MULTIPARTICLE COLLISIONS

I. ANGULAR MOMENTUM EIGENSTATES

By J. L. Cook*

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Summary

The current situation with regard to relativistic representation of multiparticle angular momentum eigenstates is reviewed and it is concluded that no generally satisfactory formalism exists. Difficulties with the formalism are outlined and a general method of construction of partial wave amplitudes is put forward.

I. Introduction

One of the most important problems that confront workers in the theory of multiparticle reactions for both low and high energy physics is to understand the role that the law of conservation of angular momentum plays in determining the analytic structure of scattering and production amplitudes. For example, some authors have advocated that the inclusion of this law in the Fermi statistical theory of high energy multiparticle production processes might remove discrepancies between this theory and experiment. Of course, this is directly related to the general difficulties encountered when trying to relate scattering amplitudes to production amplitudes. However, the moderate overall successes of the statistical theory imply that purely kinematical constraints are a significant factor in determining the behaviour of production amplitudes, so it appears reasonable to expect that the inclusion of the conservation laws for rotational motion would provide further insight into the structure of matrix elements. It would be especially provident if the general structure of inelastic scattering theory could be ascertained by doing so, without having to delve into the extremely difficult question of interactions. This could be done by deriving a general multiparticle partial wave formalism.

In the present paper, the various constraint conditions are taken into account and, after considering generalizations of non-relativistic representations, a formal structure for the partial wave amplitudes is put forward. In a paper of this kind it is difficult to separate the original content from what constitutes a review of other works. This problem has been partly overcome by confining remarks to shortcomings found in other formalisms, or their generalizations. No claim is made of having constructed a detailed general theory, but the consequences of the postulates listed here are explored in the following paper (Part II, pp. 355–61 present issue).

* Australian Atomic Energy Commission Research Establishment, Lucas Heights, N.S.W.

II. Kinematical Constraints

Consider the constraints imposed by conservation of linear energy–momentum upon a multiparticle production process. The dynamical variables defined in this section refer to the overall barycentric system, and the masses are taken to be arbitrary within the limitations imposed by the process being a physical one. Let the direction of the ith particle in the initial state define the z direction of one polar coordinate system, and the direction of the jth particle in the final state define the z direction of a second similar set of coordinates. Altogether, there are $4n$ components of four-momentum for a total of n particles. Let p_i and p_j be the three-momentum values of the ith particle in the initial state and the jth particle in the final state respectively. In the barycentric system

$$\sum_i^{N_i} \mathbf{p}_i = 0\,, \qquad \sum_j^{N_f} \mathbf{p}_j = 0\,, \qquad N_i + N_f = n\,, \tag{1}$$

where N_i and N_f are the numbers of particles in the initial and final states respectively. These conditions remove six degrees of freedom from the system. Let E_i, E_j and M_i, M_j be the energies and masses respectively of particles in the initial and final states. The constraint imposed by the mass conditions,

$$E_i^2 - \mathbf{p}_i^2 = M_i^2\,, \qquad E_j^2 - \mathbf{p}_j^2 = M_j^2\,, \tag{2}$$

removes a further n degrees of freedom. The equation expressing conservation of energy,

$$\sum_{i=1}^{N_i} E_i = \sum_{j=1}^{N_f} E_j = W\,, \tag{3}$$

removes one degree of freedom, while the z axis and two azimuthal angles can be chosen to remove the dependence of a transition amplitude upon three more angular variables. This leaves a total of $3n-10$ degrees of freedom.

III. Angular Momentum Representations

In this section some of the difficulties in generalizing the various proposed angular momentum representations to a covariant formalism are discussed. The first example encountered in inelastic scattering theory is the special case of the process (2 particles $\to$ 3 particles). Quite a number of non-relativistic techniques have been put forward as a means of analysing this reaction. The most fully developed to date is that propounded by Smith (1960), based upon a fundamental paper by Delves (1958).

By looking at the manner in which variables separate from the three-particle kinetic energy operator in the Schrödinger equation, Smith was able to perceive a hierarchy of multiparticle barycentric angular momentum operators in the non-relativistic approximation. However, when one tries to generalize his scheme, whether in terms of barycentric coordinates defined relative to physical or to proper masses, the construction of Lorentz invariant amplitudes runs into unresolved difficulties.

