A METHOD OF CORRECTION FOR UNRESOLVED LEVELS IN FISSILE NUCLEI

By A. R. Musgrove*

[*Manuscript received October* 13, 1966]

Summary

A method of correcting for unresolved levels in fissile nuclei is described. The method is based on the distribution of the ratio of the reduced neutron width to the total width of a level. Calculations are carried out on three sets of data for ^{233}U, and it is shown that approximately one-quarter of the resonance levels are undetected.

I. Introduction

In many applications of low energy neutron cross-section data it is not necessary to know the resonance parameters of particular levels, but only average parameters such as the *s*-wave strength function and the mean level spacing. To determine these quantities an important correction must be made to the experimental data for resonances that are unresolved in the measurement.

A method for calculating this correction based on the Porter–Thomas distribution of reduced neutron widths has recently been described by Fuketa and Harvey (1965). An experimental resolution limit varies as a power of the neutron energy, and levels lying below this limit are undetected in the measurement. Integration of the Porter–Thomas distribution up to this limit, over the energy range under consideration, gives the relative probability for missing levels. However, when overlapping of levels occurs a level may remain unresolved with a reduced neutron width in excess of the limit of detectability. For levels of the same spin sequence, overlapping between resonances is negligible, owing to the level repulsion effect, but the probability of two levels overlapping becomes appreciable when two spin sequences are superimposed, since the spacing distribution for each state is independent. The probability of finding a level spacing less than $\bar{\Gamma}$ for a superposition of two independent level sequences with average level spacings $\bar{D}_1$, $\bar{D}_2$ and average level width $\bar{\Gamma}$ is given by

$$P(D \leqslant \bar{\Gamma}) = \frac{\bar{D}}{\bar{D}_1}\,\mathrm{erf}\left(\frac{\pi^{1/2}\bar{\Gamma}}{2\bar{D}_2}\right)\exp\left(\frac{-\pi\bar{\Gamma}^2}{4\bar{D}_1^2}\right) + \frac{\bar{D}}{\bar{D}_2}\,\mathrm{erf}\left(\frac{\pi^{1/2}\bar{\Gamma}}{2\bar{D}_1}\right)\exp\left(\frac{-\pi\bar{\Gamma}^2}{4\bar{D}_2^2}\right), \qquad (1)$$

where $1/\bar{D} = 1/\bar{D}_1 + 1/\bar{D}_2$ and $\mathrm{erf}\, z = \dfrac{2}{\pi^{1/2}}\displaystyle\int_0^z e^{-t^2}\,dt\,.$

The function P is shown in Figure 1 plotted against the ratio $\bar{D}/\bar{\Gamma}$ for the case $\bar{D}_1 = \bar{D}_2$. For nonfissile nuclides for which $\bar{D}/\bar{\Gamma}$ is large the probability of levels overlapping is still small, and, therefore, the method of correction described above is quite adequate. We cannot ignore levels missed by overlapping in the fissile nuclides, however, because of their much smaller value of $\bar{D}/\bar{\Gamma}$, and the method of correction described below was developed specifically to deal with these cases.

* Australian Atomic Energy Commission Research Establishment, Lucas Heights, N.S.W.

In the single-level Breit–Wigner approximation to the neutron cross section the peak height of a resonance is given by $\sigma_0 = \lambda^2 g\Gamma_n/\pi\Gamma$, where λ is the de Broglie wavelength of the neutron (in the centre-of-mass system), Γ_n is the neutron width and Γ the total width of a resonance, and g is the spin weighting factor. When σ_0 is small compared with the background cross section there will be a large probability of failure to resolve the level. We base our method of correction on the distribution of the resonance peak heights, comparing the energy-independent ratio Γ_n^0/Γ, where Γ_n^0 is the reduced neutron width, with the predicted distribution. This gives a more powerful test for missed levels in those nuclei where the correction is expected to be greatest.

