941).—*Astrophys. J.* **93**, 483.

See, T. J. J. (1905).—*Astr. Nachr.* **169**, 322.

Shanks, D. (1955).—*J. Math. Phys.* **34**, 1.

Van der Borght, R., and Meggitt, S. (1963).—*Aust. J. Phys.* **16**, 415.