Consider the mass structure of the coordinate transformation operator for a system of three particles with masses m_1, m_2, and m_3,

$$\boldsymbol{M} = \begin{vmatrix} \frac{m_1}{M} & \frac{m_2}{M} & \frac{m_3}{M} \\ \frac{m_1+m_2}{M} & \frac{m_1+m_2}{M} & \frac{-m_3}{M} \\ \frac{m_1}{m_1+m_2} & \frac{-m_2}{m_1+m_2} & 0 \end{vmatrix} \quad \left\{ \begin{array}{ll} M = m_1+m_2+m_3, & \\ \zeta = \boldsymbol{M}\boldsymbol{r} & \text{(barycentric coordinates),} \\ \boldsymbol{q} = (\boldsymbol{M})^{-1}\boldsymbol{p} & \text{(barycentric momenta).} \end{array} \right. \tag{4}$$

To generalize Smith's treatment, we require functions that are eigenstates of the invariant operator

$$\begin{aligned} T &= \tfrac{1}{2}\left(\frac{p_1^2}{m_1}+\frac{p_2^2}{m_2}+\frac{p_3^2}{m_3}\right) \\ &= \tfrac{1}{2}\left(\frac{q_1^2}{\mu_1}+\frac{q_2^2}{\mu_2}+\frac{q_3^2}{\mu_3}\right), \end{aligned} \tag{5}$$

where $\mu_1 = M$, $\mu_2 = (m_1+m_2)m_3/M$, $\mu_3 = m_1m_2/(m_1+m_2)$, and (p_1, p_2, p_3), (q_1, q_2, q_3) are sets of four-momenta.

We could choose the masses to be either physical or proper masses, giving two distinct sets of reference coordinates.

(*a*) *Physical Barycentric System*

The advantages are:

(i) The correct number of degrees of freedom (five) is obtained directly from the coordinate transformation and constraint conditions (1), (2), and (3).

(ii) The eigenstates are of the same construction as the non-relativistic ones.

(iii) The centres of momenta are the physically observable ones and their equations of motion describe the actual motion of the centres of mass of the system.

The disadvantages are:

(i) The invariant T of equation (5) is physically meaningless and represents neither the kinetic energy nor the total mass.

(ii) The statistical phase space factor required to obtain cross sections proves hopelessly complicated to evaluate.

(iii) The centre-of-mass energy (3) is not a direct eigenvalue of any conceivable component of the operator (5), and is extremely difficult to extract from the eigenvalue equation.

(iv) In quantum mechanics, the matrix $\boldsymbol{M}$ would become a time-dependent operator because physical masses are proportional to total energies.

(*b*) *Proper Barycentric System*

The advantages are:

(i) The phase space factors are readily evaluated.

(ii) The eigenvalue equation (5) has the meaning that T is half the total proper mass of the system.

(iii) The problem is a complete four-space analogue of the non-relativistic case.

(iv) The matrix $\boldsymbol{M}$ is a C-number and not a differential operator.

The disadvantages are:

(i) The appropriate number of degrees of freedom is not obtained directly from the transformations and constraints. In particular, the mass shell constraints (2) are most difficult to apply, and make the eigenvalue equation very involved.

(ii) The system described is not directly observable because the barycentric coordinates do not describe the motion of actual centres of mass.

(iii) The eigenstates of the four-component pseudo-Cartesian operators obtained contain coupled Gegenbauer polynomials, whose arguments are the four-space equivalent of an angle. These polynomials, because of the hyperbolic nature of the pseudo-angle argument, do not possess the orthogonality properties of the non-relativistic analogue over the physical angular region. Furthermore, the quantum number obtained as the relativistic analogue of orbital angular momentum is little understood and unrelated to experimental observations.

(iv) The transformation (4) yields relative time coordinates whose physical interpretation remains obscure.

Each approach has its own limitations, which remove the possibility of application to multiparticle systems, so let us now consider some of the alternatives. Macfarlane (1961) has given a derivation of a partial wave representation in which there are $n-3$ energies as arguments of the projected eigenstates. Hence one cannot relate scattering to production in a simple way because the scattering partial wave amplitude depends only on W.

Furthermore his coordinates may not yield the normal spherical harmonics as eigenstates when the corresponding operators are separated from a kinetic energy or mass operator, and his states have not been shown to represent eigenstates of physical orbital angular momentum operators.

To summarize, the available methods of coupling do not generalize readily to a covariant formalism, and no satisfactory alternative exists at present. Let us deduce some of the properties such eigenstates should possess.