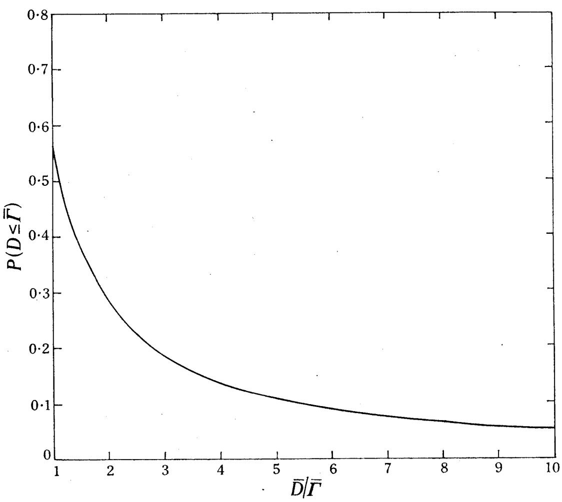

Fig. 1.—Probability P for a level spacing D less than the mean level width $\bar{\Gamma}$ for two superimposed level sequences with equal mean level spacings, plotted against the ratio $\bar{D}/\bar{\Gamma}$. $\bar{D}$ is the mean spacing of the double sequence.

II. The Method of Correction

A detectability limit is selected, and the number of levels for which Γ_n^0/Γ lies below the limit is compared with the number predicted from the theoretical distribution. The probable number of levels missed below the chosen detectability limit is the difference between the expected and the observed numbers. The limit of detectability is then increased until the number of unresolved levels below this limit remains constant (except for statistical fluctuations), and this then gives the total number of levels missed in the measurement. Since we cover a wide range of limits we no longer need to take into account the energy dependence of the cutoff value. Furthermore, the correction now includes levels that are undetected as a result of overlapping with neighbouring levels, since we use a variable detection limit.

Suppose that N_{obs} levels are observed experimentally in an energy range $E_N - E_0$, where E_N and E_0 are respectively the energies of the highest and lowest levels in the sequence. The ith resonance has reduced neutron width $\Gamma^0_{\text{n}i}$, fission width $\Gamma_{\text{f}i}$, and total width Γ_i. Further, suppose that the total number of missed levels in the energy range is r and the number of levels missed below a detectability limit k is r_k. Then the actual average level spacing $\bar{D}$, the average reduced neutron width $\bar{\Gamma}^0_{\text{n}}$, and the average total width $\bar{\Gamma}$ may be written

$$\bar{D} = \frac{E_N - E_0}{N_{\text{obs}} + r - 1}, \tag{2}$$

$$\bar{\Gamma}^0_{\text{n}} = \frac{\sum\limits_{i=1}^{N_{\text{obs}}} \Gamma^0_{\text{n}i} + r\bar{\Gamma}^{0\prime}_{\text{n}}}{N_{\text{obs}} + r}, \tag{3}$$

$$\bar{\Gamma} = \frac{\sum\limits_{i=1}^{N_{\text{obs}}} \Gamma_i + r\bar{\Gamma}'}{N_{\text{obs}} + r}, \tag{4}$$

where $\bar{\Gamma}^{0\prime}_{\text{n}}$ is the average reduced neutron width of the missed levels and $\bar{\Gamma}'$ is the average total width of the missed levels. The expected number of levels missed below the detectability limit k is given by

$$r_k = \{N_{\text{obs}} + r\}\, P_\gamma(k) - N_k, \tag{5}$$

where N_k is the number of levels observed below the cutoff k and $P_\gamma(k)$ is the probability of finding a level with $\Gamma^0_{\text{n}}/\Gamma \leqslant k$. For k not too small we may approximate r by r_k in equation (5) and solve by iteration:

$$r_k^{(m)} \approx \{N_{\text{obs}} + r_k^{(m-1)}\}\, P_\gamma^{(m-1)}(k) - N_k, \tag{5a}$$

where m is the order of the iteration; $r_k^{(0)}$ is put equal to zero and the iteration continued until $r_k^{(m)} = r_k^{(m-1)}$; then r may be estimated by averaging r_k over a number of detectability limits.