IV. Phase Space

We can now take account of the kinematical constraints. To conserve angular momentum and to ensure that the eigenstates of angular momentum are well defined and observable, they must be projected from an amplitude in such a way as to make the partial waves relativistically orthogonal and normalizable. To illustrate

that this requirement is not readily obtained we can evaluate the volume element for the three-body final state. In the notation of Lardner (1961*a*, 1961*b*),

$$\begin{aligned} \mathrm{d}\Omega = {} & \mathrm{d}^4\boldsymbol{p}_3\,\mathrm{d}^4\boldsymbol{p}_4\,\mathrm{d}^4\boldsymbol{p}_5\,\delta(\boldsymbol{p}_3^2-m_3^2)\,\delta(\boldsymbol{p}_4^2-m_4^2)\,\delta(\boldsymbol{p}_5^2-m_5^2) \\ & \times\delta(\boldsymbol{p}_1+\boldsymbol{p}_2-\boldsymbol{p}_3-\boldsymbol{p}_4-\boldsymbol{p}_5)\,\theta(p_3^0)\,\theta(p_4^0)\,\theta(p_5^0)\,, \end{aligned} \tag{6}$$

where $\boldsymbol{p}_i = (\boldsymbol{q}_i, p_i^0)$.

Elimination of the mass shell δ-functions and the conservation of energy and momentum conditions yields

$$\mathrm{d}\Omega = 2^{-3}\,\frac{q_3^2\,q_4^2\,q_5^2}{E_3\,E_4\,E_5}\,\frac{\mathrm{d}z_{35}\,\mathrm{d}z_{45}\,\mathrm{d}\phi_{35}\,\mathrm{d}z_{15}\,\mathrm{d}\phi_{15}}{q_3(1-z_{35}^2)^{\frac{1}{2}}\,.\,\Delta}\,,$$

where

$$\Delta = \begin{vmatrix} \dfrac{q_3}{E_3} & \dfrac{q_4}{E_4} & \dfrac{q_5}{E_5} \\ z_{35} & z_{45} & 1 \\ (1-z_{35}^2)^{\frac{1}{2}} & (1-z_{45}^2)^{\frac{1}{2}} & 0 \end{vmatrix},$$

which is of the general form

$$\mathrm{d}\Omega = J_3(z_{35}, z_{45}, \phi_{35}, z_{15}, \phi_{15}, W)\,\mathrm{d}z_{35}\,\mathrm{d}z_{45}\,\mathrm{d}\phi_{35}\,\mathrm{d}z_{15}\,\mathrm{d}\phi_{15}\,, \tag{7}$$

q_5, z_{15}, and ϕ_{15} being the polar coordinates of $\boldsymbol{q}_5$ relative to $\boldsymbol{q}_1$ along the z axis. In general, a Jacobian J_f ($3n-10$ variables) is obtained after eliminating the conservation laws from the volume element for an f-particle state. It turns out that ϕ_{15} is a redundant variable in the above example and that the matrix element is independent of it.

Some properties of the J_f are as follows.

(i) They are invariant under general Lorentz transformations. In particular, under ordinary rotations one can see that J_3 (equation (7)) is independent of $(z_{15}, \phi_{15}, \phi_{35})$ and all relative orientations of rotated axes can be specified by these three angles.

(ii) No appeal to special Lorentz frames of reference can remove the dependence of J_f on particle velocities, and under such transformations $J_f\,\mathrm{d}\Omega = J_f'\,\mathrm{d}\Omega'$, where the dash denotes reference to the transformed set of coordinates.

Let A_{fi} be the amplitude for all transitions between a state of i particles, to a state of f particles. In order to make the eigenfunctions of total angular momentum orthogonal with respect to integrations over the variables in the above example, one should expand the amplitude $(J_i J_f)^{\frac{1}{2}} A_f$ directly into orthogonal partial waves. The Jacobians J_i, J_f are what is known as "phase space factors".

V. Projections of Eigenstates

Taking account of all of the above constraints, intuitive arguments are put forward as to the general structure of a partial wave amplitude. It is assumed that a set of angles and pseudo angles may be defined that, together with W, specify all degrees of freedom.

Angles that refer to relative orientations of particles and cannot be changed without altering the energy of one or more particles are described as non-Eulerian. The partial wave eigenstate of orbital angular momentum, however derived, is denoted by $R^l_{m'm}(w)$, where w is a set of two angles defining relative orientations. When all such w are fixed and rotations of axes in three-space are carried out, the w do not vary, while the effect on the set of coordinates is that three degrees of freedom that describe the orientation of the entire system in space are altered. These degrees of freedom are represented by Euler angles (α, β, γ) after Edmunds (1957), and we employ his rotation group operators $D^l_{m'm}(\alpha, \beta, \gamma)$ when determining the effect of rotations; α rotates the axes in the xy plane, β rotates the z axis to align the old with the new z axis, and γ further rotates the axes in the xy plane to complete the alignment.

It is assumed that for non-Eulerian angles, a generalized spherical harmonic $Y^M_L(\Omega)$ can be defined in terms of the individual $R^l_{m'm}$, in the same manner as occurs for non-relativistic theory, as follows

$$Y^M_L(\Omega) = \prod_{v=1}^{f} \sum_{m'_v} (l_1 m_1 l_2 m_2 \ldots | LM)\, S^{l_v}_{m_v m'_v}(0, \theta_v, \phi_v)\,, \tag{8}$$

where

$$\theta_f = 0\,, \quad \phi_f = 0\,, \qquad S^l_{m'm}(\Omega) = (i)^l \{(2l+1)/4\pi\}^{\frac{1}{2}} R^l_{m'm}(\Omega)\,.$$

This is achieved by coupling the various substates $S^l_{m'm}(\Omega)$ with the aid of vector coupling coefficients $(l_1 m_1 l_2 m_2 \ldots | LM)$ to give a state of well-defined orbital angular momentum L, and z component M. In the treatment given by Macfarlane, one would have

$$S^l_{0m}(0, \theta, \phi) = (i)^l Y^m_l(\theta, \phi)\,,$$

where $Y^m_l(\beta, \gamma)$ is the usual spherical harmonic and θ, ϕ are invariantly defined pseudo angles.

It is essential that eigenstates such as (8) should be projected from transition amplitudes in such a way that total angular momentum is conserved between initial and final states. Therefore, the set of reference axes oriented in a special way with respect to particle trajectories in the final state may be related to a similar set oriented with respect to the initial state, by the rotation group operators, such that

$$S^l_{mm'}(\theta_i, \phi_i) = D^l_{m'm}(\alpha, \beta, \gamma)\, S^l_{m'm}(\theta_f, \phi_f)\,. \tag{9}$$

(θ_i, ϕ_i) and (θ_f, ϕ_f) are coordinates specifying the orientation of vectors characterizing initial and final particles respectively. However, all degrees of freedom referring to relative particle orientations and momenta remain unaltered by the rotation (9).

To complicate matters, real space angular arguments are insufficient in number to fix the system in three-momentum space when there are more than three particles present, and they must be supplemented by pseudo angles. As already pointed out, a precise structure of these coordinates has not been determined.

Following the above general properties, we write the component of the amplitude, which is invariant under three-space rotations of axes, as the resultant

eigenvector that arises from the coupling of all those spherical harmonics (8) that have arguments invariant under rotations,

$$\psi_n(L,M,\Omega,\ldots\Omega_k) = J_n^{-\frac{1}{2}} \prod_{j=1}^{k} \sum_{m''_j=-|l_j+L_{j+1}|}^{|l_j+L_{j+1}|} (l_j m_j L_{j-1}|L_j M_j) \times S^{l_j}_{m_j,m''_j}(\Omega_j), \quad (10)$$

where (l_j, m_j) refer to the orbital angular momentum and its z component respectively, of a set of uncoupled eigenstates, while (L_j, M_j) refer to the resultant states, ordered in a specific hierarchy of coupling. The index k refers to the invariant degrees of freedom, of which there are $3n-6$, where n is the number of particles in the state. There may be one or more angles involved in the Ω_j, but only one angle in each is a nonredundant degree of freedom. For scattering one has $k = 0$; hence

$$\psi_2(L,M,\Omega) = J_2^{-\frac{1}{2}}.$$

The production amplitude ${}_fA_i$ is written as

$$A_{fi} = \sum_L \psi_f(LM;\mathbf{\Omega})\, {}_fA_i(LM;W)\, D^L_{MM'}(W_{fi})\, \psi_i(LM';\mathbf{\Omega}'), \quad (11)$$

where we have set (θ_f, ϕ_f) and (θ_1, ϕ_1) all to zero in (9) and where $\mathbf{\Omega}$ contains $3f-6$ components and $\mathbf{\Omega}'$ contains $3i-6$ components. ${}_fA_i(LM;W)$ is the production partial wave amplitude. For particles with spin, a matrix representation must be used. The argument W_{fi} is (α,β,γ) as in (9).

One further vital property of the ψ_n must be enforced before further progress can be made. They must be orthogonal functions when integrated over the phase space interval for the state, that is

$$\int d\mathbf{\Omega}.\psi^*_n(LM;\mathbf{\Omega}).\psi_n(L'M';\Omega) = \prod_{j=1}^{k} \delta(l_j,l'_j)\,\delta(m_j,m'_j)$$
$$= \delta(L,L')\,\delta(M,M'). \quad (12)$$

This property will ensure that angular momentum is conserved between vertices in an interaction involving intermediate states. Thus the final constraint of conservation of angular momentum has been taken into account.

VI. Concluding Remarks

General properties of a partial wave production formalism have been outlined and the difficulties encountered in finding representations discussed. In Part II the unitary principle is investigated using methods described in the present paper, and a compound state formalism is derived, similar to the one obtained from R matrix theory.

VII. References

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