It is possible by this method of correction to detect overall imperfections in the original data. If the number of missed levels continues to decrease as the detectability limit increases it is an indication that spurious levels have been included in the analysis. These levels typically are given small values of the ratio $\Gamma^0_{\text{n}}/\Gamma$ and are included in order to improve the fit to the experimental cross section.

III. The Distribution Function $P_\gamma(k)$

It is well known that the distribution of reduced neutron widths about their mean may be represented by a Porter–Thomas distribution with one degree of freedom

$$P(x)\,\mathrm{d}x = (2\pi)^{-1/2}\,\mathrm{e}^{-x/2}\,x^{-1/2}\,\mathrm{d}x \qquad (0, \infty), \tag{6}$$

where $x = \Gamma_n^0/\bar{\Gamma}_n^0$. The fission-width distribution is usually represented by a member of the same family of curves with n degrees of freedom

$$P(w)\,\mathrm{d}w = \frac{n^{n/2}}{2^{n/2}\,\Gamma(\frac{1}{2}n)}\,\mathrm{e}^{-nw/2}\,w^{(n/2)-1}\,\mathrm{d}w \qquad (0,\,\infty), \tag{7}$$

where $w = \Gamma_f/\bar{\Gamma}_f$. In this case n is not normally an integer, since the partial fission widths in each channel do not necessarily have the same mean, and, also, when two spin states are present the average fission width for each state may be different. If we assume that the radiation width Γ_γ is constant, the probability element for $\Gamma/\bar{\Gamma}_f$ may be obtained from equation (7) by substituting $y = w+c$, where $y = \Gamma/\bar{\Gamma}_f$ and $c = \bar{\Gamma}_\gamma/\bar{\Gamma}_f$, a constant, to obtain

$$P(y)\,\mathrm{d}y = \frac{n^{n/2}\,\mathrm{e}^{nc/2}}{2^{n/2}\,\Gamma(\frac{1}{2}n)}\,\mathrm{e}^{-ny/2}\,(y-c)^{(n/2)-1}\,\mathrm{d}y \qquad (c,\,\infty). \tag{8}$$

The joint probablility element in the distribution of x and y is, therefore,

$$P(x,y)\,\mathrm{d}x\,\mathrm{d}y = K\,\mathrm{e}^{-(x+ny)/2}\,x^{-1/2}\,(y-c)^{(n/2)-1}\,\mathrm{d}x\,\mathrm{d}y, \tag{9}$$

where

$$K = \frac{n^{n/2}\,\mathrm{e}^{nc/2}}{2^{n/2}\,\Gamma(\frac{1}{2}n)(2\pi)^{1/2}}.$$

The element for the ratio x/y is then obtained by making the substitution $x = uv$, $y = v/n$ in equation (9). The distribution function for the variate x/y becomes

$$P(x/y \leqslant t) = \frac{\mathrm{e}^{nc/2}}{2^{n/2}\,\Gamma(\frac{1}{2}n)\,(2\pi)^{1/2}} \int_0^{t/n} u^{-1/2}\,\mathrm{d}u \int_{nc}^{\infty} \mathrm{e}^{-(1+u)v/2}\,v^{1/2}\,(v-nc)^{(n/2)-1}\,\mathrm{d}v. \tag{10}$$

The integration over v cannot be performed analytically, and to facilitate computation of the integral in applications we integrate over u to obtain

$$P(x/y \leqslant t) = \frac{\mathrm{e}^{nc/2}}{2^{n/2}\,\Gamma(\frac{1}{2}n)} \int_{nc}^{\infty} \mathrm{e}^{-v/2}\,(v-nc)^{(n/2)-1}\,\mathrm{erf}(tv/2n)^{1/2}\,\mathrm{d}v. \tag{11}$$

It is now a simple matter to obtain $P_\gamma^n(k)$ as

$$P_\gamma^n(k) = \mathrm{P}(\Gamma_n^0/\Gamma \leqslant k) = \frac{\mathrm{e}^{nc/2}}{2^{n/2}\,\Gamma(\frac{1}{2}n)} \int_{nc}^{\infty} \mathrm{e}^{-v/2}\,(v-nc)^{(n/2)-1}\,\mathrm{erf}\left(\frac{k\bar{\Gamma}_f\,v}{2n\bar{\Gamma}_n^0}\right)^{1/2}\mathrm{d}v. \tag{12}$$

The superscript n refers to the number of degrees of freedom in the fission-width distribution.

IV. Calculation of Missed Levels for ^{233}U

Calculations were made for the number of missed levels in ^{233}U using three different sets of experimental data. These were the compilation of recommended values of Stehn *et al.* (1965) and two analyses of Nifenecker (Nifenecker 1964; Nifenecker and Perrin 1965). The energy interval selected was 1·8–26·0 eV.

Recently, Lynn (1964) has suggested that resonance parameters obtained from the cross sections of fissile nuclei may be greatly in error due to the level–level interference term. To test this suggestion Monte Carlo methods were used to generate cross sections with both multilevel and single-level formulae. More realistic average parameters were inserted than those used by Lynn, and it was then found that, although large differences between multilevel and single-level cross sections occurred in valleys between resonances, the discrepancy at the peak height of a resonance was rarely greater than 10%. However, the resonance half-widths were often altered by a greater amount. The same levels were "missed" in both methods of generation.

Since our method in effect simply corrects the distribution of peak heights, it is thought to be independent of the method used in obtaining parameters, provided interference has no systematic effect on the fission widths and, hence, on $\bar{\Gamma}_f$. Indirectly, however, level–level interference may have a significant effect on the average parameters. This is caused by the inclusion of broad levels in some analyses to lessen the discrepancies caused by interference in the valleys between resonances. The effect of these spurious levels is to increase $\bar{\Gamma}$ and N_{obs}, thereby giving too large a number of missed levels calculated for small detectability limits. The number of levels missed then decreases with increase in the limit of detectability as spurious levels are included in N_k, the number of levels observed below the limit.

Table 1

AVERAGE DATA VALUES USED IN CALCULATION

	Stehn *et al.* (1965)	Nifenecker (1964)	Nifenecker and Perrin (1965)
N_{obs}	25	27	30
$\sum_{i=1}^{N_{obs}} \Gamma_{ni}^{0}$ (mV)	4·79	4·76	5·18
$\bar{\Gamma}_f$ (mV)	283	304	415
$\bar{\Gamma}_\gamma$ (mV)	45	45	45

In the more recent data of Nifenecker, single-level parameters were obtained by performing a least-squares fit to the fission cross section. Asymmetric resonances were assumed to be due to the overlap of a broader level with a narrow resonance, and, furthermore, broad levels were included to improve the fit between resonances. The average fission width for this set of data is therefore much larger than that in the other two. In order to bring this set more closely in agreement with the earlier compilations we omit the resonances at 2·19, 6·72, and 15·56 eV for this calculation. Table 1 gives the data averages used in the calculation. In all cases the average reduced neutron width for the missed levels was assumed to be 0·02 mV. When other values were tried it was found that 100% variations produced differences of less than one level in the number of levels missed. Missed levels were assumed to have the same average total width as that for the observed levels. The number of degrees of freedom assumed for the fission-width distribution was 3, and a range of detectability limits from 1×10^{-5} to 3×10^{-4} was chosen. Figure 2 illustrates the number of levels missed below each detectability limit for each set of data.

There is a tendency for the correction to the data of Stehn *et al.* to show a continuous rise with increasing detectability limit, while that for the more recent Nifenecker data falls after peaking for small values of the cutoff. This may be taken as evidence of significant departures of the appropriate sets of parameters from the

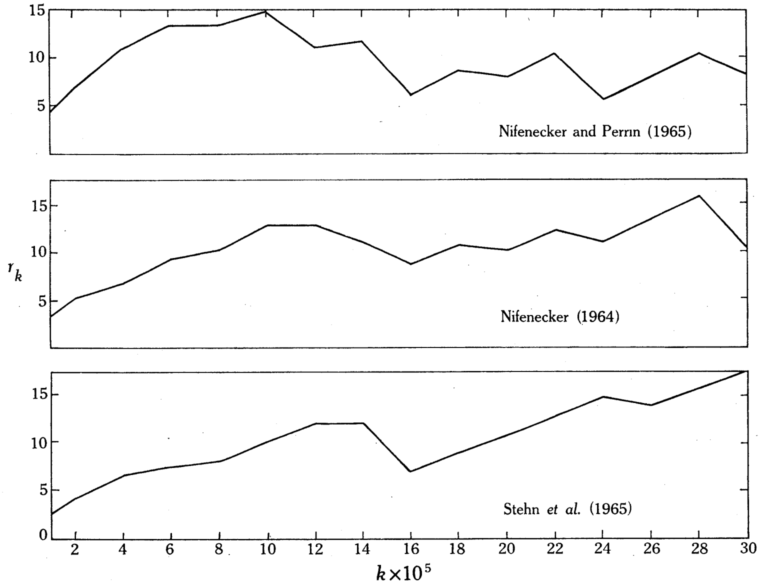

Fig. 2.—Most probable number of missed levels r_k below detectability limit k, shown for the three data compilations used.

TABLE 2

CORRECTED AVERAGE PARAMETERS

	Stehn *et al.* (1965)	Nifenecker (1964)	Nifenecker and Perrin (1965)
Number of levels missed	12·6	11·4	8·7
Variance of number missed	10·0	4·0	3·5
Total number of levels present	38 ± 3	38 ± 2	39 ± 2
$\bar{D}$ (eV)	$0\cdot65\pm0\cdot05$	$0\cdot65\pm0\cdot03$	$0\cdot64\pm0\cdot03$
$2g\bar{\Gamma}_n^0$ (mV)	0·13	0·13	0·14

expected distribution shape, either because of the inclusion of spurious levels or from the poor quality of the data. However, when we average r_k over detectability limits greater than or equal to $1\cdot4\times10^{-4}$, good agreement is obtained among the three data compilations for the total number of levels in the energy range under consideration. Table 2 shows the average parameters when the correction produced by missed levels is included. The *s*-wave strength function $\bar{\Gamma}_n^0/\bar{D} = 2\times10^{-4}$.

V. Conclusion

The method of correction for missed levels described here in effect corrects an experimental distribution of resonance parameters by comparing this with the expected distribution. Since the method is insensitive to the precise experimental resolution limit, unlike the method of Fuketa and Harvey, in which a careful determination of the limit must be made, levels missed above the resolution limit owing to overlapping are included in this correction. Furthermore, provided the number of levels considered is large enough to permit reasonable statistical accuracy, this type of test may show the presence of other systematic deviations from the expected distribution of resonance parameters.

VI. Acknowledgment

The author wishes to thank Dr. J. R. Bird for his interest and for many helpful suggestions and comments.

VII. References

Fuketa, T., and Harvey, J. A. (1965).—*Nucl. Instrum. Meth.* **33**, 107.

Lynn, J. E. (1964).—*Phys. Rev. Lett.* **13**, 412.

Nifenecker, H. (1964).—*Jnl Physique* **25**, 877.

Nifenecker, H., and Perrin, G. (1965).—"Proceedings of the Symposium on Physics and Chemistry of Fission." p. 245. (International Atomic Energy Agency: Salzburg.)

Stehn, J. R., Goldberg, M. D., Wiener-Chasman, R., Mughabghab, S. F., Magurno, B. A., and May, V. M. (1965).—"Neutron Cross Sections." Brookhaven National Laboratory Report (B.N.L. 325 Supplement No. 2, Vol. III).